Toric Varieties *

Zhiyuan Bai

Compiled on March 14, 2023

This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Toric Varieties* in Lent 2023. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

Contents

*Based on the lectures under the same name taught by Dr. T. Gräfnitz in Lent 2023.

1 Toric Varieties

Definition 1.1. The n-dimensional algebraic torus is $T = T_n = (\mathbb{C}^\times)^n$.

Remark. 1. $T = \operatorname{Spec} \mathbb{C}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}] \cong \mathbb{V}(X_1 \cdots X_{n+1} - 1) \subset \mathbb{A}^{n+1}$ is an affine variety.
2. T has the structure of an abelian group induced by coordinate-wise multiplication. Since this multiplication and its inverse are morphisms, this makes T an algebraic group.

Definition 1.2. A(n n-dimensional) toric variety is a variety X that contains an open dense subset isomorphic to T, equipped with an (algebraic) action $T \times X \to X$ that restricts to the multiplication action $T \times T \to T$.

In fancier terms, a toric variety is an "equivariant enlargement of a torus".

Example 1.1. 1. T itself is a toric variety.
2. $\mathbb{P}^n, \mathbb{A}^n$ and products of them are toric varieties, with the obvious tori.
3. The cusp $\{x^3 = y^2\} \subset \mathbb{A}^2$ is a toric variety, with the torus given by removing the origin.
4. $\{xy = zw\} \subset \mathbb{A}^4$ is a toric variety (find the torus for an easy exercise, I suppose).

Remark. Toric varieties (of the same dimension) only differ along their boundary $\partial X = X \setminus T$.

We'll show that toric varieties are defined by binomial equations, i.e. stuff like $xy = zw$. They are also stratified by torus orbits, which gives rise to some interesting combinatorics: It turns out that normal affine toric varieties correspond to "strictly convex polycones", which we will define in the time to come; Normal projective toric varieties, on the other hand, correspond to "polytopes"; In general, normal toric varieties correspond to these objects called "fans".
Toric varieties compose a simple class of varieties whose geometry corresponds to combinatorics. That makes them relatively simple for computations, and therefore a great testing ground for new stuff. More importantly, many more complicated varieties can be degenerated to (a union of) toric varieties.

2 Affine Toric Varieties; Lattices; Binomial Ideals; Semigroups

Definition 2.1. A lattice M is a free abelian group of finite rank.

For a torus $T = (C^\times)^r$, we have some natural lattices assoicated to it.

Definition 2.2. A character of T is a morphism of algebraic groups (a morphism that happens to be a group homomorphism at the same time) $\chi : T \to T_1 = \mathbb{C}^\times$.

Every $m = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ defines a character $\chi^m : T \to T_1, (t_1, \ldots, t_r) \mapsto t^m = t_1^{a_1} \cdots t_r^{a_r}$. In fact, every character of T arises this way (nontrivial but not hard, i.e. an exercise).

Definition 2.3. The set M of all characters of T is known as the character lattice of T. It is isomorphic to $\mathbb{Z}^r$.

Definition 2.4. A one-parameter subgroup of T is a morphism of algebraic groups $T_1 \to T$.

Any $n = (b_1, \ldots, b_r) \in \mathbb{Z}^r$ gives rise to $\lambda^n : T_1 \to T, t \mapsto (t^{b_1}, \ldots, t^{b_r})$, and guess what, all one-parameter subgroups arise this way.

Definition 2.5. The set N of all one-parameter subgroups of T is known as the subgroup lattice of T.

We have a natural bilinear pairing $\langle -, - \rangle : M \times N \to \mathbb{Z}, \langle (a_i), (b_i) \rangle = \sum_i a_i b_i$. We can recover T from N by $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times \to T, n \otimes t \mapsto \lambda^n(t)$.

Definition 2.6. For $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$, we can define a morphism of algebraic groups $\Phi_{\mathcal{A}} : T_N \to \mathbb{C}^s, t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_r}(t))$. We define $X_{\mathcal{A}}$ to be the (Zariski) closure of $\operatorname{Im} \Phi_{\mathcal{A}}$.

Remark. $X_{\mathcal{A}}$ is always a toric variety (as we'll probe later). We call $\mathbb{Z}\mathcal{A}$ its character lattice.

We'll show that $X_{\mathcal{A}}$ is the vanishing locus of a "lattice ideal", which correspond to binomial ideals. This will establish the fact that every affine toric variety is of the form $\operatorname{Spec} \mathbb{C}[S]$ for an affine semigroup S. Furthermore, X is normal if and only if S is saturated, in the sense that $S = \sigma^\vee \cap N$ for some cone $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$.

Definition 2.7. A polyhedral cone $\sigma \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$ is a finite intersection of half-spaces (i.e. one side of a linear subspace of codimension 1). It is convex if $m_1, m_2 \in \sigma \implies m_1 + m_2 \in \sigma$, strictly convex if it contains no linear subspace.
The dual of σ is $\sigma^\vee \subset N_{\mathbb{R}}$ given by $\{n \in N_{\mathbb{R}} : \langle n, m \rangle \geq 0\}$.

Definition 2.8. The toric variety associated to σ is $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap N]$.

We should recall the definition of a semigroup ring: $\mathbb{C}[S] = \langle x^n : n \in S \rangle$.

Example 2.1. If σ were the first quadrant on the plane, then its dual is itself and we simply get $X_\sigma = \mathbb{A}^2$.

Definition 2.9. A semigroup is affine if it is finitely generated, integral (i.e. its group-ification map is injective), and torsion-free (i.e. its group-ification is a free abelian group).

Theorem 2.1. *The followings are equivalent:*
(a) X *is an affine toric variety.*
(b) $X = X_{\mathcal{A}}$ *for some* $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$.
(c) $X = \operatorname{Spec} \mathbb{C}[\mathbb{N}^s]/I$ *for some prime ideal* I *generated by binomials.*
(d) $X = \operatorname{Spec} \mathbb{C}[S]$ *where* S *is an affine semigroup.*

Example 2.2. Take the affine toric variety $X = \{x^3 = y^2\} = \{x^3 - y^2 = 0\} \subset \mathbb{A}^2$. Then $\mathcal{A} = \{2, 3\} \subset M = \mathbb{Z}$, then $X = X_{\mathcal{A}}$. On the other hand, if $S = \langle 2, 3 \rangle \subset \mathbb{N}$, then $X = \operatorname{Spec} \mathbb{C}[S]$.

Proof. (b) $\implies$ (a): $\Phi_{\mathcal{A}} : T_N \to (\mathbb{C}^\times)^s$ is a map of tori, so its image $T = \Phi_{\mathcal{A}}(T_N)$ is a torus which is closed in $(\mathbb{C}^\times)^s$. In particular, it acts on $\mathbb{C}^s$.
Since $X_{\mathcal{A}}$ is the Zariski closure of T in $\mathbb{C}^s$, we immediately see that T is an open dense torus in $X_{\mathcal{A}}$. The action of $t \in T$ on $\mathbb{C}^s$ takes varieties to varieties and $T = tT \subset tX_{\mathcal{A}}$, so $X_{\mathcal{A}} \subset tX_{\mathcal{A}}$. Same argument with t replaced by t^{-1} reveals that $tX_{\mathcal{A}} = X_{\mathcal{A}}$, so the action does extend.
(b) $\implies$ (c): $\Phi_{\mathcal{A}}$ induces a map of character lattices $\hat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \to M, e_i \mapsto m_i$. The subgroup $L = \ker \hat{\Phi}_{\mathcal{A}} \le \mathbb{Z}^s$ encodes linear relations among the m_i's, so that's where we're gonna attempt to find I.
For $l = (l_1, \dots, l_s) \in L$, we have $\sum_i l_i m_i = 0$. Define $l_+ = \sum_{l_i > 0} l_i e_i, l_- = \sum_{l_i<0} l_i e_i$. We claim that $\mathcal{I}(X_{\mathcal{A}}) = I_L = \{x^{l_+} - x^{l_-} : l \in L\} = \{x^\alpha - x^\beta : \alpha - \beta \in L\}$. It's clear that $\mathcal{I}(X_{\mathcal{A}}) \supset I_L$. Suppose $I_L \neq \mathcal{I}(X_{\mathcal{A}})$, then choose $f \in I(X_{\mathcal{A}}) \setminus I_L$ with minimal leading monomial $x^\alpha = \prod_i x_i^{\alpha_i}$ (with respect to a chosen order on the monomials, e.g. graded lexicographic). Rescale so that the leading coefficient is 1. But $f(t^{m_1}, \dots, t^{m_s}) = 0$, so there is some $x^\beta = \prod_i x_i^{\beta_i} < x^\alpha$ such that $\prod_i (t^{m_i})^{\alpha_i} = \prod_i (t^{m_i})^{\beta_i}$. So $\alpha - \beta \in L$ and therefore $x^\alpha - x^\beta \in I_L$. So $f - x^\alpha + x^\beta \in I(X_\alpha) \setminus I_L$, contradicting mininmality.
(c) $\implies$ (b): Suppose $I = \{x^{\alpha_i} - x^{\beta_i} : i \in I\} \subset \mathbb{C}[\mathbb{N}^s]$ is prime. $\mathbb{V}(I) \cap (\mathbb{C}^\times)^s$ is an nonempty subgroup fo $(\mathbb{C}^\times)^s$. Projecting down to the i-th coordinate of $(\mathbb{C}^\times)^s$ gives rise to a character $\chi^{m_i} : T \to (\mathbb{C}^\times)^s \to \mathbb{C}^\times$ for some $m_i \in M$. Then $\mathbb{V}(I) = X_{\mathcal{A}}$ where $\mathcal{A} = \{m_1, \dots, m_s\}$.
(b) $\iff$ (d): S is an affine semigroup iff $S = \mathbb{N}\mathcal{A} \subset \mathbb{Z}\mathcal{A} = S^{\mathrm{gp}} = \mathbb{Z}^r$. $\Phi_{\mathcal{A}}$ corresponds to $\pi : \mathbb{C}[\mathbb{N}^s] \to \mathbb{C}[M], x_i \mapsto \chi^{m_i}$. And $\mathbb{C}[S] = \operatorname{Im} \pi \cong \mathbb{C}[\mathbb{N}^s]/\ker \pi = \mathbb{C}[\mathbb{N}^s]/I(X_{\mathcal{A}})$.
(a) $\implies$ (d): Suppose $T_N \subset X$ is a dense torus. This makes $\mathbb{C}[X]$ a subalgebra of $\mathbb{C}[M]$ which is stable under the T_N-action. We claim that $\mathbb{C}[X] = \bigoplus_{\chi^m \in \mathbb{C}[X]} \mathbb{C}\chi^m$. Once we know this, we see thar $\mathbb{C}[X] = \mathbb{C}[S]$ for $S = \{m \in M : \chi^m \in \mathbb{C}[X]\}$. Since $\mathbb{C}[X]$ is finitely generated, there exists $f_1, \dots, f_s \in \mathbb{C}[X]$ such that $\mathbb{C}[X] = \mathbb{C}[f_1, \dots, f_s]$. Express f_i in terms of characters shows that S is finitely generated, and it's immediate that S is integral and torsion-free.
The idea of proving the claim is the observation that $\mathbb{C}[X]$ consists of those functions on T_N that extends to polynomials on X. Since X is toric, such functions are characterised by its characters. As this is pretty cool, we state it as a separated lemma. $\square$

Lemma 2.2. *Suppose $A \subset \mathbb{C}[M]$ is a subspace stable under T_N-action, then $A = \bigoplus_{\chi^m \in A} \mathbb{C}\chi^m$.*

Proof. It's clear that $A \supset \bigoplus_{\chi^m \in A} \mathbb{C}\chi^m$. Conversely, pick any nonzero $f = \sum_{m \in \mathcal{B}} c_m \chi^m \in A \subset \mathbb{C}[M]$ for $\mathcal{B} \subset M$ finite. Then $f \in A \cap B$ where $B = \operatorname{Span}\{\chi^m : m \in \mathcal{B}\}$.
But $A \cap B$ is again stable under T_N-action, and it's finite-dimensional. Since the characters are the simultaneous eigenvectors of the toric action, $A \cap B$ must be spanned by them and therefore $f \in \bigoplus_{\chi^m \in A} \mathbb{C}\chi^m$. $\square$

3 Normal Affine Toric Varieties and Cones

Definition 3.1. A (convex polyhedral) cone in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$ is a finite intersection of half-spaces.

For a finite subset $S \subset N_{\mathbb{R}}$, its cone is $\text{Cone}(S) = \{\sum_{n \in S} \lambda_n n \geq 0 : \lambda_n \geq 0\}$. A cone is rational if it has the form $\text{Cone}(S)$.
For a cone σ, its dual cone is $\sigma^\vee = \{m \in M_{\mathbb{R}} : \forall n \in \sigma, \langle m, n \rangle \geq 0\}$. The cone σ is called strictly convex if $\dim \sigma^\vee = r$.

Equivalently, σ is strictly convex if it contains no positive dimension subspace of $N_{\mathbb{R}}$, or eqivalently $\{0\}$ is a face of σ (i.e. the intersection of σ with a hyperplane).

Definition 3.2. Let σ be a rational complex polyhedral cone. Its associated toric variety is $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$.

Example 3.1. Take $\sigma = \text{Cone}\{(1,0),(1,2)\}$, then $\sigma^\vee = \text{Cone}\{(0,1),(2,-1)\}$. So $\sigma^\vee \cap M = \langle a, b, c \rangle$ where $a = (0,1), b = (1,0), c = (2,-1)$. Another description would be $\sigma^\vee \cap M = \mathbb{N}^3 / \langle a + c - 2b \rangle$, so $X_\sigma = \operatorname{Spec} \mathbb{C}[x^{(0,1)}, x^{(1,0)}, x^{(2,-1)}] = \operatorname{Spec} \mathbb{C}[x,y,z]/(xz - y^2)$.

Theorem 3.1. *The followings are equivalent:*
(a) X is a normal affine toric variety.
(b) $X = \operatorname{Spec} \mathbb{C}[S]$ for some affine semigroup $S \subset M$ which is saturated, in the sense that for all $k \in \mathbb{N}_{>0}$ and $m \in M$, we have $km \in S \implies m \in S$.
(c) $X = X_\sigma$ for some strictly convex (rational polyhedral) cone σ.

Proof. (a) $\implies$ (b): For an affine semigroup S, $\operatorname{Spec} \mathbb{C}[S]$ is normal iff $\mathbb{C}[S]$ is integrally closed. Set $R = \mathbb{C}[S]$ and $Q = \text{FF}(R)$. Then $\mathbb{C}[S]$ is integrally closed iff whenever $q^n + r_1 q^{n-1} + \cdots + r_n = 0$ for some $q \in Q, r_i \in R$ we have $q \in R$.
Now, suppose $km \in S$ for some $k \in \mathbb{N}_{>0}$ and $m \in M$. The character χ^m is a polynomial function on T_N, hence a rational function on X. But $\chi^{km} \in \mathbb{C}[S]$ since $km \in S$, so $\chi^m \in \mathbb{C}[S]$ since it is a root of $x^k - \chi^{km}$. This means that $m \in S$.
(b) $\implies$ (c): Since S is affine, $S = \mathbb{N}\mathcal{A}$ for some finite $\mathcal{A} \subset M$. So $S \subset \sigma^\vee \cap M$ for $\sigma^\vee = \text{Cone}(\mathcal{A})$. Clearly $\mathbb{Z}\mathcal{A} = r$, so $\dim \sigma^\vee = r$ and therefore σ is strictly convex. Also, since S is saturated and $\dim \sigma^\vee = r$, we must have $S = \sigma^\vee \cap M$.
(c) $\implies$ (a): Let $\rho_1, \ldots, \rho_k$ be the rays (i.e. one-dimensional faces) of σ. Then σ is generated by its rays, so $\sigma^\vee = \bigcap_i \rho_i^\vee$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M] = \bigcap_i \mathbb{C}[S_{\rho_i}]$. So $\mathbb{C}[S_\sigma]$ is integrally closed iff each $\mathbb{C}[S_{\rho_i}]$ is.
Let $n \in N$ be the generator of ρ_i which is primitive in the sense that we can find a basis $e_1, \ldots, e_r$ of N with $e_1 = n$. Then $\mathbb{C}[S_{\rho_i}] = \mathbb{C}[x_1, x_2^{\pm n}, \ldots, x_r^{\pm n}]$, which is a localisation of $\mathbb{C}[x_1, \ldots, x_r]$, hence normal. □

What's the inuition behind these? The "new points" in X_σ are $\lim_{t \to 0} \lambda^n(t)$ for $n \in \sigma$, which corresponds to strict faces of σ, which in turn corresponds to $\sigma^\vee \setminus \{0\}$.

4 Normal Toric Varieties and Fans

We'll obtain more general, not necessarily affine, toric varieties by gluing affine varieties. These correspond to gluing cones to a fan.

Definition 4.1. A fan in $N_{\mathbb{R}}$ is a finite set Σ such that:
1. Every $\sigma \in \Sigma$ is a strictly convex rational polyhedral cone.
2. For all $\sigma \in \Sigma$, each face of σ too is in Σ.
3. For all $\sigma \cap \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1, σ_2.

Definition 4.2. Given a fan Σ, $\Sigma^{[k]}$ is the set of k-dimensoinal cones of Σ. The support of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subset N_{\mathbb{R}}$. Σ is complete if $|\Sigma| = N_{\mathbb{R}}$.

Proposition 4.1. *Suppose τ is a face of σ with $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee \cap M$, where $H_m = \{n \in N_{\mathbb{R}} : \langle m, n\rangle = 0\}$. Then $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$.*

Proof. We certainly have $S_\sigma \subset S_\tau$. Furthermore, since $\langle m, n\rangle = 0$ for all $n \in \tau$, we have $\pm m \in \tau^\vee$. So $S_\sigma + \mathbb{Z}m \subset S_\tau$. This is in fact an equality (which shows the proposition). Indeed, fix $S \subset N$ with $\sigma = \mathrm{Cone}(S)$. Pick $m' \in S_\tau$, then $m' + Cm \in S_\sigma$ for $C = \max\{|\langle m', n\rangle| : n \in S\} \in \mathbb{N}$. ☐

Proposition 4.2. *If $\tau = \sigma_1 \cap \sigma_2$, then $S_\tau = S_{\sigma_1} + S_{\sigma_2}$.*

Proof. Since $\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee$, we have $S_\tau \supset S_{\sigma_1} + S_{\sigma_2}$. Conversely, note that $\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$ for any $m \in \mathrm{Int}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$. Take such an m, then by what we did in the preceding proposition, any $p \in S_\tau$ has some $q \in S_{\sigma_1}$ and $l \in \mathbb{N}$ such that $p = q - lm$. But $-m \in \sigma_2^\vee$ implies $-m \in S_{\sigma_2}$, so $p \in S_{\sigma_1} + S_{\sigma_2}$. ☐

Suppose we now just have a fan Σ. We have a collection $\{X_\sigma\}$ of affine toric varieties. And for any pair σ_1, σ_2, X_{σ_1} and X_{σ_2} share an open subset $X_{\sigma_1\cap\sigma_2}$. We also get isomorphisms $g_{\sigma_1\sigma_2} : X_{\sigma_1\cap\sigma_2} \to X_{\sigma_2\cap\sigma_1}$ such that $g_{\sigma_1\sigma_2} = g^{-1}_{\sigma_2\sigma_1}$, $g_{\sigma_1\sigma_2}(X_{\sigma_1\cap\sigma_2} \cap X_{\sigma_1\cap\sigma_3}) = X_{\sigma_2\cap\sigma_1} \cap X_{\sigma_2\cap\sigma_3}$ and $g_{\sigma_1\sigma_3} = g_{\sigma_2\sigma_3} \circ g_{\sigma_1\sigma_2}$. These give a set of gluing data.

Definition 4.3. The toric variety associated to Σ is $X_\Sigma = \coprod_{\sigma\in\Sigma} X_\sigma / \sim$ such that $x \sim y$ iff there is some σ_1, σ_2 such that $x \in X_{\sigma_1}, y \in X_{\sigma_2}$ and $y = g_{\sigma_1\sigma_2}(x)$.

Example 4.1. 1. Consider the fan whose cones are the division of $\mathbb{R}^2$ by the rays to the directions $(1, 0), (0, 1), (-1, -1)$. We get affine toric varieties $X_{\sigma_0} = \mathrm{Spec}\,\mathbb{C}[x, y], X_{\sigma_1} = \mathrm{Spec}\,\mathbb{C}[x^{-1}, x^{-1}y], X_{\sigma_2} = \mathrm{Spec}\,\mathbb{C}[xy^{-1}, y^{-1}]$ corresponding to the largest cones.
Then the transition functions are given by the isomorphisms $g^*_{01} : \mathbb{C}[x, y]_x \to \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}}, g^*_{02} : \mathbb{C}[x, y]_y \to \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}}, g^*_{12} : \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \to \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}$.
The identification $x \mapsto x_1/x_0, y \mapsto x_2/x_0$ then reveals that $X_\Sigma \cong \mathbb{P}^2$.
2. Similarly, the fan obtained by dividing the plane with rays to the directions $(1, 0), (0, 1), (-a, -b)$ gives us the weighted projective plane $\mathbb{P}(1, a, b)$.

Proposition 4.3. $X_{\Sigma_1\times\Sigma_2} = X_{\Sigma_1} \times X_{\Sigma_2}$.

Proof. Exercise. ☐

Example 4.2. 1. $\mathbb{P}^1$ corresponds to the fan obtained by dividing the line at the origin.
2. By the proposition, we know that $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the fan on the plane describing the quadrants.

Theorem 4.4. *The followings are equivalent:*
(a) X is a separated normal toric variety.
(b) $X = X_\Sigma$ for some fan Σ.

Proof. (b) $\implies$ (a): Clear.
(a) $\implies$ (b): Omitted (Sumihiro '74). ☐

Proposition 4.5. *For a cone σ, we have $n \in \sigma$ iff $\lim_{t\to 0} \lambda^n(t)$ exists in X_σ.*

Proof. Consider the composition of maps

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\lambda^n"] \arrow[rr, bend right, "t\mapsto t^{\langle m,n\rangle}"'] & T \arrow[r, "\chi^m"] & \mathbb{C}^\times \end{tikzcd}$$

Now the limit exists iff for any $m \in \mathbb{C}$, $\lim_{t\to 0} \chi^m(\lambda^n(t))$ exists in $\mathbb{C}$, which happens iff $\langle m, n\rangle \geq 0$ for any $m \in S_\sigma = \sigma^\vee \cap M$, i.e. that $n \in (\sigma^\vee)^\vee = \sigma$. $\square$

Corollary 4.6. *There is a bijection between cones in Σ and T_N-orbits in X_Σ.*

Definition 4.4. The closure of a torus orbit is called a toric stratum.

By what we've discussed, they correspond to cones in a fan.

Example 4.3. Consider the usual fan for $\mathbb{P}^2$, then the cone σ which is the ray to the direction $(1, 0)$ gives rise to the torus orbit $\{(x_0, 0, x_2) : x_0, x_2 \neq 0\}$, whose limit under any $n \in \sigma$ is $(0, 1, 0)$.

Now let's talk about properness of toric varieties. Needless to say, the valuative criterion is booked for this occasion.
For an DVR R (e.g. $\mathbb{C}[[t]]$) and $K = \mathrm{FF}(R)$, the inclusion $R \hookrightarrow K$ gives rise to $\operatorname{Spec} K \to \operatorname{Spec} R$. This is interpreted by thinking of R as a disc near 0 and $\operatorname{Spec} K$ the disc punctured at 0. The extenede valuation on K records asymptotics of functions on the punctured unit disc near 0.
Suppose we have a morphism $f : X \to Y$ of finite type with X Noetherian.

Proposition 4.7 (Valuative Criterion)**.** *f is separated if and only if for any DVR R with fraction field K and a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is at most one map filling in the dashed arrow. X is proper if and only if there exists exactly one such map.

Theorem 4.8. *X_Σ is proper iff Σ is complete.*

Proof. Suppose Σ is not complete. Pick $n \in N_\mathbb{R} \setminus |\Sigma|$, then $\mathbb{C}^\times \to T_N \hookrightarrow X_\Sigma$ (where the first arrow is λ^n) has no limit as $t \to 0$, so the valuative criterion fails.
Conversely, suppose Σ is complete. WLOG the image of $\operatorname{Spec} K$ lies in a torus T_N. We have the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & T_N \arrow[r, hook] & X_\Sigma \\ \operatorname{Spec} R \arrow[urr, dashed] & & \end{tikzcd}$$

We'll need to show that the dashed arrow can be filled in. The map $\operatorname{Spec} K \to T_N = \operatorname{Spec} \mathbb{C}[M]$ gives a function $M \to K$. Since M is a group, this function lands in $K^\times$. Consider

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\text{some } n \in N"'] & K^\times \arrow[r, "\text{val}"] & \mathbb{Z} \end{tikzcd}$$

As Σ is complete, $n \in \sigma$ for some $\sigma \in \Sigma$. So we further refine the morphism as $S_\sigma \hookrightarrow M \to K$, which dually looks like $\operatorname{Spec} K \hookrightarrow T_N \to X_\sigma$. If $S_\sigma \to K \to \mathbb{Z}$ lands in $\mathbb{Z}_{\geq 0}$ then we're done. But this is the same map as $S_\sigma \to M \to \mathbb{Z}$ where the second map is given by $n \in \sigma$, hence must be nonnegative. $\square$

5 Toric Morphisms

Definition 5.1. For toric varieties X, Y, a morphism $f : X \to Y$ is a toric morphism if it restricts to a group homomorphism $T_X \to T_Y$.

Consequently, $f(tx) = f(t)f(x)$ for any $t \in T_X$ whenever f is a toric morphism. Note that morphisms of algebraic groups $T_X \to T_Y$ corresponds to homomorphisms $N_X \to N_Y$. The question now is to decide when $T_X \to T_Y$ extends to a toric morphism $X \to Y$.

Definition 5.2. A morphism of fans $(\Sigma_1, N_1) \to (\Sigma_2, N_2)$ is a group homomorphism $N_1 \to N_2$ such that for any cone $\sigma \in \Sigma_1$, there is some $\sigma' \in \Sigma_2$ such that $\phi(\sigma) \subset \sigma'$.

Theorem 5.1. *For $\phi : N_X \to N_Y$, the corresponding $f : T_X \to T_Y$ extends to a toric morphism $X \to Y$ iff ϕ is a fan morphism.*

So there is an equivalence of categories between the category of normal separated toric varieties with toric morphism and fans with fan morphisms.

Proof. Suppose $\phi : (\Sigma_X, N_X) \to (\Sigma_Y, N_Y)$ is a fan morphism. Let $\sigma_X \in \Sigma_X$, then $\phi(\sigma_X) \subset \sigma_Y \in \Sigma_Y$ for some $\sigma_Y \in \Sigma_Y$. So $\phi^\vee : M_Y \to M_X$ maps S_{σ_Y} into S_{σ_X}.
This gives a map $\mathbb{C}[S_{\sigma_Y}] \to \mathbb{C}[S_{\sigma_X}]$ and therefore $X_{\sigma_X} \to X_{\sigma_Y} \subset Y$. This clearly does not depend on the choice of σ_Y. We can then glue them together to $f : X \to Y$ restricting correctly to the tori.
Conversely, suppose f extends to a toric morphism $X \to Y$. Let $\sigma_X \in \Sigma_X$. Consider $x_{\sigma_X} = \lim_{t\to 0} \lim_{t \to 0} \lambda^n(t) \in X_{\sigma_X}$. for any $n \in (\operatorname{Int} \sigma_X) \cap N_X$. Since f is continuous, $f(x_{\sigma_X}) = \lim_{t\to 0} f(\lambda^n(t))$. On the other hand, $\lambda^n(t) \in T_X$ means that $f(\lambda^n(t)) = \lambda^{\phi(n)}(t) \in T_Y$, so $f(x_{\sigma_X}) = \lim_{t\to 0} \lambda^{\phi(n)}(t)$. Suppose $f(x_{\sigma_X}) \in \sigma_Y \in \Sigma_Y$, then this means that $\phi(n) \in \operatorname{Int} \sigma_Y$ for all $n \in (\operatorname{Int} \sigma_X) \cap N_X$. But this is just saying that $\phi(\operatorname{Int} \sigma_X) \subset \operatorname{Int} \sigma_Y$, so $\phi(\sigma_X) \subset \sigma_Y$. $\square$

Example 5.1. 1. If we take the standard fan of $\mathbb{P}^1 \times \mathbb{P}^1$ given by dividing the plane with quadrants, then the projections to $\mathbb{P}^1$ corresponds to the fan morphisms given by projections to the axes. Similarly, if we rotate the arm to the direction $(-1, 0)$ to, say, $(-1, -n)$, then we get what's called a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, which maps to $\mathbb{P}^1$ via the projections to axes too.

2. Consider the standard fan of $\mathbb{P}^2$ which is given by dividing the plane at $(0,1),(1,0),(-1,-1)$. The x-projection to the fan of $\mathbb{P}^1$ is not a fan morphism: Indeed, the cone cut out by $(1,0)$ and $(-1,-1)$ does not map to a fan. It corresponds to a rational map $\mathbb{P}^2 \to \mathbb{P}^1, (x_0 : x_1 : x_2) \mapsto (x_1 : x_2)$.

Despite the second example not being a morphism at all, observe that we can do the following: If we add an arm to the direction $(0,-1)$, then suddenly the map to the fan of $\mathbb{P}^1$ would be a fan morphism.

Definition 5.3. A subdivision (or refinement) is a fan morphism $(\Sigma_1, N_1) \to (\Sigma_2, N_2)$ such that $N_1 \to N_2$ is the identity and $|\Sigma_1| = |\Sigma_2|$.

Example 5.2. The subdivision we described now gives a new toric variety which is essentially $\mathbb{F}_1$, and now the x-projection gives a toric morphism to $\mathbb{P}^1$. Observe also that $\mathbb{F}_1 = \mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$.

Example 5.3. Consider the fan Σ which is the first quadrant, and a subdivision $\Sigma^1 \to \Sigma$ by adding a ray to the ditrection $(1,1)$. Note that $X_\Sigma = \mathbb{A}^2$. We also have $X_{\Sigma'} = \mathrm{Bl}_0 \mathbb{A}^2$. Indeed, let's write σ_1, σ_2 to denote the cones of Σ^1 closer to the x and y axes, respectively. Then $S_{\sigma_1} \cong \operatorname{Spec} \mathbb{C}[x,y,s]/(x-ys) \cong \operatorname{Spec} \mathbb{C}[y, xy^{-1}]$. Similarly, $S_{\sigma_2} \cong \operatorname{Spec} \mathbb{C}[x,y,s^{-1}]/(xs^{-1}-y) \cong \operatorname{Spec} \mathbb{C}[x, x^{-1}y]$. So X_Σ is the vanishing locus of $x\eta - y\xi$ in $\mathbb{A}^2 \times \mathbb{P}^1$ (on which we have coordinates $((x,y),(\xi:\eta))$).

So "adding rays" to a fan corresponds to blow-ups.
Ok now let's state some facts.

Proposition 5.2. *$f : X_{\Sigma'} \to X_\Sigma$ is proper iff $\phi^{-1}(|\Sigma|) = |\Sigma'|$.*

Proof. Valuative criterion. □

Proposition 5.3. *If $\phi : \Sigma \to \Sigma'$ is a subdivision, then the induced map $f : X_\Sigma \to X_{\Sigma'}$ is surjective and birational.*

Proof. It restricts to the identity on the tori, so it is birational. It is also proper by the preceding proposition, in particular it is closed, therefore surjective (as it has dense image). □

now let's look at another situation: What if we kept the fan, but changed the lattice?

Example 5.4. 1. Consider the fan Σ consisting of the cone $\sigma = \mathbb{R}_{\geq 0}$ on $\mathbb{R}$ and the lattices $N = 2\mathbb{Z}, N' = \mathbb{Z}$ (with duals $M = (1/2)\mathbb{Z}, M' = \mathbb{Z}$). We get a map $(N, \Sigma) \to (N', \Sigma' = \Sigma)$ given by inclusion of lattices. This corresponds to a map of rings $\mathbb{C}[S_\sigma] = \mathbb{C}[t] \hookleftarrow \mathbb{C}[S_{\sigma'}] = \mathbb{C}[t^2]$. Geometrically, this is squaring map $\mathbb{A}^1 \to \mathbb{A}^1$, which can be viewed as the ramified cover $\mathbb{A}^1 \to \mathbb{A}^1/\{\pm 1\}$.
2. Take the fan of $\mathbb{A}^2$ generated by the cone spanned by $(1,0),(1,1)$ and the lattice $\mathbb{Z}^2$. Consider now the same fan but on a finer lattice $\mathbb{Z} \oplus (1/2)\mathbb{Z}$, which gives the affine toric variety $\operatorname{Spec} \mathbb{C}[x,y,z]/(xz - y^2)$. This can again be viewed as a quotient $\mathbb{A}^2/\{\pm 1\}$ where $-1(x,y) = (-x,-y)$. Similarly, if we had taken the lattice $\mathbb{Z} \oplus (1/m)\mathbb{Z}$, then we get the quotient $\mathbb{A}^2/\mu_m$ where $\zeta(x,y) = (\zeta x, \zeta^{-1} y)$.

Probably a good idea to (re)introduce some quotient thingies.

Definition 5.4. Suppose G is a finite group acting on an affine scheme $X = \operatorname{Spec} R$, the quotient is $X/G = \operatorname{Spec} R^G$ where $R^G = \{r \in R : \forall g \in G, gr = r\}$.

Example 5.5. Let μ_m act on $\mathbb{A}^2$ as in the earlier example. Then $\mathbb{C}[x,y]^{\mu_m} = \mathbb{C}[x^m, xy, y^m] = \mathbb{C}[x,y,z]/(xz - y^m)$, justifying our intuitive interpretation in the previous example. Note that (when $m > 1$) these are not smooth, but what happens when one adds more lines to the picture that connect the new dots?

6 Toric Singularities

Proposition 6.1. *Suppose $N' \hookrightarrow N$ has finite index and $\sigma \subset N_{\mathbb{R}} = N_{\mathbb{R}'}$ is a (strictly convex rational polyhedral, as usual) cone.*
(a) $G = N/N' \cong \operatorname{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times)$.
(b) G acts on $\mathbb{C}[\sigma^\vee \cap M']$ and $\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$.
(c) $X_{\sigma,N'}/G \cong X_{\sigma,N}$.

Proof. (a) $\mathbb{C}^\times$ is an injective $\mathbb{Z}$-module since it is divisible, so $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ is exact, whence the exact sequence

$$1 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times) \longrightarrow T_{N'} \longrightarrow T_n \longrightarrow 1$$

More explicitly, the isomorphism is given by $N/N' \to \operatorname{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times), [n] \mapsto ([m'] \mapsto \exp(2\pi i \langle m', n\rangle))$.
(b) $T_{N'}$ acts on $X_{\sigma,N'}$, so $G \le T_{N'}$ does as well. This becomes as action of G on $\mathbb{C}[X_{\sigma,N'}] = \mathbb{C}[\sigma^\vee \cap M']$ where the action is essentially $g \cdot \chi^{m'} = g([m'])^{-1}\chi^{m'}$. By definition, $g([m']) = 1$ for all g iff $m' \in M \subset M'$. Therefore $\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$.
(c) Immediate. ☐

Definition 6.1. A cone σ is smooth if it has a minimal generating set that forms part of a $\mathbb{Z}$-basis of N. σ is simplicial if they it has a minimal generating set that forms part of an $\mathbb{R}$-basis of $N_{\mathbb{R}}$.

Example 6.1. The cone given by the area between $(1,0)$ and $(1,m)$ is smooth when $m = 1$ and not smooth when $m \in \mathbb{Z}_{>1}$.

Proposition 6.2. *Let σ be a cone of dimension k over a lattice N of dimension n. Then the followings are equivalent:*
(a) X_σ is smooth.
(b) σ is smooth.
(c) $X_\sigma = \mathbb{C}^k \times (|mathbbC^\times)^{n-k}$.

Proof. (b) $\implies$ (c) $\implies$ (a): Exercise.
(a) $\implies$ (b): Suppose first that $k = n$, then $\sigma^\perp = \{m \in M_{\mathbb{R}} : \langle m, n\rangle = 0\} = 0$. Let $\mathfrak{m}$ be the maximal corresponding to $x_\sigma = \lim_{t\to 0} \lambda^n(t) \in X_\sigma$ for $n \in (\operatorname{Int}\sigma) \cap N$. It must be generated by monomials z^w for $w \in S_\sigma \setminus \{0\}$.
$\mathfrak{m}/\mathfrak{m}^2$ be the cotangent space at x_σ. Since X is smooth, $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = n$.
Now $\mathfrak{m}/\mathfrak{m}^2$ is generated by z^w for $w \in S_\sigma$ such that w cannot be written as $w_1 + w_2$ for $w_1, w_2 \in S_\sigma \setminus \{0\}$.
Since $\dim \sigma = n$, $\sigma^\vee$ is strictly convex. Since $\dim \sigma^\vee = n$, there are at least n generators $w_1, \dots, w_n$. The fact that $\mathfrak{m}/\mathfrak{m}^2$ has dimension n then shows that

these generate S_σ. But S_σ generates M as a group, so $w_1, \ldots, w_n$ generates M. Their duals then give a basis for N that generate σ.
What about the general case? Take $N_1 \leq N$ the smallest saturated sublattice containing a set of generators for σ. As N/N_1 is torsion-free, $N = N_1 \oplus N_2$ for some N_2. Therefore $X_\sigma \cong X_{\sigma,N_1} \times T_{N_2}$ and $\dim \sigma = \dim N_1$. As T_{N_2} is smooth, we conclude the result. $\square$

Proposition 6.3. *Suppose σ is simplicial, then $X_\sigma = (\mathbb{C}^k/G) \times (\mathbb{C}^\times)^{n-k}$ for some finite abelian group G acting on $\mathbb{C}^k$ (algebraically, of course).*

Proof. Suppose $N' \leq N$ is a lattice spanned by generators of σ. Then $\sigma \subset N'_\mathbb{R}$ and $X_{\sigma,N'} = \mathbb{C}^n$. We are done by the Proposition 6.1. $\square$

Definition 6.2. A morphism $\phi : Y \to X$ is a resolution of singularities if ϕ is proper, X is smooth, and ϕ restricts to an isomorphism $Y \setminus \phi^{-1}(X_{\text{sing}}) \cong X \setminus X_{\text{sing}}$.

Definition 6.3. A fan Σ is smooth if every cone $\sigma \in \Sigma$ is smooth.

Corollary 6.4. *X_Σ is smooth iff Σ is smooth.*

Example 6.2. Let σ be the cone spanned by $(1, 0), (1, d)$ for $d \in \mathbb{Z}_{>1}$. Then $X_\sigma = \mathbb{A}^2/\mu_d$ is not in general smooth. But if we form the fan $\tilde{\Sigma}$ obtained by subdividing σ at $(1, k)$ for $1 \leq k \leq d - 1$, then $X_{\tilde{\Sigma}} \to X_\sigma$ is a resolution of singularities.

In general, singularies are hard to classify especially in high dimensions. But we can do something with toric surfaces.

Proposition 6.5. *Suppose $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^2$ is a two-dimensional cone (strictly convex rational polyhedral, as always). Then there is a $\mathbb{Z}$-basis for N such that $\sigma = \text{Cone}(\{(0, 1), (d, -k)\})$ for $0 \leq k < d$.*

Proof. Suppose v_1, v_2 generate σ. Taking one of the basis vectors as v_2, we get $\sigma = \text{Cone}(\{(0, 1), (d, x)\})$ for some x. Now

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ cd + x & 1 \end{pmatrix}$$

So we can cast the Euclidean algorithm to reduce x down to the correct range. $\square$

Remark. $(d, -k)$ and $(d, -k')$ give isomoprhic X_σ if $kk' \equiv 1 \pmod{d}$.

Proposition 6.6. *Let σ be as above. Then $X_\sigma = \mathbb{A}^2/\mu_d$ where μ_d acts by $\zeta \cdot (x, y) = (\zeta x, \zeta^k y)$.*

Proof. $G = N/N' = \mu_d$ acts on $\mathbb{C}[M']$ via $[n] \cdot \chi^{m'} = e^{2\pi i \langle m', n\rangle} \chi^{m'}$. Let m_1, m_2 be the dual basis of $\sigma^\vee$. For $m' \in \sigma^\vee \cap M'$ and $n = je_1$ (for $0 \leq j < d$), we have $\langle m_1, e_1\rangle = 1/d, \langle m_2, e_1\rangle = k/d$.
Consider the isomorphism $\mu_d \to G, e^{2\pi ijk/d} \mapsto [je_1]$. Then $\zeta = e^{2\pi ij/d}$ acts by $\zeta \cdot (x, y) = (e^{2\pi ij/d} x, e^{2\pi ijk/d} y) = (\zeta x, \zeta^k y)$. $\square$

7 Classification of Toric Surfaces

Proposition 7.1. *For each 2-dimensional fan Σ, there exist a smooth refinement $\Sigma' \to \Sigma$.*

Proof. Induction on the quantity $s = s(\Sigma) = \sum_{\sigma\in\Sigma}([N : N_\sigma] - 1)$. If $s = 0$ then Σ is smooth by definition. For $s > 0$, let σ be such that $[N : N_\sigma] > 1$, i.e. σ is not smooth. Then by Proposition 6.5, σ is isomorphic to something of the form $\mathrm{Cone}\{(0,1),(d,-k)\}$. But we can just subdivide at $(1,0)$. This gives a smooth cone $\mathrm{Cone}\{(1,0),(0,1)\}$ a cone $\mathrm{Cone}\{(1,0),(d,-k)\}$ which contributes to a smaller s. □

We can resolve $\sigma = \mathrm{Cone}\{(0,1),(d,-k)\}$ explicitly using what's known as the Hirzebruch-Jung continued fractions. Write $d = b_1k - k_1$ with $0 \le k_1 < k$, $k = b_1k - k_2$, $k_1 = b_3k_2 - k_3$, and so on until we are able to write $k_{r-2} = b_nk_{r-1}$. Note that we always have $b_i \ge 2$.
This procedure gives an expression

$$\frac{d}{k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots}}}$$

and we shall denote the right hand side by $[[b_1,\dots,b_r]]$.

Example 7.1. $7/3 = [[3,2,2]]$.

Write $p_0 = 1, p_1 = b_1, p_i = b_ip_{i-1} - p_{i-2}$ and $q_0 = 0, q_1 = 1, q_i = b_iq_{i-1} - q_{i-2}$, then it's easy to see that

Lemma 7.2. *(a) The sequences $(p_i),(q_i)$ are increasing.*
(b) $p_i/q_i = [[b_1,\dots,b_i]]$.
(c) $p_{i-1}q_i - p_iq_{i-1} = 1$.
(d) $d/k = p_r/q_r < \cdots < p_1/q_1$.

Example 7.2. Again take $7/3 = [[3,2,2]]$, then $p_1 = 3, p_2 = 5, p_3 = 7$ and $q_2 = 2, q_3 = 3$, therefore $p_1/q_1 = 3, p_2/q_2 = 5/2, p_3/q_3 = 7/3$.

Write $u_0 = (0,1)$ and $u_i = (p_{i-1}, -q_{i-1})$. We can form the cones $\sigma_i = \mathrm{Cone}\{u_{i-1},u_i\}$. Set Σ be the fan obtained from these σ_i's, which is indeed a fan by part (a) of the lemma.

Proposition 7.3. *(i) Each σ_i is smooth.*
(ii) $\sigma_i \cap \sigma_{i+1} = \mathrm{Cone}\{u_i\}$.
(iii) $\bigcup_i \sigma_i = \sigma$.
Hence $\Sigma \to \sigma$ is a subdivision by smooth fans, i.e. $X_\Sigma \to X_\sigma$ is a resolution of singularity.

Proof. All follows from the preceding lemma. □

Remark. Recall that $(d,-k)$ and $(d,-k')$ define the same X_σ iff $kk' \equiv 1 \pmod d$. Now this happens iff $d/k = [[b_1,\dots,b_r]], d/k' = [[b_r,\cdots,b_1]]$ for some $b_1,\dots,b_r$. The isomorphisms in fact extend to isomorphisms for their respective Hirzebruch-Jung subdivisions.

We know how to resolve singularities. But do we know if the resolution is given by blow-ups? In general, what does blow-ups look like on fans?

Example 7.3. We saw that $\mathrm{Bl}_0 \mathbb{A}^2$ (can) have the fan given by the cones $\mathrm{Cone}\{(1,0),(1,1)\}$ and $\mathrm{Cone}\{(1,1),(0,1)\}$. However, if we replace $(1,1)$ by some other integral vector, then what we get isn't even necessarily smooth. And if it is not smooth it certainly cannot be $\mathrm{Bl}_0 \mathbb{A}^2$.

Definition 7.1. A star subdivision of a fan Σ at a primitive $v \in |\Sigma| \cap N$ is the subdivision $\Sigma^*(v) \to \Sigma$ given by the cones:
(a) σ for $v \notin \sigma \in \Sigma$;
(b) $\mathrm{Cone}(\tau, v)$ for $v \notin \tau \in \Sigma$ and $\{v\} \cup \tau \leq \sigma \in \Sigma$.
A star division of Σ at $\sigma \in \Sigma$ is $\Sigma^*(\sigma) = \Sigma^*(v)$ where $v = n_1 + \cdots + n_k$ whre n_i are generators of σ.

Lemma 7.4. *$\Sigma^*(v)$ is a subdivision of Σ (in particular a fan).*

Proof. Omitted or exercise. □

Proposition 7.5. *$X_{\Sigma^*(\sigma)} \to X_\Sigma$ is the blow-up of Σ at $X(\sigma)$, where $X(\sigma)$ is the toric stratum (closure of the torus orbit) of σ.*

Proof. WLOG Σ consists a single cone σ. If σ is smooth, then $X_\sigma = \mathbb{A}^k$ where the statement follows from easy computation. The general case is omitted. Go meditate on proving it. □

Remark. 1. More generally, for $a_i \in \mathbb{N}$, we can form the weighted star division where we take $\sum_i a_i n_i$.
2. We can also blow-up along union of toric strata, where we just take more than one v (note that this is different from blowing them up one-by-one).

Example 7.4. Consider the cone $\mathrm{Cone}\{e_1, e_2, e_1 + e_3, e_2 + e_3\}$. This defines the threefold $X_\sigma = \{xy = zw\} \subset \mathbb{A}^4$. We take a subdivision of this by adding in the vertex $v = e_1 + e_2 + e_3$. This gives a fan Σ which is smooth, and the new "ray" in the middle corresponds to the exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. We can remove one of the two diagonals of the cross formed by v and the original generators. The exceptional divisor then get contracted to $\mathbb{P}^1$ in both cases. This is a phenomenon known as a "flop", which is an isomorphism outside a subvariety of codimension 2.

Theorem 7.6. *Every smooth proper toric surface X_Σ is obtained by a blow-up from either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{F}_r$ for $r \geq 2$, at torus fixed points.*

Remark. Recall that $\mathbb{F}_r$ has fan given by dividing the plane at $(1,0), (0,\pm 1)$ and $(-1, r)$.

Lemma 7.7. *Suppose Σ is a smooth complete 2-dimensional fan in $M_\mathbb{R} = \mathbb{R}^2$. Let $u_0, \ldots, u_r$ be the ray generators of Σ (with $u_r = u_0$). Then $u_{i-1} + u_{i+1} = b_i u_i$ for $b_i \in \mathbb{Z}$.*

Proof. u_{i-1}, u_i, u_{i+1} are linearly dependent, so we can write $\alpha u_{i-1} + \beta u_{i+1} = \gamma u_i$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Then

$$1 = \det(u_i | u_{i+1}) = \det(u_i | \beta^{-1}(\gamma u_i - \alpha u_{i-1})) = -\frac{\alpha}{\beta} \det(u_i | u_{i-1}) = \frac{\alpha}{\beta}$$

So $\alpha = \beta$. Play the same game

$$1 = \det(u_{i-1}|u_i) = \det(u_{i-1}|\gamma^{-1}(\alpha u_{i-1} + \beta u_{i+1})) = \frac{\beta}{\gamma} \det(u_{i-1}|u_{i+1}) \in \frac{\beta}{\gamma}\mathbb{Z}$$

So $\beta b_i = \gamma$ for some $b_i \in \mathbb{Z}$. □

Lemma 7.8. *Suppose Σ is a smooth 2-dimensional fan that refines a smooth cone σ, then Σ is obtained from σ from a sequence of star subdivisions.*

Proof. Induction on r which is the number of rays in Σ. When $r = 1$ there's nothing to prove. For $r > 1$, suppose $u_0, \dots, u_r$ are the rays. By the proof of the preceding lemma, we have $u_{i-1} + u_{i+1} = b_i u_i$ for some $b_i \in \mathbb{Z}$. If all b_i are at least 2, then $[[b_1, \dots, b_r]] = d/k$ with $d \geq 2$, contradicting smoothness of σ. So some b_i is 1. Then $\operatorname{Cone}\{u_{i-1}, u_{i+1}\}$ is smooth and adding u_i gives the star subdivision. □

Proof of Theorem 7.6. Let $\Sigma^{[1]} = \{u_0, \dots, u_r\}$ be the rays of Σ, then $u_{i-1} + u_{i+1} = b_i u_i$ for all i. If $b_i = 1$, then X_Σ is the blow-up of a smooth $X_{\Sigma'}$ with fewer rays. So suppose $b_i \neq 1$ for all i.
If $|\Sigma^{[1]}| = 3$, then $X_\Sigma = \mathbb{P}^2$. If $|\Sigma^{[1]}| > 3$, then there is some u_i, u_j with $u_j = -u_i$. WLOG $i = 1$. Note that $j > 2$ since our cones are supposed to be strictly convex.
We have $u_0 + u_2 = b_1 u_1$, so $u_0 = -u_2 + b_1 u_1$ for $b_1 \neq 1$. For $b_1 \geq 2$, Σ is a star subdivision of $\mathbb{F}_r$ with $r = b_1$; For $b_1 \leq -2$, Σ is a star subdivision of $\mathbb{F}_r$ with $r = |b_1|$; For $b_1 = 0$, Σ is a star subdivision of $\mathbb{P}^1 \times \mathbb{P}^1$; For $b_1 = -1$, Σ is a star subdivision of $\mathbb{P}^2$. □

8 Divisors and Line Bundles

Recall:

Definition 8.1. A Weil divisor $D = \sum_i a_i D_i, a_i \in \mathbb{Z}$ on X is a finite formal sum of irreducible hypersurfaces $D_i \subset X$. The group of divisors on X is denoted $\operatorname{Div}(X)$.
For a rational function $f \in H^0(X, \mathscr{K}_X^\times)$, its divisor is the (necessarily finite) sum $\operatorname{div}(f) = \sum_{D \subset X \text{ irred. hypersurface}} \operatorname{ord}_f(D) D$. Divisors of this form are known as principal divisors, which form a subgroup $\operatorname{Div}_0(X) \leq \operatorname{Div}(X)$.
The divisor class group of X is $\operatorname{Cl}(X) = \operatorname{Div}(X) / \operatorname{Div}_0(X)$.

Lemma 8.1. *Suppose R is a UFD, then* $\operatorname{Cl}(\operatorname{Spec} R) = 0$.

Proof. Irreducible hypersurfaces in $\operatorname{Spec} R$ correspond to height 1 primes $I \leq R$. It then has to be principal. Indeed, suppose $r \in I \setminus \{0\}$ has a prime factorisation $r = p_1 \cdots p_r$. Since I is prime, some p_i lives in I, so $0 \leq (p_i) \leq I$ and yet I has height 1, therefore $I = (p_i)$. □

Corollary 8.2. *The class group of a torus is trivial.*

Now fix X a toric variety.

Definition 8.2. An irreducible hypersurface D of X is toric if the toric action on X restricts to a toric action on D. A toric divisor is a finite formal sum of these.

So toric hypersurface correspond to elements of $\Sigma^{[1]}$. For $\rho \in \Sigma^{[1]}$, we write D_ρ for the asscociated toric hypersurface.

Proposition 8.3. $\mathrm{Cl}(X)$ *is generated by toric hypersurfaces.*

Proof. We have the excision sequence

$$\mathrm{Cl}(X_\Sigma \setminus T) \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T) \longrightarrow 0$$

But $\mathrm{Cl}(T) = 0$ and $\mathrm{Cl}(X_\Sigma \setminus T) = \bigoplus_{\rho \in \Sigma^{[1]}} \mathbb{Z} D_\rho$. □

Proposition 8.4. *All relations between toric divisors in* $\mathrm{Cl}(X_\Sigma)$ *comes from* $z^m \in H^0(T, \mathcal{O}_T)^\times \subset H^0(X, \mathscr{K}_X^\times)$ *for* $m \in M$. *Moreover,*

$$\mathrm{div}(z^m) = \sum_{\rho \in \Sigma^{[1]}} \langle m, n_\rho \rangle D_\rho$$

where n_ρ *is a primitive generator for* ρ.

Proof. The relations come from $H^0(X, \mathscr{K}_X^\times)$ with $\bigcup_{\mathrm{ord}_f(D) \neq 0} D \subset \partial X_\Sigma = X_\Sigma \setminus T$. f has no zeros or poles on T, so $f|_T \in H^0(T, \mathcal{O}_T)^\times$.
Now consider a basis of N given by $n_1 = n_\rho, n_2, \ldots, n_r$. Then we can write $X_\rho = \operatorname{Spec} \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_r^{\pm 1}] = \mathbb{C} \times (\mathbb{C}^\times)^{r-1}$. So $z^m = z_1^{\langle m, n_\rho \rangle} z_2^{\langle m, n_2 \rangle} \cdots z_r^{\langle m, n_r \rangle}$. Consequently the coefficient of D_ρ in $\mathrm{div}(z^m)$ is exactly $\langle m, n_\rho \rangle$. □

Corollary 8.5. *We have a short exact sequence*

$$0 \longrightarrow M \xrightarrow{m \mapsto \mathrm{div}(z^m)} \bigoplus_{\rho \in \Sigma^{[1]}} \mathbb{Z} D_\rho \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

In particular, $\mathrm{Cl}(X)$ *is finitely generated.*

Example 8.1. 1. Consider $\mathrm{Bl}_0 \mathbb{A}^2$ which has the fan given by the cones between the three rays $(1,0), (0,1), (1,1)$, which we shall call D_1, D_2, E respectively. Then the relations are $0 \sim \mathrm{div}(z^{e_1}) = D_1 + E$ and $0 \sim \mathrm{div}(z^{e_2}) = D_2 + E$. Therefore $D_1 = -E = D_2$.
2. Consider $\mathbb{P}^n$. Then the relations are $0 \sim \mathrm{div}(z^{e_i}) = D_i - D_0$ where D_i means what you think it means.

Definition 8.3. A Cartier divisor on a normal separated variety X is a Weil divisor D that is locally principal, in the sense that X has an open cover $\{U_i\}_i$ such that $D|_{U_i} = \mathrm{div}(f|_{U_i})$ is principal on U_i.

The system $\{(U_i, f_i)\}_i$ is known as the local data for D. Such a system correspond to an element $f \in H^0(X, \mathscr{K}_X^\times / \mathcal{O}_X^\times)$, from which we can recover the divisor $D_f = \sum_Y \mathrm{ord}_f(Y) Y$. We write the subgroup of Cartier divisors as $\mathrm{CDiv}(X)$, which contains $\mathrm{Div}_0(X)$.
This collection of data specifying a Cartier divisor is exactly the transition functions we need to form a line bundle. And f can be viewed as a rational section of it.

Definition 8.4. The Picard group of X is $\mathrm{Pic}(X) = \mathrm{CDiv}(X) / \mathrm{Div}_0(X)$.

Remark. To spell out our transition-function argument, we observe the short exact sequence of cohomology

$$H^0(X, \mathscr{K}_X^\times) \longrightarrow H^0(X, \mathscr{K}_X^\times/\mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow 0$$

The first term is $\mathrm{Div}_0(X)$, second $\mathrm{CDiv}(X)$ and third $\mathrm{Pic}(X)$ in the usual definition with line bundles.

Definition 8.5. A Cartier divisor is toric if it is toric as a Weil divisor.

Corollary 8.6. *We have a short exact sequence*

$$0 \longrightarrow M \longrightarrow \mathrm{CDiv}_T(X) \longrightarrow \mathrm{Pic}_T(X) \longrightarrow 0$$

Proposition 8.7. *Suppose X_σ is an affine toric variety. Every toric Cartier divisor on X_σ has the form* $\mathrm{div}(z^m)$ *for some $m \in M$.*

Proof. If $D \geq 0$, we write I for its ideal. Since the toric action on X restricts to a toric action on D, we can write $I = \bigoplus_{m \in S_\sigma, z^m \in I} \mathbb{C}z^m$ (recall Lemma 2.2). Since D is Cartier, it is locally principal and therefore principal in a neighbourhood of $x_\sigma = \lim_{t\to 0} \lambda^n(t) \in X_\sigma$ for $n \in (\mathrm{Int}\,\sigma) \cap N$. Let $\mathfrak{m}$ be the maximal ideal corresponding to x_σ, then $\mathfrak{m} = \bigoplus_{m \in S_\sigma, m \neq 0} \mathbb{C}z^m$. Then essentially $I/\mathfrak{m}I$ is one-dimensional.
There is a unique minimal z^{m_0} which cannot be written as $z^{m_1}z^{m_2}$ for $m_i \in S_\sigma \setminus \{0\}$ and $z^{m_i} \in I$. As σ is full-dimensional $\sigma^\vee$ is strictly convex. By induction every $z^m \in I$ is divisible by z^{m_0}, and therefore $I = (z^{m_0})$, nice.
In the general case (not necessarily effective, that is), we choose $m \in (\mathrm{Int}\,\sigma^\vee) \cap M$, then $\langle m, m_\rho \rangle \geq 0$ for all $\rho \in \sigma^{[1]}$, so $D + \mathrm{div}(z^{km})$ is effective for large k. We therefore conclude by the effective case. ☐

Corollary 8.8. $\mathrm{Pic}_T(X_\sigma) = 0$

Proposition 8.9. $\mathrm{CDiv}_T(X_\sigma) = M/M(\sigma) = \mathrm{Hom}_\mathbb{Z}(N_\sigma, \mathbb{Z})$*.*

Recall that $M(\sigma) = \sigma^\perp \cap M$ and N_σ is the sublattice of N generated by the generators of σ.

Proof. $\mathrm{div}(z^m) = \mathrm{div}(z^{m'})$ iff $m - m' \in M(\sigma)$. ☐

Definition 8.6. A linear function $\phi : \sigma \to \mathbb{R}$ is the restriction of a $\mathbb{Z}$-linear map $N_\sigma \to \mathbb{Z}$.

Remark. Linear functions correspond to toric Cartier divisors on X_σ. For a fan Σ, toric Cartier divisors on X_Σ correspond to Cartier divisors on each X_σ compactible with restriction.

Definition 8.7. A piecewise-linear function $\phi : \Sigma \to \mathbb{R}$ is a continuous function that is linear on each $\sigma \in \Sigma$.

Theorem 8.10. $\mathrm{CDiv}_T(X_\Sigma)$ *is in bijection with the collection with piecewise linear functions $\phi : \Sigma \to \mathbb{R}$, where ϕ gives rise to the Cartier divisor $D_\phi = \sum_{\rho \in \Sigma^{[1]}} \phi(n_\rho) D_\rho$.*
Moreover, under this identification $\mathrm{Pic}_T(X_\Sigma)$ *is the set of piecewise linear $\Sigma \to \mathbb{R}$ modulo $\mathbb{Z}$-linear $\mathbb{R}^n \to \mathbb{R}$.*

Example 8.2. I can't be bothered to draw the pictures. Work out the example of $\mathbb{P}^2$ as an exercise, which will be what you expect. Also work out the example of $\{xy = zw\} \subset \mathbb{A}^4$ where $\mathrm{Pic}_T(X_\sigma) = 0$ but $\mathrm{Cl}_T(X_\sigma) = \mathbb{Z}$.

Back to the case where X_Σ is a smooth proper surface. For $\rho, \rho' \in \Sigma^{[1]}$, we define

$$D_\rho \cdot D_{\rho'} = \begin{cases} 1 & \text{if there is some } \sigma \in \Sigma \text{ containing } \rho, \rho' \text{ as faces} \\ 0 & \text{otherwise} \end{cases}$$

We can also attempt to understand a self-intersection $D_\rho^2 = D_\rho \cdot D_\rho$ by relations in $\mathrm{Cl}(X_\Sigma)$.

Example 8.3. Subdivide the fan for $\mathbb{P}^2$ (cones dividing the plane at $e_1, e_2, -e_1 - e_2$) at $-e_2$. Write D_0, D_1, D_2, E for the classes of the cones $-e_1 - e_2, e_1, e_2, -e_2$, respectively. Then $E \sim D_2 - D_0$ since $\mathrm{div}(z^{e_2}) = D_2 - D_0 - E$. Therefore $E^2 = E \cdot (D_2 - D_0) = E \cdot D_2 - E \cdot D_0 = 0 - 1 = -1$.

9 Polytopes and Projective Toric Varieties

Recall that for a finite subset $\mathcal{A} = \{m_1, \ldots, m_s\} \subset M$ we have defined $\Phi_\mathcal{A} : T_N \to (\mathbb{C}^\times)^s, t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t))$. Let $\hat{X}_\mathcal{A}$ be the Zariski closure of the image of $\Phi_\mathcal{A}$ in $\mathbb{C}^s$.

Definition 9.1. We write $X_\mathcal{A}$ to denote the Zariski closure of the image of $\Phi_\mathcal{A}$ in $\mathbb{P}^{s-1} \supset T_{\mathbb{P}^{s-1}}$.

If we take $\mathcal{A} \times \{1\} \subset M \oplus \mathbb{Z}$, then $\hat{X}_{\mathcal{A}\times\{1\}}$ is the affine cone over $X_{\mathcal{A}\times\{1\}} = X_\mathcal{A}$.

Proposition 9.1. *$X_\mathcal{A}$ has lattice $M' = \mathbb{Z}'\mathcal{A} = \{\sum_i a_i m_i : a_i \in \mathbb{Z}, \sum_i a_i = 0\}$.*

Proof.

$$\begin{tikzcd} T_N \arrow[r] \arrow[dr] & T_{\mathbb{P}^{s-1}} \arrow[r, hook] & \mathbb{P}^{s-1} \\ & T_{X_\mathcal{A}} \arrow[u, hook] & \end{tikzcd}$$

Dualise this, we obtain

$$\begin{tikzcd} M & M_{s-1} \arrow[l] \arrow[d] \\ & M' \arrow[ul, hook] \end{tikzcd}$$

where $M_{s-1} = \{a \in \mathbb{Z}^s : \sum_i a_i = 0\}$. Now $M_{s-1} \to M$ is induced by $\mathbb{Z}^s \to M, e_i \mapsto m_i$, hence the result. □

Corollary 9.2. $\dim X_\mathcal{A}$ *is the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing $\mathcal{A}$.*

Proof. Both equals the rank of $M' = \mathbb{Z}'\mathcal{A}$. □

Definition 9.2. A polytope $P \subset \mathcal{M}_{\mathbb{R}}$ is the convex hull of a finite set $S \subset M$. P is called a lattice polytope if S can be chosen to be a subset of $M \subset M_{\mathbb{R}}$. The dimension $\dim P$ of a polytope P is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P.

Example 9.1. The standard simplex $\Delta_r \subset M_{\mathbb{R}} = \mathbb{R}^r$ is the convex hull of $0, e_1, \dots, e_r$.

Definition 9.3. A polytope P is a simplex if it is the convex hull of $\dim P + 1$ points ("vertices").

Given a polytope P, we want to define a toric variety out of it. The obvious way to do this is to consider $X_{P \cap M}$. However, this is not a very useful construction at all.

Example 9.2. 1. $X_{\Delta_3 \cap M} = \operatorname{Proj} \mathbb{C}[z^{(0,0,0)}, z^{(1,0,0)}, z^{(0,1,0)}, z^{(0,0,1)}] = \mathbb{P}^3$. Looks pretty good? Hold your breath.
2. Suppose P is the convex hull of $0, e_1, e_2, e_1+e_2+3e_3$. Then we have $X_{\Delta_3 \cap M} = \operatorname{Proj} \mathbb{C}[z^{(0,0,0)}, z^{(1,0,0)}, z^{(0,1,0)}, z^{(1,1,3)}]$, which is again isomorphic to $\mathbb{P}^3$?!

So we need enough lattice points in P for $X_{P \cap M}$ to capture some properties of P.

Definition 9.4. P is normal if $(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M$ for all $k, l \in \mathbb{N}$, where the addition here is the Minkowski sum $A + B = \{a + b : a \in A, b \in B\}$ (and of course kP means $P + \cdots + P$ (k times)).

Note that it is always true that $(kP) \cap M + (lP) \cap M \subset ((k+l)P) \cap M$

Example 9.3. 1. Δ_r is normal.
2. The convex hull of $0, e_1, e_2, e_1 + e_2 + 3e_3$ is not normal: $e_1 + e_2 + e_3$ lives in $2P$ but not $P \cap M + P \cap M$.

Let P be a full-dimensional lattice polytope P of dimension $r = \dim P$.

Lemma 9.3. *P is a finite union of r-dimensional lattice simplices with no interior lattice points.*

Proof. Caratheodory proved that if $\mathcal{A} \subset M$ is finite, then the convex hull of $\mathcal{A}$ is the union of convex hulls of $\mathcal{B} \subset \mathcal{A}$ consisting of $r + 1$ affine independent elements (so they are in particular r-dimensional lattice simplices).
Now if a r-dimensional lattice simplex Q with vertices $w_0, \dots, w_r$ has a interior lattice point v, then Q is the union of Q_i which is the convex hull of $w_0, \dots, \hat{w}_i, \dots, w_r, v$. And each Q_i has fewer interior points, whence induction. $\square$

Proposition 9.4. *Suppose $P \subset M_{\mathbb{R}} = \mathbb{R}^r$ is a full-dimensional lattice polytope, then kP is normal for $k > r - 1$.*

Proof. By induction, it suffices to show that $(kP) \cap M + P \cap M = ((k+1)P) \cap M$ for $k \geq r - 1$. By the preceding lemma, we can assume WLOG that P is a lattice simplex with no interior lattice point. Let $m_0, \dots, m_r$ be the vertices. Then $(k+1)P$ has vertices $(k+1)m_0, \dots, (k+1)m_r$. For $m = \sum_i \mu_i (k+1) m_i$ with $\mu_i \geq 0, \sum_i \mu_i = 1$, we set $\lambda_i = (k+1)\mu_i$. Then $m = \sum_i \lambda_i m_i$ and $\lambda_i \geq 0, \sum_i \lambda_i = k + 1$.

If $\lambda_i \geq 1$ for some i, then $m - m_i \in (kP) \cap M$, so $m = (m - m_i) + m_i \in (kP) \cap M + P \cap M$. If $\lambda_i < 1$ for all i, then $\sum_i \lambda_i < r + 1$. But also $\sum_i \lambda_i = k + 1 \geq r$, so $k = r - 1$. Consider $\tilde{m} = \sum_i \tilde{\lambda}_i m_i$ where $\tilde{\lambda}_i = 1 - \lambda_i$. So $\sum_i \tilde{\lambda}_i = 1$ and $\tilde{\lambda} > 0$. Therefore $\tilde{m}$ is a lattice point in the interior of D, contradiction. □

Definition 9.5. P is very ample if for all $m \in P \cap M$, the semigroup $S_{P,m}$ generated by $P \cap M - m$ is saturated in M.

Proposition 9.5. *Any normal polytope is very ample.*

Proof. Take $m_0 \in P \cap M$. Take $m \in M$ such that $km \in S_{P,m}$ for some $k \geq 1$. We can write $km = \sum_{m' \in P \cap M} a_{m'}(m' - m_0)$ for $a_{m'} \geq 0$. Pick $d \in \mathbb{N}$ such that $kd \geq \sum_{m' \in P \cap M}$. Then $km + kdm_0 = \sum_{m' \in P \cap M} a_{m'} m' + (kd - \sum_{m' \in P \cap M} a_{m'}) m_0 \in kdP$. Therefore $m + dm_0 \in dP$.
Since P is normal, $m + dm_0 = \sum_{i=1}^{d} m_i$ for some $m_i \in P \cap M$. Therefore $m = \sum_i (m_i - m_0) \in S_{P,m_0}$. □

Corollary 9.6. *kP is very ample for $k \geq r - 1$.*

Definition 9.6. Suppose P is a full-dimensional lattice polytope. The projective toric variety associated to P is $X_P = X_{kP \cap M}$ for any k such that kP is very ample.

We'll see later that this is well-defined and P and kP have the same "normal fan". The choice of k is a choice of projective embedding.
To understand this construction, let's try to understand affine pieces of $X_{\mathcal{A}}$ for $\mathcal{A} = \{m_1, \ldots, m_s\}$. Let $U_i = \mathbb{P}^{s-1} \setminus \{x_i = 0\} \cong \mathbb{C}^{s-1}$ be the affine pieces of the projective space. Then $X_{\mathcal{A}} \cap U_i$ is the Zariski closure of $T_{X_{\mathcal{A}}}$ in U_i, which is the Zariski closure of the image of

$$T_N \to \mathbb{C}^{s-1}, t \mapsto (\chi^{m_1 - m_i}(t), \ldots, \chi^{m_{i-1} - m_i}, \chi^{m_{i+1} - m_i}, \ldots, \chi^{m_s - m_i})$$

So this is $\hat{X}_{\mathcal{A}_i} = \operatorname{Spec} \mathbb{C}[S_i]$ where $\mathcal{A}_i = \mathcal{A} - m_i$ and $S_i = \mathbb{N}\mathcal{A}_i$.

Theorem 9.7. *Suppose P is a full-dimensional lattice polytope.*
(i) X_P is normal.
(ii) X_P is projectively normal (in the sense that its affine cone is normal) under the embedding given by kP if and only if kP is normal.

Proof, some details omitted. (i) X_P is covered by $X_{kP \cap M} \cap U_i = X_{\sigma_i}$ for $\sigma_i = \operatorname{Cone}(kP \cap M - m_i)^\vee \subset N_{\mathbb{R}}$. This is normal.
(ii) The affine cone is normal iff $\mathbb{N}((kP \cap M) \times \{1\})$ is saturated in $M \times \mathbb{Z}$ iff $\operatorname{Cone}(P) \cap (M \times \mathbb{Z})$ is generated by $((kP) \cap M) \times \{1\}$ iff kP is normal. □

10 Line Bundles and Polytopes

For this section we always assume the toric variety $X = X_\Sigma$ is proper.

Definition 10.1. For a toric Cartier divisor $D = \sum_{\rho \in \Sigma^{[1]}} a_\rho D_\rho$, we define its associated polytope $P_D = \{m \in M : \forall \rho \in \Sigma^{[1]}, \langle m, n_\rho \rangle \geq -a_\rho\}$ where n_ρ is a primitive generator for ρ.

Proposition 10.1. *$P_D \cap M$ forms a basis for $H^0(X, \mathcal{O}(D))$. That is, we have a direct sum decomposition $\Gamma(X_\Sigma, \mathcal{O}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C}z^m$.*

Proof. $H^0(X_\Sigma, \mathcal{O}(D)) = \{f \in H^0(X_\Sigma, \mathscr{K}^\times_{X_\Sigma}) : D + \operatorname{div} f \geq 0\} \subset \mathbb{C}[M]$ as any such f must have $\operatorname{div} f|_{T_N} \geq 0$. Since D is toric invariant, $H^0(X_\Sigma, \mathcal{O}(D))$ is also toric invariant, and hence it equals $\bigoplus_{z^m \in H^0(X_\Sigma, \mathcal{O}(D))} \mathbb{C}z^m$.
Recall that $\operatorname{div}(z^m) = \sum_\rho \langle m, \rho \rangle D_\rho$, so $D + \operatorname{div} z^m \geq 0$ iff $m \in P_D \cap M$. □

Example 10.1. Let D be the class of a line in $\mathbb{P}^2$, then $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(D)) = \mathbb{C}1 \oplus \mathbb{C}t_1^{-1} \oplus \mathbb{C}t_1^{-1}t_2$ where $t_1 = x/z, t_2 = y/z$.

Remark. There are two ways one might think about global sections of $\mathcal{O}(D)$, either as the space of $f \in H^0(X, \mathscr{K}^\times_X)$ with $D + \operatorname{div} f \geq 0$, or as maps from X to the total space of $\mathcal{O}_X(D)$ with $\operatorname{pr} \circ s = \operatorname{id}_X$. They are related by $\mathbb{V}(s) = D + \operatorname{div} f$.

Definition 10.2. For a line bundle L on X, we choose basis $s_0, \ldots, s_k \in H^0(X, L)$. The Kodaira map is a rational map $\kappa = \kappa_L : X \dashrightarrow \mathbb{P}^k$ defined by $[s_0 : \cdots : s_k]$. The base locus for L is the locus $\mathbb{V}(s_0, \ldots, s_k)$. L is basepoint-free if $B(L) = \varnothing$.

Proposition 10.2. *$\mathcal{O}_X(D)$ is basepoint-free iff $m_\sigma = (-\phi_D)_\sigma \in P_D$ for all maximal cone $\sigma \in \Sigma^{[r]}$, where ϕ_D is the piecewise linear function associated to D.*

Proof. "If": Fix $\sigma \in \Sigma^{[r]}$. We get a global section $z^{m_\sigma} \in H^0(X, \mathcal{O}_X(D))$. Now z^{m_σ} is nonvanishing on $X_\sigma \subset X_\Sigma$:

$$\begin{aligned}
\mathbb{V}(z^{m_\sigma})|_{X_\sigma} = (\operatorname{div}(z^{m_\sigma}) + D)|_{X_\sigma} &= \sum_{\rho \in \sigma^{[1]}} (\langle m_\sigma, n_\rho \rangle + a_\rho) D_\rho \\
&= \sum_{\rho \in \Sigma^{[1]}} (-\phi_D(n_\rho) + a_\rho) D_\rho = 0
\end{aligned}$$

Since X_Σ is covered by these X_σ, we are done.
"Only if": Suppose there is some $\sigma \in \Sigma^{[r]}$ such that $m_\sigma \notin P_D$. Recall that we have $H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C}z^m$. So the base locus contains $\bigcap_{m \in P_D \cap M} \mathbb{V}(z^m)$.
As $m_\sigma \notin P_D$, for all $m \in P_D$ there is some $\rho \in \sigma^{[1]}$ such that $\langle m, n_\rho \rangle > -a_\rho$. Therefore $\mathbb{V}(z^m) \supset D_\rho \supset \bigcap_{\rho \in \sigma^{[1]}} D_\rho$. □

Proposition 10.3. *The condition in the preceding proposition is further equivalent to ϕ being convex.*

For a basepoint-free line bundle, we get a morphism $\kappa : X \to \mathbb{P}^k$.

Definition 10.3. L is very ample if κ is a closed embedding. It is ample if $L^{\otimes k}$ is very ample for some $k \geq 1$.

Remark. X is projective if and only if it admits an ample line bundle.

Proposition 10.4. *D is ample if and only if ϕ_D is strictly convex.*

Now let P be a full-dimensional polyhedron. We shall recover a toric variety equipped with a toric divisor.
A codimension 1 face of P is called a facet of P. For a facet F, let $n_F \in N$ be a primitive normal lattice vector to F. We can always write P in the form $\{m \in M_{\mathbb{R}} : \langle m, n_F\rangle \geq -a_F\}$ for various $a_F \in \mathbb{Z}$.

Definition 10.4. The inner normal fan Σ_P to P is a fan with cones $\sigma_Q = \mathrm{Cone}\{n_F : F \supset Q\}$.

Proposition 10.5. *Σ_P is a fan and $X_P \cong X_{\Sigma_P}$.*

Sketch of proof. Suppose P is very ample, then $X_{P\cap M} \cap U_v \cap U_w = X_{\sigma_Q}$ where Q is the smallest face containing both v and w. The inclusions $X_{P\cap M} \cap U_v \supset X_{P\cap M} \cap U_v \cap U_w \subset X_{P\cap M} \cap U_w$ give $X_{\sigma_Q} = (X_{\sigma_v})_{\chi^{w-v}} = (X_{\sigma_w})_{\chi^{v-w}}$. □

Definition 10.5. Write $D_P = \sum_{F\subset P \text{ facet}} a_F D_F$.

Proposition 10.6. *D_P is a (non-principal) Cartier divisor.*

Proof. Omitted. □

Definition 10.6. The piecewise linear function $\phi_P : \Sigma_P \to \mathbb{R}$ associated to P is defined by sending n to $-\min\{\langle m, n\rangle : m \in P\}$.

Proposition 10.7. *ϕ_P corresponds to D_P.*

Proof. Omitted, again. □

Example 10.2. Let P be the triangle with vertices $(0,0),(1,0),(0,1)$. Then the facet representation for P is $P = \{m \in M : \langle m,(1,0)\rangle = m_1 \geq 0, \langle m,(0,1)\rangle \geq 0, \langle m,(-1,-1)\rangle \geq -1\}$. $D_P = D_0$ is a line in $\mathbb{P}^2$. On the other hand, $\phi(1,0) = 0, \phi(0,1) = 0, \phi(-1,-1) = 1$. So indeed they correspond.

Proposition 10.8. *(i) D_P is always ample and basepoint-free.*
(ii) kD_P is very ample for $k \geq r-1$.
(iii) D_P is very ample iff P is.

Proof. Take a guess. □

Theorem 10.9. *The set of full-dimensional polytopes $P \subset M_{\mathbb{R}}$ is in bijection with the set of pairs (X_Σ, D) where Σ is a complete fan in $N_{\mathbb{R}}$ and D a torus-invariant ample divisor on X_Σ.*

Proof. For a polytope P, we get (X_{Σ_P}, D_P). Conversely, for such a D we recover a polytope P_D. □

11 Intersections and Chow Groups

Suppose $D \subset X$ is a Cartier divisor and $C \subset X$ is an irreducible complete curve. Let $\phi : \tilde{C} \to C$ be the normalisation of C.

Definition 11.1. We define the intersection product of D and C to be $D\cdot C = \deg_{\tilde{C}}(\phi^*\mathcal{O}_X(D)) \in \mathbb{Z}$.

More generally, if $D \subset X$ is a $\mathbb{Q}$-Cartier divisor (i.e. lD is Cartier for some integer $l > 0$; for toric varieties this is to say that its fan is simplicial), we define $D \cdot C = l^{-1}(lD) \cdot C \in \mathbb{Q}$.

Proposition 11.1. *Let $\tau = \sigma \cap \sigma'$ be a codimension 1 cone. Write $C = \mathbb{V}(\tau)$ for the toric stratum corresponding to it, which is a curve. Let D be a Cartier divisor with local data $m_\sigma, m_{\sigma'} \in M$.*
Pick $n \in \sigma' \cap N$ that maps to a minimal generator of the image $\bar{\sigma}' \subset N(\tau)_\mathbb{R}$ of σ'. Then $D \cdot C = \langle -m_\sigma + m_{\sigma'}, n\rangle$.

Sketch of proof. Assume that $X_\Sigma = X_\sigma \cap X_{\sigma'}$. We have $D|_{X_\sigma} = \mathrm{div}(\chi^{m_\sigma})|_{X_\sigma}$ and $D|_{X_{\sigma'}} = \mathrm{div}(\chi^{m_{\sigma'}})|_{X_{\sigma'}}$. Then $\mathcal{O}_{X_\Sigma}(D)$ is determined by $g_{\sigma\sigma'} = \chi^{m_\sigma - m_{\sigma'}}$, so $D \cdot C = \deg \mathcal{O}_C(\bar{D})$ where $\bar{D}$ has local data $m_{\bar{\sigma}} = 0$, $m_{\bar{\sigma}'} = m_{\sigma'} - m_\sigma$. Suppose $\bar{n}$ is the image of n. Then $\bar{D} = \langle m_{\sigma'} - m_\sigma, n\rangle x_{\sigma'}$, hence the result. ☐

Definition 11.2. A Cartier divisor D on a normal variety X is nef if $D \cdot C \geq 0$ for all irreducible complete curves C on X.

Proposition 11.2. *Suppose D is a Cartier divisor on $X = X_\Sigma$. Suppose Σ has convex support of full dimension. Then the followings are equivalent:*
(i) D is basepoint-free.
(ii) D is nef.
(iii) $D \cdot C \geq 0$ for all torus-invariant C.

Proof. (i) $\implies$ (ii) $\implies$ (iii): Immediate.
(iii) $\implies$ (i): WLOG D is toric. We claim that ϕ_D is convex. Write $\tau = \sigma \cap \sigma'$, $C = \mathbb{V}(\tau)$, and $n \in \sigma' \cap N$ as before. Then $\langle -m_\sigma + m_{\sigma'}, n\rangle = D \cdot C \geq 0$. Therefore $\phi_D(n) = \langle m_{\sigma'}, n\rangle \geq \langle m_\sigma, m\rangle$ for all $n \notin \sigma'$, hence ϕ_D is convex. ☐

Proposition 11.3 (Toric Kleiman Criterion)**.** *Suppose now that Σ is complete. Then D is ample iff $D \cdot C > 0$ for all torus-invariant irreducible complete $C \subset X_\Sigma$.*

Definition 11.3. A divisor D is numerically equivalent to 0 (written $D \equiv 0$) if $D \cdot C = 0$ for all irreducible complete curves $C \subset X$. Two divisors D, E are numerically equivalent (written $D \equiv E$) if $D - E$ is numerically equivalent to 0.

Proposition 11.4. *Suppose D is a Cartier divisor on X_Σ where Σ is a full-dimensional fan with convex support. Then $D \sim 0$ iff $D \equiv 0$.*

Proof. It is always true that $D \sim 0$ only if $D \equiv 0$. Conversely, suppose $D \equiv 0$. Using the notation as in the proof of Proposition 11.2, we see $0 = D \cdot C = \langle -m_\sigma + m_{\sigma'}, n\rangle$ and so $m_\sigma = m_{\sigma'}$ since $-m_\sigma + m_{\sigma'} \in \tau^\perp$ and $n \notin \sigma$. So all m_σ are equal for $\sigma \in \Sigma^{[1]}$ and hence D is principal. ☐

Let's put things together.

Definition 11.4. $Z_1(X)$ is the free abelian group generated by irreducible complete curves on X. Its elements are called proper 1-cycles.
We say a proper 1-cycle C is numerically equivalent to 0 if $D \cdot C = 0$ for all divisor D, and two proper 1-cycles C, C' numerically equivalent (written $C \equiv C'$) if $C - C'$ is numerically equivalent to 0.

So the intersection product is a bilinear pairing $\mathrm{CDiv}(X) \times Z_1(X) \to \mathbb{Z}$.

Definition 11.5. We write $N^1(X)$ for $(\mathrm{CDiv}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$, and $N_1(X)$ for $(Z_1(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$.

So the intersection product becomes a bilinear map $N^1(X) \times N_1(X) \to \mathbb{R}$.

Definition 11.6. The nef cone $\mathrm{Nef}(X) \subset N^1(X)$ of X is the (not necesssarily polyhedral) cone generated by the set of nef Cartier divisors. We also write $\mathrm{NE}(X) \subset N_1(X)$ for the cone generated by the set of nef irreducible complete curves. Its closure $\overline{\mathrm{NE}}(X) \subset N_1(X)$ is known as the Mori cone.

Proposition 11.5. *(a) $\mathrm{Nef}(X), \overline{\mathrm{NE}}(X)$ are closed convex cones dual to each other.*
(b) $\mathrm{NE}(X)$ is full-dimensional.
(c) $\mathrm{Nef}(X)$ is strictly convex.

Proposition 11.6. *Suppose Σ is a full-dimensional fan with convex support.*
(a) $\mathrm{Nef}(X_\Sigma)$ is a rational polyhedral cone in $N^1(X_\Sigma) = \mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$.
(b) $\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{NE}(X_\Sigma)$ is a rational polyhedral cone in $N_1(X_\Sigma)$. Moreover, $\mathrm{NE}(X_\Sigma) = \sum_\tau \mathbb{R}_{\geq 0}[\mathbb{V}(\tau)]$ where τ goes over all walls, i.e. one-dimensional faces not on the boundary.

Proof. (a) Follows from (b).
(b) It suffices to show the formula for $\mathrm{NE}(X_\Sigma)$. Let $\Gamma = \sum_\tau \mathbb{R}_{\geq 0}[\mathbb{V}(\tau)]$, then $\mathrm{Nef}(X_\Sigma) = \Gamma^\vee$, and so $\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{Nef}(X_\Sigma)^\vee = \Gamma^{\vee\vee} = \Gamma \subset \mathrm{NE}(X)$ and hence all inclusions are equalities. $\square$

Proposition 11.7. *Suppose X_Σ is projective.*
(a) $\mathrm{Nef}(X_\Sigma)$ and $\overline{\mathrm{NE}}(X_\Sigma)$ are dual strictly convex rational polyhedral cones of full dimension.
(b) A Cartier divisor D is ample iff its class in $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ lies in the interior of $\mathrm{Nef}(X_\Sigma)$.

Proof. (b) D is ample iff $D \cdot C > 0$ for all C iff D is in the interior.
(a) By (b), $\mathrm{Nef}(X_\Sigma)$ is full-dimensional and so $\overline{\mathrm{NE}}(X_\Sigma)$ is strictly convex. $\square$

Example 11.1. Can't be bothered to draw pictures. Compute these things for the Hirzebruch surface F_k.

Proposition 11.8. *Suppose Σ is simplicial, $\sigma, \sigma' \in \Sigma^{[r]}$, $\tau = \sigma \cap \sigma'$. Suppose $\sigma = \mathrm{Cone}(\tau, \rho_1), \sigma' = \mathrm{Cone}(\tau, \rho_{r+1})$ and $\tau = \mathrm{Cone}(\rho_2, \ldots, \rho_n)$. Then:*
(a) $D_\rho \cdot \mathbb{V}(\tau) = 0$ for $\rho \notin \{\rho_1, \ldots, \rho_{r+1}\}$.
(b) If we let $\mathrm{mult}(\sigma)$ be the index of $\mathrm{Span}(\sigma) \cap N \subset N$, then $D_{\rho_1} \cdot \mathbb{V}(\tau) = \mathrm{mult}(\tau)/\mathrm{mult}(\sigma)$ and $D_{\rho_{r+1}} \cdot \mathbb{V}(\tau) = \mathrm{mult}(\tau)/\mathrm{mult}(\sigma')$.
(c) $D_{\rho_i} \cdot \mathbb{V}(\tau) = b_i \,\mathrm{mult}(\tau)/(\alpha \,\mathrm{mult}(\sigma)) = b_i \,\mathrm{mult}(\tau)/(\beta \,\mathrm{mult}(\sigma'))$ where $\alpha n_{\rho_1} + \sum_{i=2}^r b_i n_{\rho_i} + \beta n_{\rho_{r+1}} = 0$.

In general, we set $Z_k(X)$ to be the free abelian group on the set of irreducible subvarieties of X of dimension k, and $\mathrm{Rat}_k(X)$ the subgroup generated by the divisors of rational functions on an irreducible subvariety of X of dimension $k+1$. The k-th Chow group of X is the quotient $\mathrm{CH}_k(X) = Z_k(X)/\mathrm{Rat}_k(X)$ and we set $\mathrm{CH}^k(X) = \mathrm{CH}_{\dim X - k}(X)$.

Remark. Suppose X is normal, then $\mathrm{CH}^1(X) = \mathrm{Cl}(X)$.
Suppose X is smooth and projective, then there is an "intersection product" $\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \to \mathrm{CH}^{k+l}(X)$, which is given by the naïve intersection of cycles in the case of transverse intersections (but more complicated in other situations). This makes $\mathrm{CH}^*(X) = \bigoplus_k \mathrm{CH}^k(X)$ a commutative graded ring, called the Chow ring of X.

Proposition 11.9. *Suppose X is a toric variety. Then $\mathrm{CH}^*(X)$ is generated by toric strata.*

Proposition 11.10. *Suppose Σ is complete and simplicial and $\rho_1, \ldots, \rho_d \in \Sigma^{[1]}$ are distinct rays. Then*

$$[D_{\rho_1}] \cdots [D_{\rho_d}] = \begin{cases} \mathrm{mult}(\sigma)^{-1}[\mathbb{V}(\sigma)] & \textit{if } \sigma = \rho_1 + \cdots + \rho_d \in \Sigma \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 11.11. *Suppose Σ is complete and simplicial. Then we have an exact sequence*

$$\bigoplus_{\tau \in \Sigma^{[r-k-1]}} M(\tau)_{\mathbb{Q}} \longrightarrow \bigoplus_{\sigma \in \Sigma^{[r-k]}} \mathbb{Q}[\mathbb{V}(\sigma)] \longrightarrow \mathrm{CH}_k(X_\Sigma)_{\mathbb{Q}} \longrightarrow 0$$

where the first map sends m to $[\mathrm{div}(\chi^m)] = \sum_{\sigma \in \Sigma^{[n-k]}} \langle m, n_{\sigma,\tau} \rangle [\mathbb{V}(\sigma)]$ with $n_{\sigma,\tau}$ the generator of $N_\sigma / N_\tau \cong \mathbb{Z}$.

12 Canonical Divisor

Let Ω_X be the cotangent sheaf of X. Write $\Omega_X^p = \bigwedge^p \Omega_X$ for the sheaf of p-forms. It's know that if X is smooth then Ω_X is locally free of rank $r = \dim X$.

Definition 12.1. The canonical bundle on X is the line bundle Ω_X^r.

In general, for a normal variety X, we define $\hat{\Omega}_X^p = j_* \Omega^p_{X_{\mathrm{smooth}}}$ where $j : X_{\mathrm{smooth}} \to X$ is the inclusion of the smooth locus. $\omega_X = \hat{\Omega}_X^r$ is called the canonical sheaf of X, which is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor $D = K_X$.

Theorem 12.1.

$$\omega_{X_\Sigma} = \mathcal{O}_{X_\Sigma}\left(- \sum_{\rho \in \Sigma^{[1]}} D_\rho \right)$$

Definition 12.2. X is called Gorenstein if K_X is Cartier, and Fano if $-K_X$ is ample.

Definition 12.3. A polytope P is reflexive if it can be represented in the form $P = \{m \in M_{\mathbb{R}} : \forall F \text{ facet}, \langle m, n_F \rangle \geq -1\}$. We can form its polar dual P° which is the convex hull of $\{n_F : F \text{ facet}\}$.

Theorem 12.2. *P_{-K_X} is reflexive if and only if X_Σ is projective, Gorenstein and Fano.*

Example 12.1. In the 2-dimensional case, there are exactly 16 projective Gorenstein del Pezzo (i.e. Fano) toric surfaces, corresponding to 16 reflexive polytopes.

13 Quotients and Homogeneous Coordinates

Our goal here is to make precise the identification of $\mathbb{P}^n$ with a "quotient" of $\mathbb{C}^{n+1} \setminus \{0\}$, in a more general setting. For a toric variety X_Σ, we want to reconstruct it as an "almost generic" quotient $X_\Sigma = (\mathbb{C}^m \setminus Z)//G$.
Assume X_Σ has no torus factors, i.e. we cannot write $X_\Sigma = (\mathbb{C}^\times)^k \times X_{\Sigma'}$ for some k and fan Σ'. Then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_\rho \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

and hence the exact sequence

$$1 \longrightarrow \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^\times) \longrightarrow (\mathbb{C}^\times)^{\Sigma^{[1]}} \longrightarrow T_N \longrightarrow 1$$

We take $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^\times)$.

Lemma 13.1. *(i)* $\mathrm{Cl}(X_\Sigma)$ *is the character group of* G.
(ii) G *is the product of a torus and a finite abelian group, hence is in particular reductive.*
(iii) $G = \{(t_\rho) \in (\mathbb{C}^\times)^{\Sigma^{[1]}} : \forall m \in M, \prod_{\rho\in\Sigma^{[1]}} t_\rho^{\langle m,n_\rho\rangle} = 1\} = \{(t_\rho) \in (\mathbb{C}^\times)^{\Sigma^{[1]}} : \forall i, \prod_{\rho\in\Sigma^{[1]}} t_\rho^{\langle e_i,n_\rho\rangle} = 1\}$.

Definition 13.1. Set $S = \mathbb{C}[x_\rho : \rho \in \Sigma^{[1]}]$, $B(\Sigma) = \langle x^{\hat\sigma} = \prod_{\rho\notin\sigma^{[1]},\rho\in\Sigma^{[1]}} x_\rho : \sigma \in \Sigma^{[r]}\rangle \subset S$ (note that $\Sigma^{[1]}$ and $B(\Sigma)$ determine Σ). Set $Z(\Sigma) = \mathbb{V}(B(\Sigma)) \subset \mathbb{C}^{\Sigma^{[1]}}$, which is a union of coordinate subspaces.

Theorem 13.2. $X_\Sigma = (\mathbb{C}^{\Sigma^{[1]}} \setminus Z(\Sigma))//G$ *is an almost geometric quotient, which is geometric if* Σ *is simplicial.*

Example 13.1. 1. The fan of $\mathbb{P}^r$ has rays $(-1,\ldots,-1), e_1, e_2, \ldots$. So $G = \{(t_i) \in (\mathbb{C}^\times)^{r+1} : t_0^{-1}t_1 = \cdots = t_0^{-1}t_r = 1\} = \mathbb{C}^\times$, $B(\Sigma) = \langle x_0,\ldots,x_r\rangle$, $Z(\Sigma) = \{0\}$ and $\mathbb{P}^r = (\mathbb{C}^{r+1} \setminus \{0\})/\mathbb{C}^\times$.
2. Consider the weighted projective space $\mathbb{P}(1, a_1, \ldots, a_r)$ with $\gcd\{a_i\} = 1$. Then Σ has rays $(-a_1,\ldots,-a_r), e_1, e_2, \ldots$. We have $G = \{(\lambda, \lambda^{a_1}, \ldots, \lambda^{a_r}) : \lambda \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$, $Z(\Sigma) = \{0\}$ and $\mathbb{P}(1, a) = (\mathbb{C}^{r+1} \setminus \{0\})/\mathbb{C}^\times$. 3. In general, $\mathbb{P}(a_0,\ldots,a_n)$, $\gcd\{a_i\} = 1$ has rays $n_0 = -a_0^{-1}(1,\ldots,1)$ and $n_i = e_i/a_i$.

14 A Message from our Sponsor, the Tropical Geometry Gang

Turns out, the technique of toric degeneration has cool applications to enumerative geometry.
Consider the map $\log_t : (\mathbb{C}^\times)^r \to \mathbb{R}^r, (x_i)_i \mapsto (-\log_t |x_i|)_i$ for $0 < t < 1$.

Example 14.1. For $t = e^{-1}$ and $C = \{e^3x_1 + e^2x_2 = e^0\} \subset (\mathbb{C}^\times)^2$. Then $\log_t(C)$ looks like…something something amoeba, can't be bothered to draw it. Anyways, as $x_1 \to 0$, $\log_t(x_1) \to -\infty$ and $\log_t(x_2) \to -2$.
Sending $t \to 0$, we get a union of three rays all starting at $(-3, -2)$ and going towards the directions $(1, 1)$, $(-1, 0)$ and $(0, -1)$.

Formally, one take the limit $t \to 0$ by using the formalism of Paiseux series $\mathbb{C}\{\{t\}\} = \{f = \sum_{q \in \mathbb{Q}} a_q t^q : v(f) = \min\{q : a_q \neq 0\} \text{ exists}\}$, which is a valued field under v. Whatever, this $\log = \lim_{t \to 0} \log_t$ turns $((\mathbb{C}^\times)^r, +, \cdot)$ into the "tropical semiring" $(\mathbb{R}^r, \oplus, \odot) = (\mathbb{R}^r, \max, +)$.

Definition 14.1. A tropical hypersurface is the corner locus of a tropical polynomial.

Example 14.2. $3 \odot x \oplus 2 \odot x_2 \oplus 0 = \max\{3 + x_1, 2 + x_2, 0\}$ whose corner locus is given by $\{x_1 = -3, x_2 \leq -2\}, \{x_2 = -2, x_3 \leq -3\}, \{x_2 = x_1 + 1 \geq -2\}$, which is what we got in the previous example.

Let's extend this to toric varieties. $(\mathbb{C}^\times)^r \subset X_\Sigma$ should give a compactification $\mathbb{R}^r \subset \overline{\mathbb{R}^r}^\Sigma$. The new stuff at the boundary correspond to rays of Σ, and boundary of P.

Example 14.3. Um. Work this out for $\mathbb{P}^2$.

Combinatorial type of tropical hypersurfaces correspond to subdivisions of "Newton polytopes", i.e. convex hulls of $\{m_i\}$ if our equation is $\sum_i a_i x^{m_i}$ with $a_i \neq 0$.

Example 14.4. For lines in $\mathbb{P}^2$, the corresponding Newton polytope is simply a triangle (recall Example 14.1). And smooth conics correspond to a subdivision of the triangle by joining the midpoints. Smooth cubics correspond to a subdivision by joining the thirds of the triangle (in a way that produces 9 small triangles). We can see the tropical degree-genus formula, where the genus is the number of interior vertices of the polygon.

Let's dig deeper in plane curves. Let Σ be the fan of $\mathbb{P}^2$, and write $\mathbb{R}^2$ for $\overline{\mathbb{R}^2}^\Sigma$.

Definition 14.2. A parameterised (plane) tropical curve is $h : \Gamma \to \mathbb{R}^2$ where Γ is an wedge-weighted graph with tails and no bivalent vertices, and h is continuous and piecewise affine, in the sense that $h|_E : E \to \mathbb{R}^2$ is a piecewise affine embedding of a line segment for every edge E, and $h(L)$ is half-infinite for tails L. And they are required to satisfy the balancing condition. That is, for every vertex V we have $\sum_E w_E m_E = 0$, where E goes over all edges at V, w_E is the weight of E, and m_E is the primitive tangent at E (via h).
The degree of a such a tropical curve is the sum of its weights.

So, you must have drawn some pictures of tropical curves corresponding to plane curves and observed that they seem to (generically) satisfy Bézout's theorem, right? Might be helpful to define multiplicity, idk.

Definition 14.3. Suppose we have tropical curves C_1, C_2, not sharing any positive-dimensional part. The intersection multiplicity at one of their intersections x is $\mathrm{mult}_x(C_1, C_2) = \mathrm{vol}\, P_x = w_1 w_2 \det(m_1 \mid m_2)$ where P_x is the polygon corresponding to x (I am trying here okay), and w_i, m_i are the weights and primitive tangents of the edges meeting at x. And define $C_1 \cdot C_2 = \sum_{x \in C_1 \cap C_2} \mathrm{mult}_x(C_1, C_2)$.

Lemma 14.1 (Tropical Moving Lemma)**.** *$C_1 \cdot C_2$ is invariant under translations.*

So for general C_1, C_2, we can always translate to a position where they do not share any positive-dimensional part and compute $C_1 \cdot C_2$ there.

Proposition 14.2 (Tropical Bézout)**.** *Suppose C_i are plane tropical curves with degrees d_1, d_2, respectively. Then $C_1 \cdot C_2 = d_1 d_2$.*

Proof #1. $\sum_x \operatorname{vol} P_x = \operatorname{vol} P_{C_1 \cup C_2} - \operatorname{vol} P_{C_1} - \operatorname{vol} P_{C_2} = (1/2)(d_1 + d_2)^2 - (1/2)d_1^2 - (1/2)d_2^2 = d_1 d_2$ (okay I stopped trying). □

Proof #2. Move 'em to nice position (really?). □

This has applications to curve counting. For example, in $\mathbb{P}^2$, let N_d be the number of degree d rational curves through $3d-1$ general points, then $N_1 = N_2 = 1$, $N_3 = 12$, and so on. And perhaps it's easy to see this on tropical curves, with N_d^{trop} defined with some multiplicity concerns.

Definition 14.4. Let $h : \Sigma \to \mathbb{R}^2$ be a tropical curve. Its multiplicity is $\sum_V m_V$ where $m_V = w_1 w_2 |\det(w_1 \mid m_2)|$ if V is trivalent (where w_i, m_i are the data associated to two of the edges), well-defined by the balancing condition.
We write N_d^{trop} for the sum of all m_h where h is a tropical curve of degree d passing through $3d-1$ general points.

Everything is trivalent in general position, so we are not gonna define things more generally.

Theorem 14.3 (Mikhalkin '03, Nishinou-Siebert '04)**.** $N_d = N_d^{\text{trop}}$.

The idea is to find a degeneration of $X = \mathbb{P}^2$ to a union of toric surfaces ("toric degeneration"), glued together along toric divisors, such that some curves split into a union of simpler pieces. More precisely, we want to find a toric family $\mathfrak{X} \to \mathbb{A}^1$ such that $\mathfrak{X}_{t \neq 0} \cong X$, $\mathfrak{X}_0 = X_0$ is a union of toric surfaces and N_d can be computed on X_0.
Start with the union of all degree d tropical curves through $3d-1$ points. And add Σ at each p_i, which gives a polyhedral subdivision $\mathcal{P}$ of $\mathbb{R}^2$. The structure of X_0 is then described by $\mathcal{P}$ (or, equivalently, its dual Newton polygon subdivision).
Let's describe $\mathfrak{X}$ more explicitly. Put $\mathcal{P}$ in $\mathbb{R}^2 \times \mathbb{R}$ by placing it on the plane of height 1, and take cones over the (possibly unbounded) polytopes. This leads to a fan Σ, and we set $\mathfrak{X} = X_\Sigma$ equipped with a map $\mathfrak{X} \to \mathbb{A}^1$ given by the projection $\Sigma \to \mathbb{R}_{\geq 0}$. Then $\mathfrak{X}_{t \neq 0} \cong \mathbb{P}^2$ collects to cones supported at height 0, i.e. the infinite elements of $\mathcal{P}$. And $\mathfrak{X}_0$ sees everything, so it is a union of toric surfaces corresponding to elements of $\mathcal{P}$, glued together according to the combinatorics of $\mathcal{P}$.
Let's also see what the dual picture ("polytope picture") look like. Consider an ample line bundle corresponing to a piecewise linear function $\Sigma \to \mathbb{R}$. We get a Newton polygon subdivision B, on which we also have a piecewise linear function ϕ (i.e. continuous function that is linear on each cell) to build $\mathfrak{X}$. Given such a ϕ, we can form its upper convex hull, which is the convex hull of everything above the graph of ϕ. This has a map to $\mathbb{R}_{\geq 0}$ which is the polytope for $\mathbb{A}^1$.
Anyways, let's get back to curve counting. Turns out that curves on $X = \mathbb{P}^2$ correspond (up to some multiplicity subtlety) to reducible curves on X_0, which is one-to-one with tropical curves $h : \Gamma \to \mathbb{R}^2$. The first step is done by associating each to smooth stable log maps.

Definition 14.5. A log structure on X is a morphism of sheaves of monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ such that $\alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism.

Example 14.5 (Divisorial log sturcture)**.** Suppose D is an effective divisor on X and let $j : D \to X$ be the inclusion. Then $\mathcal{M}_{(X,D)} = (j_*\mathcal{O}_{X\setminus D}^\times) \cap \mathcal{O}_X \to \mathcal{O}_X^\times$ is a log structure. $\mathcal{M}_{(X,D)}$ is essentially the functions invertible away from D. Associated to it is a "ghost sheaf" $\bar{\mathcal{M}}_{(X,D)} = \mathcal{M}_{(X,D)}/\mathcal{O}_X^\times$ which keeps track of vanishing orders along D.

The advantage of a log structure is that it allows us to treat mildly singular ("log-smooth") varieties as being smooth. For our purpose, $\mathcal{M}_{(\mathfrak{X},X_0)}$ keeps track of infinitesimal information about $X_0 \subset \mathfrak{X}$, and therefore about $\mathfrak{X} \to \mathbb{A}^1$. So any log-smooth curve C_0 on X_0 can be extended to curves on $\mathfrak{X} \to \operatorname{Spec} \mathbb{C}[[t]]$, and hence to a curve C on X by looking at the general fibre. Locally, at a node x of C_0, it is supposed to correspond to a compact edge E of Γ, and C_0 has multiplicity w_E at x. This is somehow a choice of a w_E-th root of unity in the log structure. If one globalises this, one gets m_h choices of smooth log structure on C_0.

Theorem 14.4 (Kontsevich-Manin)**.**

$$N_d = \sum_{d_1+d_2=d, d_1,d_2>0} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

Tropical proof. Write $n = 3d$. Let $\bar{\mathcal{M}}_{0,n}^{\text{trop}}(\mathbb{P}^2, d)$ be the moduli space of tropical curves of degree d with n marked points. Forgetting the 5-th point onwards and the embedding into $\mathbb{R}^2$, we get a map $\text{ft}_\phi : \bar{\mathcal{M}}_{0,n}^{\text{trop}}(\mathbb{P}^2, d) \to \bar{\mathcal{M}}_{0,4}^{\text{trop}}$. If we let ev_i^j be the j-th coordinate of the i-th marked point, and ev_i the evaluation at the i-th point, then we get a map $\pi : \text{ev}_1^1 \times \text{ev}_2^2 \times \text{ev}_3 \times \cdots \times \text{ev}_n \times \text{ft}_\phi : \bar{\mathcal{M}}_{0,n}^{\text{trop}}(\mathbb{P}^2, d) \to \mathbb{R}^{2n-2} \times \bar{\mathcal{M}}_{0,4}^{\text{trop}}$, and its number of preimages does not depend on $\mathcal{P}$. So idk, calculate $\deg \pi$ for special values of $\mathcal{P}$ or something. $\square$

Finally, let's discuss applications to non-toric settings. For a log-singular toric degeneration $\mathfrak{X} \to \mathbb{A}^1$, we may find a log resolution $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ which is not necessarily toric, and look at its fibres.

Example 14.6. Consider the cubic surface $\mathfrak{X} = \{XYZ = t^3(W^3 + f_3)\}$ where f_3 is a general degree 3 polynomial in four variables. Count lines on the dual picture, we get $3 \times 3 \times 3 = 27$ lines, idk.