Abelian Varieties *

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This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Abelian Varieties* in Lent 2023. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

Contents

1 Curves and Abel-Jacobi Maps

Let X be a smooth irreducible projective curve over $\mathbb{C}$. $X(\mathbb{C})$ has the structure of a compact connected Riemann surface, which locally looks like open subsets of $\mathbb{C}$ and globally looks like a compact orientable surface. We have the notion of a genus g of X.
Recall that we can form the group $\operatorname{Div}(X)$ of divisors on X, which is the free abelian group on $X(\mathbb{C})$. It admits a degree map $\operatorname{Div}(X) \to \mathbb{Z}, \sum_p n_p[p] \mapsto \sum_p n_p$. Denote by $\operatorname{Div}^0(X)$ its kernel.
Let $\mathcal{k}(X)$ be the field of rational functions of X (which is the same as the set

*Based on the lectures under the same name taught by Prof. A. J. Scholl in Lent 2023.

of meromorphic functions on $X(\mathbb{C})$). If we have a nonzero function $f \in k(X)^\times$, we can identify its divisor $\operatorname{div} f = \sum_{p \in X} \operatorname{ord}_p(f)[p] \in \operatorname{Div}^0(X)$. Divisors of this form are called principal divisors. The quotient of $\operatorname{Div}(X)$ by it is known as the divisor class group $\operatorname{Cl}(X)$ of X, and the quotient of $\operatorname{Div}^0(X)$ is written as $\operatorname{Cl}^0(X)$.
There is another point of view when dealing with divisors, namely by viewing each as a line bundle (locally free $\mathcal{O}_X$-module of rank 1). For a divisor D, recall that we can associate to it a line bundle $\mathcal{O}_X(D)$, and it turns out that D is principal if and only if $\mathcal{O}_X(D)$ is trivial, and that every line bundle on X has the form $\mathcal{O}_X(D)$ for some D. So we have an isomorphism between $\operatorname{Cl}(X)$ and the Picard group of X, which is the group of isomorphism classes of line bundles on X under tensor product.
Let's now introduce differentials. $X(\mathbb{C})$ is covered by open subsets of $\mathbb{C}$, and so we have holomorphic differentials locally of the form $f(z)\,\mathrm{d}z$ with the obvious change-of-coordinate rules. We write $H^0(X, \Omega_X)$ to denote the complex vector space of holomorphic differentials on X, which turns out to have dimension g. Let $\omega_1, \ldots, \omega_g$ be a basis for it.
While we're at it, let's continue with the analytic story. For a holomorphic differential ω and a path $\gamma : [0,1] \to X(\mathbb{C})$, i.e. a piecewise C^1 function, we can take an integral of ω along γ. The vectors

$$\int_\gamma \omega_i, 1 \leq i \leq g$$

reside in $\mathbb{C}^g$, viewed as the dual space to $H^0(X, \Omega_X)$.
By Cauchy's theorem, if γ, γ' are homologous with the same endpoints, then the integrals of any holomorphic differential along them agree. In particular, if γ is a closed path, then integration over ω_i's gives a map $\alpha : H_1(X(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}^g$, known as the period homomorphism.

Theorem 1.1 (Riemann, allegedly)**.** *α is injective and its image Λ is a lattice in $\mathbb{C}^g$, i.e. spanned by $2g$ $\mathbb{R}$-linearly independent vectors.*
Moreover, $\mathbb{C}^g/\Lambda$ is analytically isomorphic to a smooth algebraic variety over $\mathbb{C}$, known as the Jacobian variety $\operatorname{Jac}(X)$ of X.
In addition, the natural structure of a group on $\mathbb{C}^g/\Lambda$ is given by morphisms.

Definition 1.1. An abelian variety A over $\mathbb{C}$ is a projective variety over $\mathbb{C}$ together with a group structure on $A(\mathbb{C})$ induced by morphisms.

Fix $p_0 \in X(\mathbb{C})$. If $p \in X(\mathbb{C})$, let γ_p be a path from p_0 to p. Any two choices of γ_p differ by a closed path, so

$$\int_{\gamma_p} \omega_i, 1 \leq i \leq g$$

are well-defined modulo Λ. So they give a map $X(\mathbb{C}) \to \operatorname{Jac}(X)$. This extends to a homomorphism $\mathrm{AJ}_{p_0} : \operatorname{Div}(X) \to \operatorname{Jac}(X)$, known as the Abel-Jacobi map.
How does this depend on p_0? Suppose we had chosen $p_0' \in X(\mathbb{C})$ instead, let's get a path δ from p_0 to p_0'. Then we simply have

$$\mathrm{AJ}_{p_0'}(p) = \mathrm{AJ}_{p_0}(p) + \left(\int_\delta \omega_i\right)_i$$

In general, for $D \in \mathrm{Div}(X)$, we have

$$\mathrm{AJ}_{p_0'}(D) = \mathrm{AJ}_{p_0}(D) + (\deg D)\left(\int_\delta \omega_i\right)_i$$

This however means that the restriction of AJ_{p_0} to $\mathrm{Div}^0(X)$ does not depend on p_0. We write $\mathrm{AJ} : \mathrm{Div}^0(X) \to \mathrm{Jac}(X)$ to denote this restriction.

Theorem 1.2. AJ *induces an isomorphism* $\mathrm{Cl}^0(X) \to \mathrm{Jac}(X)$.

2 Homology of Riemann Surfaces

We are still in the setting of the preceding section. Suppose X has genus g, then $H_1(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and is generated by simple closed curves $a_j, b_j, 1 \leq j \leq g$, which are disjoint except for a_j meeting b_j, transversally at one point and positively oriented.
Let

$$A_{ij} = \int_{a_j} \omega_i, B_{ij} = \int_{b_j} \omega_i$$

Λ is the span of the period matrix $P = (A \mid B)$. Theorem 1.1 relies on the following special properties of it:

Theorem 2.1 (Riemann Period Relations)**.** *(a)* $AB^\top$ *is symmetric.*
(b) The Hermitian matrix $i^{-1}(BA^\dagger - AB^\dagger)$ *is positive-definite.*

In other words, (a) is saying that $\sum_j (A_{ij}B_{i'j} - B_{ij}A_{i'j}) = 0$ for any i, i', and (b) is saying that

$$\mathrm{Im}\left(\sum_j \int_{a_j} \bar{\omega} \int_{b_j} \omega\right) > 0$$

for any $\omega \in H^0(X, \Omega_X) \setminus \{0\}$.

Remark. We'll soon see that, due to these relations, A, B are invertible and the columns of $P = (A \mid B)$ are $\mathbb{R}$-linearly independent (and therefore Λ is a lattice). And (b) will give rise to $\mathbb{C}^g/\Lambda$ being a projective variety.

Lemma 2.2. *Let* ω, η *be closed (i.e. having zero differential), smooth but not necessarily holomorphic,* 1*-forms on* X*. Then*

$$\int_X \omega \wedge \eta = \sum_j \left(\int_{a_j} \omega \int_{b_j} \eta - \int_{b_j} \omega \int_{a_j} \eta\right)$$

Proof. Cut X along the curves a_j, b_j. Let X^* be the resulting surfaces with boundary. X^* is a sphere with g holes 'coz I said so.
The projection $\pi : X^* \to X$ induces the zero map $\pi_* : H_1(X^*, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ on homology. So on X^* there is a function f such that $\omega = \mathrm{d}f$.
If p^+, p^- are points on a_j^+, a_j^- (the slits after cutting along a_j) with the same image in X, then

$$f(p^+) - f(p^-) = \int_{p^-}^{p^+} \mathrm{d}f = \int_{b_j} \omega$$

Similarly, if $q^\pm$ are points on $b_j^\pm$ with the same image in X, then

$$f(q^+) - f(q^-) = \int_{q^-}^{q^+} \mathrm{d}f = \int_{a_j} \omega$$

Now we are basically done: By Stokes' theorem,

$$\begin{aligned}\int_X \omega \wedge \eta = \int_{X^*} \mathrm{d}(f\eta) = \int_{\partial X^*} f\eta &= \sum_j \left(\int_{b_j^+} - \int_{b_j^-} - \int_{a_j^+} + \int_{a_j^-} \right) f\eta \\ &= \sum_j \left(\int_{a_j} \omega \int_{b_j} \eta - \int_{b_j} \omega \int_{a_j} \eta \right) \qquad \square\end{aligned}$$

What this lemma actually says is that the intersection pairing on homology is dual to the pairing on closed 1-forms given by

$$(\omega, \eta) \mapsto \int_X \omega \wedge \eta$$

Proof of Theorem 2.1. (a) Apply the lemma to $\omega_i, \omega_{i'}$.
(b) Take $\omega \in H^0(X, \Omega_X)$ apply the lemma to $\bar{\omega}, \omega$. Locallhy, $\omega = f(z)\,\mathrm{d}z$ for holomorphic f, so $\bar{\omega} \wedge \omega = |f|^2\,\mathrm{d}\bar{z} \wedge \mathrm{d}z = 2i|f|^2\,\mathrm{d}x \wedge \mathrm{d}y$. So whenever $\omega \neq 0$ we have

$$0 < \frac{1}{i}\int_X \bar{\omega} \wedge \omega = \sum_{j=1}^{g} \frac{1}{i}\left(\int_{a_j} \bar{\omega} \int_{b_j} \omega - \int_{b_j} \bar{\omega} \int_{a_j} \omega \right) = 2\,\mathrm{Im}\left(\sum_j \int_{a_j} \bar{\omega} \int_{b_j} \omega \right)$$

which is what we need. $\square$

Now consider the matrix

$$J = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}$$

This is viewed as the "intersection matrix of $a_1, \ldots, a_g, b_1, \ldots, b_g$", since Theorem 2.1(a) is saying $PJ^{-1}P^\top = 0$. On the other hand, Theorem 2.1(b) is saying that the Hermitian matrix $Q = i^{-1}PJ^{-1}P^\dagger$ is positive-definite.
For $\lambda \in \mathbb{C}^g \setminus \{0\}$, $0 < \lambda^\top Q \bar{\lambda} = 2\,\mathrm{Im}(\lambda^\top BA^\dagger \bar{\lambda})$. So A, B are invertible, so

Corollary 2.3. *(i) There is a basis $(\omega_1, \ldots, \omega_g)$ such that*

$$\int_{a_j} \omega_i = \delta_{ij}$$

That is, $A = I_g$ and B is symmetric with positive-definite imaginary part.
(ii) The columns of P are linearly independent over $\mathbb{R}$, si $\alpha : H_1(X, \mathbb{Z}) \to \mathbb{C}^g$ is injective with image Λ which is a lattice.

Let's now sketch how one prove Theorem 1.2. One way to do it is by using the exponential sequence, which is the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp(2\pi i f)} \mathcal{O}_X^\times \longrightarrow 0$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, $\mathcal{O}_X^\times$ the nonvanishing ones, and $\underline{\mathbb{Z}}$ the constant sheaf $\mathbb{Z}$. Since $\mathbb{C} = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times) = \mathbb{C}^\times$ is surjective, the long exact sequence of cohomology gives

$$0 \longrightarrow H^1(X, \underline{\mathbb{Z}}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X, \underline{\mathbb{Z}})$$

This holds for any compact connected complex manifold. In the case of a Riemann surface, $H^2(X, \underline{\mathbb{Z}}) = \mathbb{Z}, H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$. By the way, we also have $H^1(X, \underline{\mathbb{Z}}) = H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})$. Under the isomorphism $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$, this map $\mathrm{Cl}(X) = H^1(X, \mathcal{O}_X^\times) \to \mathbb{Z} = H^2(X, \underline{\mathbb{Z}})$ is simply the degree map on divisors. We therefore have $\mathrm{Cl}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$.
One can check that the diagram

$$\begin{tikzcd}
\mathrm{Div}^0(X) \arrow[r] \arrow[d, "\mathrm{AJ}"'] & \mathrm{Cl}^0(X) \arrow[d, "\cong"] \\
H^0(X, \Omega_X)^\vee / \alpha(H_1(X, \mathbb{Z})) & H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \arrow[l, "\cong"]
\end{tikzcd}$$

commutes, where the bottom arrow is given by Serre duality. This establises Theorem 1.2.

3 Complex Tori

Recall that for $w_1, w_2 \in \mathbb{C} \setminus \{0\}, w_1/w_2 \notin \mathbb{R}$, the space $\mathbb{C}/(\mathbb{Z}w_1 \oplus \mathbb{Z}w_2)$ is a complex elliptic curve and can be embedded in $\mathbb{P}^2$ using the Weierstrass $\wp$-function and its derivative. By applying some more thought, we see that this gives a bijection between homothety classes of lattices in $\mathbb{C}$ and isomorphism classes of elliptic curves. Let's see what happens in higher dimensions, where things just get a lil' bit complicated.

Definition 3.1. Let V be a finite-dimensional real vector space. A lattice in V is a discrete subgroup $\Gamma \leq V$ such that V/Γ is compact.

Equivalently, a subgroup $\Gamma \leq V$ is a lattice if and only if $\Gamma = \bigoplus_i \mathbb{Z}e_i$ for $\{e_i\}_i$ is an $\mathbb{R}$-basis for V.
For a lattice $\Gamma \leq V$, the quotient V/Γ is a commutative, compact, connected Lie group, and is sometimes referred to as a real torus. By a change of basis, we get an isomorphism (of Lie groups) $V/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$ where $n = \dim V$.
The converse is also true: Any commutative, compact, connected Lie group is of this kind.
Now let V be a finite-dimensional $\mathbb{C}$-vector space, and let $\Gamma \leq V$ be a lattice (this makes sense since V is also a real vector space). In addition to being a Lie group, V/Γ is also a complex manifold: Suppose $\pi : V \to V/\Gamma$ is the quotient map. Then for any $v \in V$, we can choose some open neighbourhood $v \in U \subset V$ such that $\pi : V \to \pi(V)$ is a homeomorphism. This gives rise to a complex chart, and every complex chart obtained this way is compatible with each other. We therefore get a complex atlas.
Notably, the group structure on V/Γ is holomorphic with respect to this atlas. So it is a (compact, connected) complex Lie group.

Definition 3.2. We call V/Γ a complex torus.

Proposition 3.1. *Any compact connected complex Lie group is a complex torus. In particular, it is commutative.*

Proof. Omitted. $\square$

For any (real or complex) torus $X = V/\Gamma$, the map $\pi : V \to X$ is a regular covering. But V is simply connected, so V is the universal cover of X (with the origin always implicitly chosen to be the origin). So $\Gamma \cong \pi_1(X, 0)$. Hurewitz's theorem says that $H_1(X, \mathbb{Z}) \cong \pi_1(X, 0)^{\mathrm{ab}}$. But in this case $\pi_1(X, 0)$ is already abelian, so $\Gamma \cong \pi_1(X, 0) \cong H_1(X, \mathbb{Z})$.
Let $X = V/\Gamma$ and $X' = V'/\Gamma'$ be complex tori and suppose $\phi : V \to V'$ is a linear map such that $\phi(\Gamma) \subset \Gamma$. It induces a holomorphic map $X \to X'$ which is a homomorphism.
The converse is also true:

Proposition 3.2. *Suppose $f : X \to X'$ is a holomorphic map, then:*
(i) If $f(0) = 0$, then there exists a linear map $\tilde{f} : V \to V'$ with $\tilde{f}(\Gamma) \subset \Gamma'$ which lifts f. In particular, f is a homomorphism.
(ii) $f(x) = f_0(x) + y$ with $y = f(0) \in X'$ and f_0 is as in (i).

Proof. Since translations are holomorphic, it suffices to show (i).
As V' is simply connected, we always have a continuous lift $\tilde{f} : V \to V'$ with $\tilde{f}(0) = 0$.

$$\begin{tikzcd} V \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & V' \arrow[d, "\pi'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

Since π, π' are local isomorphisms, $\tilde{f}$ is holomorphic. For all $v \in V, \gamma \in \Gamma$ we have $\tilde{f}(v + \gamma) = \tilde{f}(v) + g_\gamma(v)$ for $g_\gamma(v) \in \Gamma'$. but Γ' is discrete and g_γ is continuous in v, so g_γ is constant. This in turn means that partial derivatives of $\tilde{f}$ are Γ-invariant, therefore they descend into a map $V/\Gamma \to V'$.
But V/Γ is compact, therefore the map is constant by the open mapping theorem, which shows that $\tilde{f}$ is affine. The fact that $\tilde{f}(0) = 0$ shows that $\tilde{f}$ is linear. $\square$

Corollary 3.3. *$V/\Gamma \cong V'/\Gamma'$ (as complex manifolds) iff $\phi(\Gamma) = \Gamma'$ for some linear isomorphism $\phi : V \to V'$.*

So any complex torus of (complex) dimension g is isomorphic to $\mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ where $\Pi \in \mathrm{Mat}_{g\times 2g}(\mathbb{C})$ whose columns are $\mathbb{R}$-linearly independent. Moreover, Π, Π' give isomorphic tori iff there is some $A \in \mathrm{GL}_g(\mathbb{C})$ and $B \in \mathrm{GL}_{2g}(\mathbb{Z})$ with $\Pi' = A\Pi B$.
As columns of Π span $\mathbb{C}^g$ over $\mathbb{R}$, some of them form a $\mathbb{C}$-basis, hence:

Proposition 3.4. *Every complex torus of dimension g is isomorphic to $\mathbb{C}^g/\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ where $\Omega \in \mathrm{Mat}_{g\times g}(\mathbb{C})$ such that the columns of $\mathrm{Im}\,\Omega$ are $\mathbb{R}$-linearly independent.*

Example 3.1. Every complex elliptic curve is isomorphic to something in the form $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathbb{C} \setminus \mathbb{R}$.

Let's turn back to topology.

Proposition 3.5. *If $X = V/\Gamma$ is a real torus of dimension $d \geq 1$, then $H^1(X,\mathbb{Z}) = \operatorname{Hom}(\Gamma,\mathbb{Z}) \cong \mathbb{Z}^d$, and for any $0 \leq n \leq d$ we have $H^n(X,\mathbb{Z}) = \bigwedge_{\mathbb{Z}}^n H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{\binom{d}{n}}$. In particular, they are free.*

Proof. For $n = 1$, $H^1(X,\mathbb{Z}) = \operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) = \operatorname{Hom}(\Gamma,\mathbb{Z})$. For $n > 1$, we do induction on d. When $d = 1$ we have $X \cong S^1$, so $H^n(X,\mathbb{Z}) = 0$ for $n > 1$. For $d > 1$, we write $\Gamma = \Gamma_1 \oplus \Gamma_2$ for $\Gamma_i \neq \{0\}$. Then $X = X_1 \times X_2$ where $X_i = V_i/\Gamma_i$ where $V_i = \mathbb{R} \otimes \Gamma_i$. By induction, $H^*(X_i,\mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}^* \operatorname{Hom}(\Gamma_i,\mathbb{Z})$ is torsion-free, so Künneth formula gives

$$\begin{aligned} H^n(X,\mathbb{Z}) &= \bigoplus_{p+q=n} H^p(X_1,\mathbb{Z}) \otimes H^q(X_2,\mathbb{Z}) \\ &= \bigoplus_{p+q=n} \left(\bigwedge^p \operatorname{Hom}(\Gamma_1,\mathbb{Z})\right) \otimes \left(\bigwedge^q \operatorname{Hom}(\Gamma_2,\mathbb{Z})\right) \\ &= \bigwedge^n (\operatorname{Hom}(\Gamma_1,\mathbb{Z}) \oplus \operatorname{Hom}(\Gamma_2,\mathbb{Z})) = \bigwedge^n \operatorname{Hom}(\Gamma,\mathbb{Z}) \end{aligned}$$

□

Remark. The isomorphism $H^*(X,\mathbb{Z}) \cong \bigwedge^* \operatorname{Hom}(\Gamma,\mathbb{Z})$ is in fact a ring isomorphism. This is because the isomorphism in the Künneth formula is given by

$$H^p(X_1) \times H^q(X_2) \xrightarrow{(\mathrm{pr}_1^*,\mathrm{pr}_2^*)} H^p(X_1 \times X_2) \times H^q(X_1 \times X_2) \xrightarrow{\smile} H^{p+q}(X_1 \times X_2)$$

If we turn to cohomology with real or complex coefficients $H^*(X,\mathbb{R}) = H^*(X,\mathbb{Z}) \otimes \mathbb{R}, H^*(X,\mathbb{C}) = H^*(X,\mathbb{Z}) \otimes \mathbb{C}$, then the cohomology can be described using differential forms.
For a smooth manifold X, we write $A^n(X)$ to denote the real vector space of smooth real-valued n-forms. Recall that there is an exterior derivative $\mathrm{d} : A^n(X) \to A^{n+1}(X)$ sending $f\,\mathrm{d}x_1 \wedge \cdots$ to $\mathrm{d}f \wedge \mathrm{d}x_1 \wedge \cdots$. The de Rham cohomology of X are $H^n_{\mathrm{dR}}(X) = A^n(X)|_{\mathrm{d}=0}/\mathrm{d}A^{n-1}(X)$.
We can do everything with complex coefficients (but still with smooth differentials), which gives us spaces of differentials $A^n_{\mathbb{C}}(X) = A^n(X) \otimes_{\mathbb{R}} \mathbb{C}$ and cohomology $H^n_{\mathrm{dR}}(X,\mathbb{C}) = A^n_{\mathbb{C}}(X)|_{\mathrm{d}=0}/\mathrm{d}A^{n-1}_{\mathbb{C}}(X) \cong H^n_{\mathrm{dR}}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. We have a comparison theorem:

Theorem 3.6 (de Rham)**.** *The integration pairing $H_n(X,\mathbb{Z}) \times H^n_{\mathrm{dR}}(X,\mathbb{R}) \to \mathbb{R}$ gives an isomorphism between $H^n(X,\mathbb{R}) = \operatorname{Hom}(H_n(X,\mathbb{Z}),\mathbb{R})$ and $H^n_{\mathrm{dR}}(X,\mathbb{R})$, compatible with the respective ring structures.*

Let's now specialise to the case of tori. Let $X = V/\Gamma$ be a real torus.

Definition 3.3. We say $\omega \in A^n(X)$ is invariant if for all $y \in X$ we have $T_y^*\omega = \omega$ where $T_y : X \to X$ is the translation $x \mapsto x + y$. We write $A^n(X)^{\mathrm{inv}} \subset A^n(X)$ to denote the space of invariant forms.

Example 3.2. $A^0(X)^{\mathrm{inv}}$ consists of constant functions.

Proposition 3.7. *If $\phi : V \to \mathbb{R}$ is a linear form, then $\mathrm{d}\phi \in A^1(X)^{\mathrm{inv}}$. This induces an isomorphism $\bigwedge^n \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}) \cong A^n(X)^{\mathrm{inv}}$ for all $n \geq 0$.*

Proof. For $y \in V$, let $\tilde{T}_y : V \to V, x \mapsto x + y$. Then $\tilde{T}_y^*\phi = \phi + \phi(y)$ as ϕ is linear. So $\tilde{T}_y^*\mathrm{d}\phi = \mathrm{d}\phi$, hence $T_y^*\mathrm{d}\phi = \mathrm{d}\phi$.

Pick coordinates x_i (so (x_i) is a basis for $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$). $\omega \in A^n(X)$ can be written (uniquely) in the form $\sum_{I=(i_1<\cdots<i_n)} f_I \,\mathrm{d}x_I$ where $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$. Then ω is invariant iff each f_I is invariant, i.e. constant. So $(\mathrm{d}x_I)_I$ is a basis for $A^n(X)^{\mathrm{inv}}$, therefore we have the claimed isomorphism. $\square$

Theorem 3.8. *$A^n(X)^{\mathrm{inv}} \subset A^n(X)|_{\mathrm{d}=0}$ and the map $A^n(X)^{\mathrm{inv}} \to H^n_{\mathrm{dR}}(X,\mathbb{R})$ is an isomorphism.*

We'll also show that the composite isomorphism

$$\bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \cong A^n(X)^{\mathrm{inv}} \cong H^n_{\mathrm{dR}}(X,\mathbb{R}) \cong H^n(X,\mathbb{R}) \cong \bigwedge^n \mathrm{Hom}(\Gamma,\mathbb{R})$$

is the obvious one, i.e. the n-th exterior power of the map $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \to \mathrm{Hom}(\Gamma,\mathbb{R})$ given by restriction, which we know is an isomorphism since Γ is a lattice.

Proof. We know that $A^n(X)^{\mathrm{inv}}$ is spanned by $\mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_n$ where $\phi_i \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$. We have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \arrow[r, "\mathrm{d},\cong"] \arrow[d, dashed, "f"'] & A^1(X)^{\mathrm{inv}} \arrow[r, hook] & A^1(X)|_{\mathrm{d}=0} \arrow[d] \\
\mathrm{Hom}(\Gamma,\mathbb{R}) \arrow[r, equal] & \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{R}) & H^1_{\mathrm{dR}}(X,\mathbb{R}) \arrow[l, "\cong"]
\end{tikzcd}$$

What does f do? It maps $\phi \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ to the homomorphism

$$\Gamma \ni \gamma \mapsto \int_\gamma \mathrm{d}\phi = \int_0^\gamma \mathrm{d}\phi = \phi(\gamma)$$

Of course this f is an isomorphism. Taking exterior powers give isomorphism on all degrees. $\square$

Remark. We get the same thing if we suddenly decide to tensor everything with $\mathbb{C}$. In this case, we get an isomorphism

$$\bigwedge^n_{\mathbb{C}} \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \cong A^n_{\mathbb{C}}(X)^{\mathrm{inv}} \cong H^n(X,\mathbb{C}) \cong \bigwedge^n_{\mathbb{C}} \mathrm{Hom}(\Gamma,\mathbb{C})$$

Now suppose X is a complex torus. V would've been a complex vector space. Observe that $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{R}} V,\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V \oplus \bar{V},\mathbb{C}) \cong V^* \oplus \bar{V}^*$, where $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$ and $\bar{V}^* = \mathrm{Hom}_{\mathrm{anti-linear}}(V,\mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$.
In other words, we have a decomposition $H^1(X,\mathbb{C}) \cong A^1_{\mathbb{C}}(X)^{\mathrm{inv}} \cong V^* \oplus \bar{V}^*$. More explicitly, $(a_i) \in V^*$ is identified with $\sum_i a_i \mathrm{d}z_i$ and $(b_j) \in \bar{V}^*$ with $\sum_j b_j \,\mathrm{d}\bar{z}_j$.
And in higher degrees,

$$H^n(X,\mathbb{C}) = \bigwedge^n_{\mathbb{C}}(V^* \oplus \bar{V}^*) = \bigoplus_{p+q=n} \bigwedge^p_{\mathbb{C}} V^* \otimes \bigwedge^q_{\mathbb{C}} \bar{V}^*$$

Definition 3.4. $\omega \in A^n_{\mathbb{C}}(X)$ is of Hodge type (p,q) (for $p+q=n$) if, locally, ω is a (smooth function) combination of $\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}$ for multi-indices $(i_1 < \cdots < i_p), (j_1 < \cdots < j_q)$.
We write $A^{p,q}(X)$ to denote the space of $\omega \in A^{p+q}_{\mathbb{C}}(X)$ of Hodge type (p,q).

Remark. This definition works for any complex manifold X, not just tori.

We always have a decomposition

$$A^n_{\mathbb{C}}(X) = \bigoplus_{p+q=n} A^{p,q}(X)$$

which does not always pass to cohomology if X is a general complex manifold. But for tori, our computation above suggests

Theorem 3.9 (Hodge Decomposition)**.** *Suppose $X = V/\Gamma$ is a complex torus, then for all $n \geq 0$,*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X) \cong A^{p,q}(X)^{\mathrm{inv}} \cong \bigwedge^p V^* \otimes \bigwedge^q \bar{V}^*$. Moreover, inside $H^n(X,\mathbb{C})$ we have the conjugation relation $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Remark. In general, we always need some extra structure to get such a decomposition. For example, this holds if we have a Kähler metric on the complex manifold, where we replace "invariant" by "harmonic".

Proposition 3.10. *Let $H^0(X,\Omega^n_X)$ be the vector space of holomorphic n-forms. Then $H^0(X,\Omega^n_X) = A^{n,0}(X)^{\mathrm{inv}} \cong \bigwedge^n_{\mathbb{C}} V^* = H^{n,0}(X)$.*

Proof. Pick basis $\mathbb{C}^g \cong V$. By the preceding theorem, $A^{n,0}(X)^{\mathrm{inv}}$ has basis $\{\mathrm{d}z_I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_n} : I = (i_1 < \cdots < i_n)\}$. On the other hand, $H^0(X,\Omega^n_X)$ consists of $\sum_I f_I \, \mathrm{d}z_I$ where each f_I is holomorphic and Γ-invariant. But this means that each f_I is constant, hence the proposition. $\square$

Theorem 3.11 (Dolbeault Isomorphism)**.** *There is a canonical isomorphism $H^{p,q}(X) \cong H^q(X,\Omega^p_X)$ where Ω^p_X is the sheaf of holomorphic p-forms on X.*

Proof. We reduce to the special case $p=0$ as follows: By the proof of the preceding proposition, $\Omega^p_X = \bigoplus_I \mathcal{O}_X \, \mathrm{d}z_I$. So we get an isomorphism $H^0(X,\Omega^p_X) \otimes_{\mathbb{C}} \mathcal{O}_X \cong \Omega^p_X$ and therefore $H^0(X,\Omega^p_X) \otimes_{\mathbb{C}} H^q(X,\mathcal{O}_X) \cong H^q(X,\Omega^p_X)$. In view of the identification $H^0(X,\Omega^p_X) \cong \bigwedge^p V^*$, it suffices to show that $H^q(X,\mathcal{O}_X) \cong \bigwedge^q \bar{V}^*$. More precisely, we have the next theorem. $\square$

Theorem 3.12. *The map $H^n(X,\mathbb{C}) \to H^n(X,\mathcal{O}_X)$ (via the inclusion $\underline{\mathbb{C}} \hookrightarrow \mathcal{O}_X$) factors as $H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \to H^{0,n}(X) \cong H^n(X,\mathcal{O}_X)$.*

Proof. We'll sketch the proof in the occasion $g=1$, just to give you a taste of the type of argument. The main perspective is that the elements of $A^0_{\mathbb{C}}(X)$ are given by Fourier series.
Assume $g=1$, then $X = \mathbb{C}/\Gamma = \mathbb{C}/(\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2)$. Set $z = x_1\gamma_1 + x_2\gamma_2$. For $f \in A^0_{\mathbb{C}}(X)$, we can write it in a form

$$f = \sum_{m_1,m_2 \in \mathbb{Z}} e^{2\pi i(m_1x_1 - m_2x_2)} c_{m_1,m_2} = \sum_{\gamma \in \Gamma} e^{\pi(\bar{\gamma}z - \gamma\bar{z})/A} c_\gamma$$

where $A = \operatorname{covol}\Gamma$ and for all N we have $|c_\gamma||\gamma|^N \to 0$ as $|\gamma| \to \infty$.
Write $\mathcal{A}^{p,q}$ to denote the sheaf of C^∞ (p,q)-forms. We have $\mathcal{O}_X = \ker\bar\partial$ where $\bar\partial : \mathcal{A}^{0,0} \to \mathcal{A}^{0,1}$ is the differential operator underlining the Cauchy-Riemann equations.
We claim that $\bar\partial$ is surjective as a map of sheaves. For an open set $U \subset X$ and $\omega = f\,\mathrm{d}\bar z \in \mathcal{A}^{0,1}$. Using bump functions, after possiby shrinking U we can find $g \in A^0_{\mathbb{C}}(X)$ such that $g|_U = f$ and the integral of g over $\mathbb{C}/\Gamma$ is zero. So g has a Fourier series with $c_0 = 0$. And we have

$$g\,\mathrm{d}\bar z = \bar\partial \sum_{\gamma\neq 0} -\frac{A}{\pi\gamma} c_\gamma e^{\pi(\bar\gamma z - \gamma\bar z)/A} \in \bar\partial(A^{0,0}(X))$$

So we now have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}^{0,0} \longrightarrow \mathcal{A}^{0,1} \longrightarrow 0$$

Using a partition of unity argument, $H^i(X, \mathcal{A}^{p,q}) = 0$ for any $i > 0$. So $H^0(X, \mathcal{O}_X) = \ker(\bar\partial : A^{0,0}(X) \to A^{0,1}(X)) = \mathbb{C}$ and $H^1(X, \mathcal{O}_X) = \operatorname{coker}(\bar\partial : A^{0,0}(X) \to A^{0,1}(X)) = A^{0,1}(X)/\bar\partial(A^{0,0}(X))$. But we just saw that $\omega \in A^{0,1}(X)$ lies in the image of $\bar\partial$ iff its zeroth Fourier coefficient is zero. So $A^{0,1}(X) = \operatorname{Im}\bar\partial \oplus \mathbb{C}\,\mathrm{d}\bar z = \operatorname{Im}\bar\partial \oplus A^{0,1}(X)^{\mathrm{inv}}$. □

Remark. In general, we have the exact sequence (from the $\bar\partial$-Poincaré Lemma)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}^{0,0} \longrightarrow \cdots \longrightarrow \mathcal{A}^{0,g} \longrightarrow 0$$

from which we get $A^{0,q}(X)|_{\bar\partial=0} = \bar\partial A^{0,q-1}(X) \oplus A^{0,q}(X)^{\mathrm{inv}}$.

4 Picard Groups of Complex Tori

Suppose X is a complex manifold.

Definition 4.1. An $\mathcal{O}_X$-module $\mathcal{L}$ is a line bundle if it is an invertible $\mathcal{O}_X$-module, i.e. if it's locally free of rank 1.

That is, X has an open cover $X = \bigcup_i U_i$ and isomorphisms (trivialisations) $s_i : \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$ (i.e. $s_i(1)$ is an nowhere vanishing section of $\mathcal{L}$ over U_i).

Definition 4.2. The Picard group of X is the group of isomorphism classes of line bundles on X under $\otimes$.

Proposition 4.1. $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$ *where* $\mathcal{O}_X^\times$ *is the sheaf of invertible holomorphic functions.*

We'll define the map and leave you to check it's an isomorphism.
Given $\mathcal{L}$ and its trivialisations (s_i) on (U_i), $c_{ij} = s_j^{-1}s_i$ lives in $\mathcal{O}_X^\times(U_i \cap U_j)$. Now $c_{ij}c_{jk} = c_{ik}$ on $U_i \cap U_j \cap U_k$, so $(c_{ij})_{i,j}$ gives a 1-Čech cocycle of $\mathcal{O}_X^\times$, therefore defines an element of $H^1(X, \mathcal{O}_X^\times)$. This depends only on $\mathcal{L}$, for if (s_i') is another trivialisation, then $t_i = s_i'(1)/s_i(1) \in \mathcal{O}_X^\times(U_i)$ has $c_{ij}' = c_{ij}t_i/t_j$ and (t_i/t_j) is a coboundary.

This bit is true for any ringed spaces. In the case of a complex manifold X, we have the additional access to the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) = \underline{2\pi i \mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

Suppose X is compact and connected. Then the H^0 of this sequence is $0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$. The long exact sequence continues as

$$\begin{tikzcd}
0 \arrow[r] & H^1(X,\mathbb{Z}(1)) \arrow[r, "j"] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^\times) \arrow[dll, "c_1"'] \\
& H^2(X,\mathbb{Z}(1)) \arrow[r] & H^2(X,\mathcal{O}_X)
\end{tikzcd}$$

Definition 4.3. Write $\operatorname{Pic}^0(X) = \operatorname{coker} j = \ker c_1$. $\operatorname{NS}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is called the Néron-Severi group of X.

Via c_1, $\operatorname{NS}(X)$ is isomorphic to the kernel of $H^2(X,\mathbb{Z}(1)) \to H^2(X,\mathcal{O}_X)$, which is a finitely-generated abelian group. It is the “discrete part” of $\operatorname{Pic}(X)$. On the other hand, $\operatorname{Pic}^0(X)$ is a quotient of complex vector spaces, hence may be interpreted as the “continuous part” of $\operatorname{Pic}(X)$.
Let’s now specialise to the case where $X = V/\Gamma$ is a complex torus. Accio diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}(1)) = \operatorname{Hom}(\Gamma,\mathbb{Z}(1)) \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "j"] & H^1(X,\mathbb{C}) = \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{C}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[d, "\cong"] \\
H^1(X,\mathbb{R}(1)) = \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R}(1)) \arrow[ur, hook] \arrow[rr, bend right, "j_{\mathbb{R}}"'] & V^* \oplus \bar{V}^* \arrow[u, "\cong"] \arrow[r, "\mathrm{pr}_2"'] & \bar{V}^*
\end{tikzcd}$$

Note that the reverse isomorphism $\operatorname{Hom}_{\mathbb{R}}(V,\mathbb{C}) \to V^* \oplus \bar{V}^*$ is given by $\ell \mapsto (\lambda,\mu)$ where $\lambda(v) = (1/2)(\ell(iv) - i\ell(iv)), \mu(v) = (1/2)(\ell(iv) + i\ell(iv))$. So $j_{\mathbb{R}}$, which is the $\mathbb{R}$-linear extension of j, is essentially $j_{\mathbb{R}}(\ell)(v) = (1/2)(\ell(v) + i\ell(iv)) = \mu(v)$. And so $j_{\mathbb{R}}$ is an isomorphism whose inverse is $\mu \mapsto \mu - \bar{\mu}$. Hence $j(H^1(X,\mathbb{Z}(1))) \subset \bar{V}^*$ is a lattice!

Theorem 4.2. $\hat{X} = \operatorname{Pic}^0(X) \cong \bar{V}^*/\operatorname{Im} j$ *is a complex torus.*

Definition 4.4. $\hat{X}$ is called the dual of X.

$j_{\mathbb{R}}^{-1}$ gives an isomorphism $X \cong \operatorname{Hom}(\Gamma,\mathbb{R}(1))/\operatorname{Hom}(\Gamma,\mathbb{Z}(1))$, and via exp it is then isomorphic to $\operatorname{Hom}(\Gamma, U(1))$.

Definition 4.5. A Riemann form for X is a Hermitian form $H : V \times V \to \mathbb{C}$ for which the alternating form $E = \operatorname{Im} H : V \times V \to \mathbb{R}$ is integer-valued on $\Gamma \times \Gamma$, i.e. $E \in \operatorname{Alt}^2_{\mathbb{Z}}(\Gamma)$.

Recall from example sheet that to give a Riemann form H is equivalent to give $E : \Gamma \times \Gamma \to \mathbb{Z}$ alternating such that its $\mathbb{C}$-bilinear extension $E_{\mathbb{C}} : (\mathbb{C}\otimes\Gamma)\times(\mathbb{C}\otimes\Gamma) = (V\oplus\bar{V})\times(V\oplus\bar{V}) \to \mathbb{C}$ satisfies $E_{\mathbb{C}}(V,V) = 0$ (equivalently, $E_{\mathbb{C}}(\bar{V},\bar{V}) = 0$). We can recover H via $H(u,v) = 2iE_{\mathbb{C}}((u,0),(0,\bar{v}))$.

Theorem 4.3. $\mathrm{NS}(X)$ *is isomorphic to the group of Riemann forms on* X*.*

Proof. We have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{NS}(X) \arrow[r, hook] & H^2(X,\mathbb{Z}(1)) \arrow[r] & H^2(X,\mathcal{O}_X) \\
0 \arrow[r] & \mathrm{NS}(X) \arrow[r, hook] \arrow[u, equal] & H^2(X,\mathbb{Z}) \arrow[r, "p"'] \arrow[u, "2\pi i,\cong"] & H^2(X,\mathcal{O}_X) \arrow[u, "2\pi i,\cong"]
\end{tikzcd}$$

with exact rows. But $H^2(X,\mathbb{Z}) = \bigwedge^2 \mathrm{Hom}(\Gamma,\mathbb{Z}) = \mathrm{Alt}^2_{\mathbb{Z}}(\Gamma)$, is the group of alternating bilinear forms $E : \Gamma\times\Gamma \to \mathbb{Z}$. On the other hand, $H^2(X,\mathcal{O}_X) = \bigwedge^2 \bar{V}^*$ is the group of alternating bilinear forms $\bar{V}\times\bar{V} \to \mathbb{C}$.
We claim that $p(E) = E_{\mathbb{C}}|_{\bar{V}\times\bar{V}}$, which implies the theorem.
Indeed, p factors as $H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \to H^2(X,\mathcal{O}_X)$. E is taken to $E_{\mathbb{C}} \in \mathrm{Alt}^2_{\mathbb{C}}(V\oplus\bar{V}) = H^2(X,\mathbb{C})$, which is mapped to $\mathrm{Alt}_{\mathbb{C}}(\bar{V}) = H^2(X,\mathcal{O}_X)$ by restricting to $\bar{V}\times\bar{V}$ due to Theorem 3.12. ☐

Remark. $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}(1))$ is essentially the first Chern class. And $c_1(\mathcal{L}) = 0$ iff the corresponding C^∞-line bundle is trivial.

We are left with a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0$$

But $\mathrm{NS}(X)$, being isomorphic to the group of Riemann forms, is free. So this short exact sequence splits (not canonically, though).
Let $P(X)$ be the group of pairs (H,α) such that H is a Riemann form and $\alpha : \Gamma \to U(1)$ satisfies $\alpha(\gamma+\delta) = \alpha(\gamma)\alpha(\delta)e^{\pi i E(\gamma,\delta)}$ where $E = \mathrm{Im}\, H$. This fits into an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\Gamma, U(1)) \longrightarrow P(X) \longrightarrow \{\text{Riemann forms}\}$$

It turns out that this is exact on the right, i.e. for any Riemann form H, there is some $\alpha : \Gamma \to U(1)$ satisfying the conditions we mentioned.

Theorem 4.4 (Appell-Humbert)**.** *There is an isomorphism* $P(X) \to \mathrm{Pic}(X)$ *making the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Gamma,U(1)) \arrow[r] \arrow[d, "\cong"] & P(X) \arrow[r] \arrow[d, "\cong"] & \{\textit{Riemann forms}\} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{NS}(X) \arrow[r] & 0
\end{tikzcd}$$

commute.

Proof. Once we've got a map $P(X) \to \mathrm{Pic}(X)$ making the diagram commute, that map must be an isomorphism by Five Lemma. So let's construct it.

Let $\pi : V \to X$ be the projection. The idea is the following: We'll write down $\mathcal{L}$ with $\pi^*\mathcal{L} \cong \mathcal{O}_V$ (in fact every invertible $\mathcal{O}_V$ sheaf is trivial). By adjunction, we need to find a subsheaf $\mathcal{L} \subset \pi_*\mathcal{O}_V$. We do this locally. For the purpose of this proof, a connected open subset $U \subset X$ is called small if $U = \pi(U')$ for some $U' \subset V$ open such that $\gamma + \overline{U'}, \gamma \in \Gamma$ are disjoint.
For a small open subset U, $\pi^{-1}U$ is the disjoint union of isomorphic copies U' of U, with isomorphisms induced by π. Γ permute the set of these copies simply transitively, and $\pi_*\mathcal{O}_X(U) = \prod_{\pi:U'\cong U} \mathcal{O}_V(U')$. Every open of X is a union of small opens. So to define a sheaf on X it suffices to define it over the small opens.
For a small open U, set $\mathcal{L}(U)$ to be

$$\left\{(s_{U'}) \in \prod_{\pi:U'\cong U} \mathcal{O}_V(U') : \forall\gamma \in \Gamma, z \in U', s_{U'+\gamma}(z+\gamma) = s_{U'}(z)c_\gamma(z)\right\}$$

For some family of holomorphic $c_\gamma : V \to \mathbb{C}^\times$ to be determined (note that for $c_\gamma \equiv 1$ then we recover $\mathcal{O}_X$).
Now, the condition $s_{U'+\gamma}(z+\gamma) = s_{U'}(z)c_\gamma(z)$ shows that $\mathcal{L}(U) \hookrightarrow \mathcal{O}_V(U')$ for each U'. If $\gamma, \delta \in \Gamma$, then condition gives $c_{\gamma+\delta}(z)s_{U'}(z) = s_{U'+\gamma+\delta}(z+\gamma+\delta) = c_\delta(z+\gamma)s_{U'+\gamma}(z+\gamma) = c_\delta(z+\gamma)c_\gamma(z)s_{U'}(z)$.
So if $\mathcal{L}(U) \neq \{0\}$, then (c_γ) satisfies the cocycle condition $c_{\gamma+\delta}(z) = c_\gamma(z)c_\delta(z+\gamma)$. Conversely, suppose (c_γ) satisfies the cocycle condition, then our inclusion becomes an isomorphism $\mathcal{L}(U) \cong \mathcal{O}_V(U')$ for any $U' \cong U$ via π. Observe also that if $g : V \to \mathbb{C}^\times$ is holomorphic and (c_γ) satisfies the cocycle condition, so does $c'_\gamma(z) = c_\gamma(z)g(z+\gamma)/g(z)$, which would define an invertible sheaf $\mathcal{L}'$ ismorphic to $\mathcal{L}$ by multiplying $s_{U'+\gamma}$ with $g(z+\gamma)$.
We'll construct (c_γ) starting from $(H, \alpha) \in P(X)$, which would define sheaves $\mathcal{L}(H, \alpha)$. Set $c_\gamma(z) = \alpha(\gamma)\exp(\pi(H(z,\gamma)+(1/2)H(\gamma,\gamma)))$. For each γ, $c_\gamma : V \to \mathbb{C}^\times$ is clearly holomorphic. It satisfies the cocycle condition, for

$$\begin{aligned}
&c_\gamma(z)c_\delta(z+\gamma)\\
&= \alpha(\gamma)\alpha(\delta)\exp\left(\pi\left(H(z,\gamma) + \frac{1}{2}H(\gamma,\gamma) + H(z,\delta) + H(\gamma,\delta) + \frac{1}{2}H(\delta,\delta)\right)\right)\\
&= \alpha(\gamma+\delta)\times\\
&\quad \exp\left(\pi\left(H(z,\gamma+\delta) + \frac{1}{2}(H(\gamma+\delta,\gamma+\delta) + H(\gamma,\delta) - H(\delta,\gamma)) - iE(\gamma,\delta)\right)\right)\\
&= c_{\gamma+\delta}(z)
\end{aligned}$$

since $H(\gamma,\delta) - H(\delta,\gamma) = H(\gamma,\delta) - \overline{H(\delta,\gamma)} = 2iE(\gamma,\delta)$, where $E = \operatorname{Im} H$ as you sure will recall from half a zillion years ago.
Let $\mathcal{L}(H,\alpha)$ be the line bundle on X given by this cocycle. If $(H,\alpha),(H',\alpha') \in P(X)$ give cocycles $(c_\gamma),(c'_\gamma)$, then $(H+H',\alpha\alpha')$ gives the cocycle $(c_\gamma c'_\gamma)$, so $\mathcal{L}(H+H',\alpha\alpha') \cong \mathcal{L}(H,\alpha)\otimes\mathcal{L}(H',\alpha')$. We therefore get a homomorphism $P(X) \to \operatorname{Pic}(X)$ sending (H,α) to the isomorphism class of $\mathcal{L}(H,\alpha)$. One can check that this does make the diagram commute. □

Let $\mathcal{L} \in \operatorname{Pic}(X)$ and $x \in X$. Let $T_x : X \to X$ be the translation map $y \mapsto x+y$. Then $T_x^*\mathcal{L}, \mathcal{L}$ have the same image in $\operatorname{NS}(X)$. Indeed, $\operatorname{NS}(X) \subset H^2(X,\mathbb{C}) \cong A^2_{\mathbb{C}}(X)^{\text{inv}}$.
So we can define $\phi_{\mathcal{L}}(x) = T_x^*\mathcal{L}\otimes\mathcal{L}^\vee \in \operatorname{Pic}^0(X)$.

Proposition 4.5. *$\phi_{\mathcal{L}} : X \to \mathrm{Pic}^0(X) = \hat{X}$ is a homomorphism of complex tori (i.e. a holomorphic homomorphism).*

Proof. Example sheet. $\square$

Theorem 4.6. *Let $\mathcal{L} = \mathcal{L}(H, \alpha)$. The followings are equivalent:*
(i) H is positive-definite.
(ii) $H^0(X, \mathcal{L}) \neq \{0\}$ and $\phi_{\mathcal{L}}$ is an isogeny.
(iii) $\mathcal{L}$ is ample.

Later, we'll prove algebraic analogues of some of these statements.

Definition 4.6. A polarisation on X is a positive-definite Riemann form on X.

So the theorem says that X is a projective variety if and only if X has a polarisation.

5 Group Schemes over a Field

Fix a field k, which we'll often assume to be algebraically closed. A k-scheme is a scheme X equipped with a morphism $a_X : X \to \mathrm{Spec}\, k$. This makes each $\mathcal{O}_X(U)$ a k-algebra. A morphism $X \to Y$ of k-schemes (a "k-morphism") is simply a morphism of schemes $X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[dl] \\ \mathrm{Spec}\, k & \end{tikzcd}$$

commutes. We'll write (Sch/k) for the category of k-schemes. Sometimes we just not explicitly mention k when it's understood that we're working with k-schemes and k-morphisms.

Definition 5.1. For k-schemes X, S, the the set $X(S) = \mathrm{Mor}_k(S, X)$ is called the set of S-valued points of X. For a k-algebra R, we write $X(R) = X(\mathrm{Spec}\, R)$ and call it the set of R-valued points of X.

Note that for any k-morphism $h : S' \to S$, we have a map $X(S) \to X(S')$ via $g \mapsto f \circ h$. This makes the assignment $S \mapsto X(S)$ a functor $(\mathsf{Sch}/k)^{\mathrm{op}} \to (\mathsf{Sets})$.

Definition 5.2. This functor, denoted $\hat{h}_X$, is called the functor of points of X.

For a k-morphism $X \to Y$ and any k-scheme S, we get a map $f_S : X(S) \to Y(S), g \mapsto f \circ g$. This gives a natural transformation $\hat{h}_X \to \hat{h}_Y$.

Example 5.1. Suppose $X = \mathbb{V}(I) \subset \mathbb{A}^n_k$ for an ideal $I \leq k[T_1, \ldots, T_n]$, then $X(R) = \{(a_i) \in R^n : \forall f \in I, f(a) = 0\}$ for any k-algebra R.

For k-schemes X, Y, we write $X \times Y$ to denote $X \times_{\mathrm{Spec}\, k} Y$ by convention. By definition we have a natural bijection $(X \times Y)(R) = X(R) \times Y(R)$.

Definition 5.3. A group scheme over k (or k-group scheme) is a k-scheme G, together with a (k-)morphism $m : G \times G \to G$ such that for all k-algebra R, $m_R : G(R) \times G(R) \to G(R)$ makes $G(R)$ a group.

Example 5.2. 1. $\mathbb{G}_a = \operatorname{Spec} k[t] \cong \mathbb{A}^1$ is a group scheme with $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ given by $k[t] \to k[t_1, t_2], t \mapsto t_1 + t_2$. Then $\mathbb{G}_a(R) = R$ and m_R is just addition. This is known as the additive group over k.
2. $\mathbb{G}_m = \operatorname{Spec} k[t^{\pm 1}] \cong \mathbb{A}^1 \setminus \{0\}$ is a group scheme with $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ given by $k[t^{\pm 1}] \to k[t_1^{\pm 1}, t_2^{\pm 1}], t \mapsto t_1 t_2$. Then $\mathbb{G}_a(R) = R^\times$ and m_R is just multiplication.
3. For $n \geq 1$, we define $\mathrm{GL}_n = \operatorname{Spec} k[T_{ij}, 1/\det(T_{ij})]$ with $m_R : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$ given by

$$\begin{aligned} \operatorname{Spec} k[T_{ij}, \det(T_{ij})^{-1}] &\to \operatorname{Spec} k[U_{ij}, V_{ij}, \det(U_{ij})^{-1}, \det(V_{ij})^{-1}] \\ T_{ij} &\mapsto \sum_l U_{il} V_{lj} \end{aligned}$$

Guess what, $(\mathrm{GL}_n)(R) = \mathrm{GL}_n(R)$.

Let's talk more about the functor of points coz we want to. As you know, the underlying set of a scheme doesn't tell you much about it. For example, the underlying set of $X = \mathbb{A}^1$ is not a group (what would one do with the generic point anyways). The problem is that we don't even have $|\mathbb{A}^1 \times \mathbb{A}^1| = |\mathbb{A}^1| \times |\mathbb{A}^1|$. However, we have a powerful workaround: The functor $S \mapsto X(S)$ captures everything, and I mean everything, about X.

Lemma 5.1 (Yoneda Lemma)**.** *The natural map between the set of k-morphisms $X \to Y$ and the set of natural transformations $\hat{h}_X \to \hat{h}_Y$ is a bijection.*

Proof. Given a natural transformation (f_S), we get a k-morphism $X \to Y$ given by $f_X(\mathrm{id}_X) \in Y(X)$, which provides an inverse. □

Lemma 5.2 (Yoneda but better)**.** *Let h_X be the restriction of $\hat{h}_X$ to the category (Aff/k) of affine schemes over k, then the preceding lemma is still true if we replace $\hat{h}_X \to \hat{h}_Y$ by $h_X \to h_Y$.*

Proof. The inverse is now given by the following: Suppose we have an affine open cover $X = \bigcup_\alpha U_\alpha$. Let $j_\alpha : U_\alpha \to X$ be the open immersions. Given a natural transformation (f_S), we have $f_{U_\alpha}(j_\alpha) \in Y(U_\alpha)$. For any open affine $V \subset U_\alpha \cap U_\beta$, $f_{U_\alpha}(j_\alpha)$ and $f_{U_\beta}(j_\beta)$ restricts to the same element of $Y(V)$ by naturality, so these $f_{U_\alpha}(j_\alpha)$ glue to an element of $Y(X) = \{X \to Y\}$. □

Proposition 5.3. *Let G be a group scheme with multiplication m. Then:*
(i) $G(S)$ is a group under m_S for each $S \in (\mathsf{Sch}/k)$ (i.e. not just for affine S).
(ii) For any $S' \to S$, the map $G(S) \to G(S')$ is a group homomorphism.

Proof. Suppose $S' = \operatorname{Spec} R', S = \operatorname{Spec} R$ are affine. We'll show (ii) in this occasion, i.e. we need to check that the diagram

$$\begin{tikzcd} (G \times G)(S) \arrow[r, "\cong"] \arrow[d] & G(S) \times G(S) \arrow[r, "m_S"] \arrow[d] & G(S) \arrow[d] \\ (G \times G)(S') \arrow[r, "\cong"'] & G(S') \times G(S') \arrow[r, "m_{S'}"'] & G(S') \end{tikzcd}$$

commutes. But it does.
(i) Cover S by open affines $(U_i)_{i \in I}$. Also cover $U_i \cap U_j = \bigcup_{k \in K_{ij}} U_{ij}^k$. Then for

any X, we have

$$X(S) = \left\{ (x_i) \in \prod_i X(U_i) : \forall i, j \in I, \forall k \in K_{ij}, x_i|_{U_{ij}^k} = x_j|_{U_{ij}^k} \right\}$$

Apply this identification to G and $G \times G$. $G(S)$ therefore embeds in the group $\prod_{i \in I} G(U_i)$, so m_S is associative. To see it is a subgroup, we simply need to know that the maps $G(S) \times G(S) \to G(S) \times G(S)$ given by $(x, y) \mapsto (xy, y)$ and $(x, y) \mapsto (yx, y)$ are bijective. But they are clear from our identification.
(ii) Use our argument for the affine case again, except now knowing (i). $\square$

In particular,

Corollary 5.4. *There exists $e \in G(k), i \in G(G)$ such that for any k-scheme S, e becomes the identity on $G(S)$ and $i_S : G(S) \to G(S)$ is the inverse on $G(S)$.*

Proof. We of course take e to be the identity on $G(k)$. And $i \in G(G)$ is taken to be the inverse of $\mathrm{id}_G \in G(G)$. $\square$

Example 5.3. Take any (abstract) group Γ. The constant group scheme over k associated to Γ is $G = \coprod_{\gamma \in \Gamma} \operatorname{Spec} k$ with the group morphism given by the group operation on Γ. Clearly G is affine iff Γ is finite.

Remark. There is an alternative way to define a group scheme, which is a set of data (G, m, e, i) with $m \in G(G \times G), e \in G(k), i \in G(G)$ satisfying the same axioms we'd expect for a group except phrased in terms of morphsims. For example, the associativity axiom becomes the commutativity of the diagram

$$\begin{tikzcd}
(G \times G) \times G \arrow[r, "m \times \mathrm{id}_G"] \arrow[d, "\cong"] & G \times G \arrow[r, "m"] & G \\
G \times (G \times G) \arrow[r, "\mathrm{id}_G \times m"'] & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

That is, G is a group object in the category of schemes.

Definition 5.4. A homomorphism of group schemes is a morphism $G \to G'$ such that for each k-scheme S (eqv. k-algebra R), $G(S) \to G'(S)$ (resp. $G(R) \to G'(R)$) is a group homomorphism.

Remark. Equivalently, $f : G \to G'$ is a homomorphism iff the diagram

$$\begin{tikzcd}
G \times G \arrow[r, "f \times f"] \arrow[d, "m"'] & G' \times G' \arrow[d, "m'"] \\
G \arrow[r, "f"'] & G
\end{tikzcd}$$

commutes.

Definition 5.5. A closed subgroup scheme H of G is a closed subscheme such that for all k-algebra R, $H(R)$ is a subgroup of $G(R)$.

In particular, a closed subgroup scheme is also a group scheme and the closed embedding is a homomorphism of group schemes. The image of $\mathrm{id}_{H\times H} \in (H\times H)(H\times H)$ in $H(H\times H)$ from the diagram

$$\begin{tikzcd} (H\times H)(S) \arrow[r, hook] \arrow[d] & (G\times G)(S) \arrow[d] \\ H(S) \arrow[r, hook] & G(S) \end{tikzcd}$$

is the multiplication morphism.

Example 5.4. $e : \operatorname{Spec} k \to G$ is a closed subgroup scheme.

Let $f : G \to G'$ be a homomorphism of group schemes.

Definition 5.6. The kernel $\ker f$ of f is the fibre of f at $e' \in G'(k)$. That is, it is the fibre product $\ker f = G \times_{f,e',G} \operatorname{Spec} k$.

As e' is a closed immersion, $\ker f$ is a closed subscheme of G. Moreover, $(\ker f)(S) = \ker(f_S : G(S) \to G'(S))$, so $\ker f$ is a closed subgroup scheme.

Example 5.5. Let $G = \mathrm{GL}_n$ and $G' = \mathbb{G}_m$. We have group homomorphisms $\det_R : \mathrm{GL}_n(R) \to \mathbb{G}_m(R)$ functorial in R, therefore defines a homomorphism of group schemes $\det : G \to G'$. We write $\mathrm{SL}_n = \ker\det$. Explicitly, it is the closed subscheme defined by $\det(T_{ij}) = 1$ in $\operatorname{Spec} k[T_{ij}, 1/\det(T_{ij})]$.

Quotients are much more subtle and is not gonna be discussed, as it seems.

Definition 5.7. Let G be a group scheme and $x \in G(k)$. The left translation by x is the unique morphism $T_x : G \to G$ such that $T_x(y) = (x \circ a_S)y$ for all $y \in G(S)$. In other words, it is the composite $m \circ (x \times \mathrm{id}_G)$, identifying G with $(\operatorname{Spec} k) \times G$.

So $T_e = \mathrm{id}_G$ and $T_{xy} = T_x T_y$.

Definition 5.8. A k-variety is a separated k-scheme of finite type over k which is geometrically integral, in the sense that $X_{\bar{k}} = X \times \operatorname{Spec} \bar{k}$ is integral.

In particular, k is algebraically closed in the function field of X.

Definition 5.9. A complete k-variety is a k-variety proper over k.
A group variety (or connected algebraic group) is a group scheme which is a variety.
An abelian variety is a complete group variety.

It's worth noting that the phrase "algebraic group" means different things for different people.

Example 5.6. $\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n$ are all (affine) group varieties. The simplest nontrivial example of an abelian variety is an elliptic curve E, which is a nonsingular plane cubic with a distinguished k-point.

The completeness assumption has very strong implications, such as commutativity.

Theorem 5.5 (Mumford's Rigidity Lemma)**.** *Suppose X, Y, Z are k-varieties with X complete. Let $y_0 \in Y$ and $f : X \times Y \to Z$ a morphism. Suppose $f(X \times \{y_0\}) = \{z_0\}$ is a point, then there is some $g : Y \to Z$ such that $f = g \circ \mathrm{pr}_2$. In particular, for all $y \in Y$, $f(X \times \{y\})$ is a single point.*

More precisely, $X \times \{y_0\}$ means $X \times \operatorname{Spec} \kappa(y_0) \hookrightarrow X \times Y$ which is the fibre of pr_2 at $y_0 \in Y$. This is in general not the set-theoretic product. But it is if $y_0 \in Y(k)$.

Remark. The completeness assumption is necessary: For the map $f : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, (x, y) \mapsto xy$, we have $f(\mathbb{A}^1 \times \{0\})$ being a single point, and yet $f(\mathbb{A}^1 \times \{1\}) = \mathbb{A}^1$.

Corollary 5.6. *Suppose X is an abelian variety and G a group variety. Then for any morphism $f : X \to G$, $T_{f(e_X)^{-1}} \circ f$ is a homomorphism of group schemes.*

Proof. WLOG $f(e_X) = e_G$. Consider $p : X \times X \to G$ given on S-valued points by $p(x, y) = f(x)f(y)f(xy)^{-1}$. Since $p(X \times \{e_X\}) = e_X = p(\{e_X\} \times X)$, the preceding theorem shows that p factors through both pr_1 and pr_2. So $p(x, y) = p(x, e) = p(e, e) = e$ for all $x, y \in X(S)$, which means that f is a homomorphism of group schemes. □

Taking $G = X$ reveals that any isomorphism of schemes $f : X \to X$ taking e to e is an isomorphism of group schemes. In particular,

Corollary 5.7. *Any abelian scheme is commutative.*

By commutative we of course mean that the group structure on any S-valued points is commutative.

Proof. Take $f = i_X$. □

From now on, we state theorems for arbitrary fields and prove them for algebraically closed ones. We'll discuss the differences between these situations whenever convenient.

Proof of Theorem 5.5. Suppose $k = \bar{k}$. Let $x_0 \in X(k)$. Define $g : Y \to Z$ via $g(y) = f(x_0, y)$ (on the level of S-valued points). In other words, this is the map

$$\begin{tikzcd} \operatorname{Spec} k \times Y \arrow[r, "x_0 \times \mathrm{id}_Y"] \arrow[d, equal] & X \times Y \arrow[r, "f"] & Z \\ Y \arrow[urr, "g"'] & & \end{tikzcd}$$

We need to show that $g \circ \mathrm{pr}_2 = f$. It is enough to do so for a dense open of $X \times Y$ since everything is separated.
Let $W \subset Z$ be an affine open neighbourhood of z_0, and let $S = Z - W$. Then $f^{-1}(S)$ is closed, so $\mathrm{pr}_2(f^{-1}(S)) \subset Y$ is closed since X is complete. Let $V = Y - \mathrm{pr}_2(f^{-1}(S))$ which is open in Y, and we have by definition that $f(X \times V) \subset W$.
Now for all $y \in V(k)$, f restricts to a morphism $X \times \{y\} \to W$, which is constant since X is complete and W is affine. So $f|_{X \times \{y\}}$ is constant with value $f(x_0, y) = g(y)$.
For all $y \in V(k)$, $f|_{X \times \{y\}} = g \circ \mathrm{pr}_2|_{X \times \{y\}}$. And so $f|_{X \times V} = g \circ \mathrm{pr}_2|_{X \times V}$. By

the way, V is nonempty: Since $z_0 \notin S$, we have $X \times \{y_0\} \cap f^{-1}(S) = \varnothing$, so $y_0 \notin \mathrm{pr}_2(f^{-1}(S))$. Therefore $y_0 \in V$ and hence $V \neq \varnothing$. □

Remark. We used the condition $k = \bar{k}$ when we chose $x_0 \in X(k)$ (if k is not algebraically closed, $X(k)$ can be empty). In the general case, for affine opens $U \subset X, V \subset Y$ sufficiently small so that $f(U \times V) \subset W \subset Z$, we can consider $\mathcal{O}_Z(W) \to \mathcal{O}_{X\times Y}(U \times V) = \mathcal{O}_X(U) \otimes O_Y(V)$. What we need is for this to factor through $k \otimes_k \mathcal{O}_Y(V)$. But we can check this after extending scalar to $\bar{k}$ since $k \otimes_k \mathcal{O}_Y(V) = (\mathcal{O}_X(U) \otimes_k \mathcal{O}_Y(V)) \cap (\bar{k} \otimes_k \mathcal{O}_Y(V))$

6 Seesaw and Cube

Suppose we have a morphism $X \to Y$ and an invertible sheaf $\mathcal{L}$ on X. For $y \in Y$, we can write down the fibres $X_y = X \times \operatorname{Spec} \kappa(y)$ and $\mathcal{L}_y = i_y^* \mathcal{L}$. It is common to ask the following questions: How does $H^0(X_y, \mathcal{L}_y)$ (or in general H^i) vary with y? And what conditions ensure that there is some $\mathcal{M}$ on Y with $\mathcal{L} \cong f^*\mathcal{M}$?
Indeed, all $\mathcal{L}_y$ are trivial if $\mathcal{L} = f^*\mathcal{M}$ for some $\mathcal{M}$. Is the converse true, though?

Example 6.1. 1. Suppose C is a complete curve over k and D a divisor on C. From the Riemann-Roch formula, we get some estimates on the dimension of $L(D) = H^0(C, \mathcal{O}_C(D))$. One can ask how this varies with the choice of D.
2. Let $Y = \operatorname{Spec} k[u, v, w]/(uv - w^2) \subset \mathbb{A}^3$ be a quadric cone (say with $\operatorname{char} k \neq 2$). Take $X \subset Y$ the complement of the origin and let $f : X \to Y$ be the open immersion. Let L be a line through the origin contained in Y and consider $\mathcal{L} = \mathcal{O}_X(L \cap X)$. As fibres of f are points, all $\mathcal{L}_y$ are trivial, but there does not exist an invertible sheaf $\mathcal{M}$ on Y such that $f^*\mathcal{M} = \mathcal{L}$ since $L \subset Y$ is not a Cartier divisor.

Theorem 6.1 ("Seesaw Theorem")**.** *Let X, Y be k-varieties with X complete. Let $\mathcal{L}$ be an invertible $\mathcal{O}_{X\times Y}$-module, then:*
(i) $F = \{y \in Y : \mathcal{L}|_{X\times\{y\}}$ trivial$\}$ is closed in Y.
(ii) If $F = Y$, then there is an invertible sheaf $\mathcal{M}$ on Y with $\mathcal{L} = \mathrm{pr}_2^* \mathcal{M}$.

Theorem 6.2. *If X is a complete k-variety and $S = \operatorname{Spec} A$ for some Noetherian k-algebra A. Suppose $\mathcal{L}$ is an invertible sheaf on $X \times S$, then:*
(i) $H^0(X \times S, \mathcal{L})$ is a finite A-module.
(ii) There exists a morphism $\alpha : K^0 \to K^1$ of finite free A-modules and functorial isomorphisms $H^0(X \times \operatorname{Spec} B, \mathcal{L}_B) \cong \ker(\alpha_B = \alpha \otimes_A \mathrm{id}_B : K^0 \otimes_A B \to K^1 \otimes_A B)$ for all A-algebras B, where $\mathcal{L}_B$ is the pullback of $\mathcal{L}$ along $X \times \operatorname{Spec} B \to X \times \operatorname{Spec} A$.

Remark. The theorem in fact holds for other H^i's, where instead we use a complex $K^0 \to K^1 \to \cdots$.

Corollary 6.3. *Under the hypotheses of the preceding theorem, there is a finite A-module M such that for all B, $H^0(X \times \operatorname{Spec} B, \mathcal{L}_B) \cong \operatorname{Hom}_A(M, B) = \operatorname{Hom}_B(M \otimes_A B, B)$.*

Proof. Take $M = \operatorname{coker} \alpha^\vee$, i.e. we have the exact sequence

$$(K^1)^\vee \xrightarrow{\alpha^\vee} (K^0)^\vee \longrightarrow M \longrightarrow 0$$

where $(K^i)^\vee = \mathrm{Hom}_A(K^i, A)$. As $\mathrm{Hom}_A((K^i)^\vee, B) = K^i \otimes_A B$ (as K^i are finite free), we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, B) \longrightarrow K^0 \otimes_A B \xrightarrow{\alpha_B} K^1 \otimes_A B \qquad \square$$

Corollary 6.4. *Under the same hypothesis, for every $d \geq 0$, $Z_d = \{s \in S : \dim_{\kappa(s)} H^0(X \times \operatorname{Spec} \kappa(s), \mathcal{L}_s) \geq d\}$ is closed in S.*

This is a special case of what's known as the "semicontinuity theorem", which is actually true for all H^i.

Proof. Suppose $K^0 =\cong A^m, K^1 \cong A^n$. Then $\alpha^\top$ is represented by an $(m \times n)$-matrix C. We then have $Z_d = \{s \in S : \operatorname{rank}(\alpha^\top \otimes \mathrm{id}_{\kappa(s)}) \leq m - d\}$. But $\operatorname{rank}(\alpha^\top \otimes \mathrm{id}_{\kappa(s)}) \leq m - d$ is equivalent to say that all $(m - d + 1)$-minors of C vanish in $\kappa(s)$, which is a closed condition. $\square$

Lemma 6.5. *Suppose V is a complete K-variety and $\mathcal{L}$ is an invertible $\mathcal{O}_V$-module. Then $\mathcal{L}$ is trivial iff both $H^0(V, \mathcal{L})$ and $H^0(V, \mathcal{L}^\vee)$ are nonzero.*

Proof. Exercise. It might be useful to mention the fact that $\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{L}, \mathcal{L}) = \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \mathcal{O}_V) = K$ as V is a complete variety, and that $\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, \mathcal{L}) = H^0(V, \mathcal{L})$. $\square$

Proof of Theorem 6.1. (i) WLOG $Y = \operatorname{Spec} A$ is affine since this is a local statement. Then the claim is immediate from the preceding lemma and Corollary 6.4.
(ii) Suppose $F = Y$. We want to show that $\mathcal{L}$ is the pullback of some $\mathcal{M}$ on Y. We will show that $\mathcal{M} = (\mathrm{pr}_2)_*\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module and the adjunction $\mathrm{pr}_2^* \mathcal{M} \to \mathcal{L}$ is an isomorphism. This is now a local statement, so we may assume $Y = \operatorname{Spec} A$ is affine. It's enough to show that for all $y \in Y$ there is an open affine $U \ni y$ on which $\mathcal{L}|_{X \times U}$ is trivial.
We know that $\dim_{\kappa(y)} M \otimes_A \kappa(y) = \dim_{\kappa(y)} H^0(\mathcal{L}_y) = 1$ since $\mathcal{L}_y$ is trivial. By Nakayama's lemma, M is locally free of rank 1. Replacing Y with a smaller affine neighbourhood of y, we may assume that M is cyclic.
Suppose m generates M. Then

$$\mathrm{Hom}_{\mathcal{O}_{X \times Y}}(\mathcal{O}_{X \times Y}, \mathcal{L}) = H^0(X \times Y, \mathcal{L}) = \mathrm{Hom}_A(M, A) \cong m^\vee A$$

So $m^\vee$ gives a map $\mathcal{O}_{X \times Y} \to \mathcal{L}$ whose restriction to each $X \times \{y\}$ is the isomorphism $\mathcal{O}_{X \times \{y\}} \to \mathcal{L}_y$, and similarly for $\mathcal{L}^\vee$. This gives an isomorphism $\mathcal{O}_{X \times Y} \to \mathcal{L}$. $\square$

Remark. The proof actually gives something stronger: There exists a maximal closed subscheme $Z \subset Y$ such that $\mathcal{L}|_{X \times Z} \cong \mathrm{pr}_2^* \mathcal{M}$ for some $\mathcal{M}$ on Z. For example, if $Y = \operatorname{Spec} A$ is affine and M is cyclic, then $Z = \operatorname{Spec} A/I$ with $I = \mathrm{ann}_A(M)$.

Suppose now that $\mathcal{L}$ is an invertible sheaf on $X \times Y$ such that $\mathcal{L}_{X \times \{y\}}$ is trivial for all $y \in Y$ and that there is some $x_0 \in X(k)$ with $\mathcal{L}_{\{x_0\} \times Y}$ trivial. Then $\mathcal{L} = \mathrm{pr}_2^* \mathcal{M}$ by Theorem 6.1, so $\mathcal{O}_Y \cong (\mathrm{pr}_2^* \mathcal{M})|_{\{x_0\} \times Y} = \mathcal{M}$, which in turn means that $\mathcal{L}$ is trivial.

Remark. There exists an example of a nontrivial $\mathcal{L}$ on $X \times Y$ with X, Y elliptic curves such that $\mathcal{L}_{\{x_0\} \times Y}$, $\mathcal{L}_{X \times \{y_0\}}$ are trivial for some $x_0 \in X(k)$ and $y_0 \in Y(k)$.

But things are better when you get to three.

Theorem 6.6 (Theorem of the Cube)**.** *Suppose X, Y, Z are k-varieties with X, Y complete. Suppose x, y, z are k-points of X, Y, Z and $\mathcal{L}$ is an invertible sheaf on $X \times Y \times Z$ which is trivial on $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$ and $X \times Y \times \{z\}$. Then $\mathcal{L}$ is trivial.*

Proof. Probably omitted or deferred. □

Corollary 6.7. *Suppose X is an abelian variety with an invertible $\mathcal{O}_X$-module $\mathcal{L}$. For any variety Y and three morphisms $f, g, h \in X(Y)$, we have $(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^\vee \otimes g^*\mathcal{L}^\vee \otimes h^*\mathcal{L}^\vee$.*

Proof. First consider the case where $Y = X \times X \times X$ and $f = \mathrm{pr}_1^3, g = \mathrm{pr}_2^3, h = \mathrm{pr}_3^3$ are the projections. We need to show that $\mathcal{M} = (\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3)^*\mathcal{L} \otimes (\mathrm{pr}_1^3 + \mathrm{pr}_2^3)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^3 + \mathrm{pr}_3^3)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_2^3 + \mathrm{pr}_3^3)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^3)^*\mathcal{L} \otimes (\mathrm{pr}_2^3)^*\mathcal{L} \otimes (\mathrm{pr}_3^3)^*\mathcal{L}$ is trivial.
Let $q : X \times X \to X \times X \times X, (x, y) \mapsto (x, y, e)$. Then $(\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = (\mathrm{pr}_1^2 + \mathrm{pr}_2^2) \circ q = m$ where $\mathrm{pr}_i^2 : X \times X \to X$ are the projections.
On the other hand, $(\mathrm{pr}_1^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_1^3 \circ q = \mathrm{pr}_1^2$, $(\mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_2^3 \circ q = \mathrm{pr}_2^2$.
And $\mathrm{pr}_3^3 \circ q = e$.
These computations yield $\mathcal{M}|_{X \times X \times \{e\}} = q^*\mathcal{M} = m^*\mathcal{L} \otimes m^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^2)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_2^2)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^2)^*\mathcal{L} \otimes (\mathrm{pr}_2^2)^*\mathcal{L} \otimes \mathcal{O}_{X \times X}$, which is trivial. By symmetry, $\mathcal{M}$ is trivial when restricted to $X \times \{e\} \times X$ and $\{e\} \times X \times X$ too. Hence by the preceding theorem we conclude that $\mathcal{M}$ is trivial.
The general case is reduced to the special case of projections by the factorisations $f = \mathrm{pr}_1^3 \circ (f, g, h)$, $g = \mathrm{pr}_2^3 \circ (f, g, h)$, $h = \mathrm{pr}_3^3 \circ (f, g, h)$. □

Corollary 6.8 (Theorem of the Square)**.** *Suppose X is an abelian variety and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then for any $x, y \in X(k)$, $T_{x+y}^*\mathcal{L} = T_x^*\mathcal{L} \otimes T_y^*\mathcal{L} \otimes \mathcal{L}^\vee$.*

Proof. Take $f = x (= x \circ a_X)$, $g = y$ and $h = \mathrm{id}_X$ in the preceding corollary. □

Corollary 6.9. *Let X be an abelian variety, $n \in \mathbb{Z}$, and $[n] : X \times X$ is the multiplication-by-n morphism. Then $[n]^*\mathcal{L} = \mathcal{L}^{\otimes(n(n+1)/2)} \otimes (i^*\mathcal{L})^{\otimes(n(n-1)/2)}$ (where $i = [-1]$ is the inversion map on X).*

Here, when n is negative, we use the convention $\mathcal{L}^{\otimes n} = (\mathcal{L}^\vee)^{\otimes |n|}$.

Proof. When $n = 0, 1$ the statement is trivial. For $n \geq 2$, we take $f = [n-1]$, $g = \mathrm{id}_X = [1]$, $h = i = [-1]$ in Corollary 6.7 and finish by induction. We obtain the case for negative n from the observation $[-n]^*\mathcal{L} = i^*[n]^*\mathcal{L}$. □

7 Picard Group of Abelian Varieties

First, an aside.

Proposition 7.1. *Suppose G/k is a group variety. Then G is nonsingular.*

Proof. Assume WLOG that $k = \bar{k}$. The set of nonsingular closed points on G is dense in G. Suppose $y \in G$ is nonsingular, then for any other closed point $x \in G$, $T_{xy^{-1}} : G \to G$ is an automorphism of G taking y to x. Hence x too is nonsingular. Therefore G is nonsingular. □

For a k-scheme X, we write $X_{\bar{k}} = X \times \operatorname{Spec} \bar{k}$.

Definition 7.1. Let X be an abelian variety.
Suppose $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. For $x \in X(\bar{k})$, we define $\phi_{\mathcal{L}}(x) \in \operatorname{Pic}(X_{\bar{k}})$ to be the isomorphism class of $T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ (after a base-change to $\bar{k}$, that is).

Theorem 6.8 tells us that $\phi_{\mathcal{L}}$ is a homomorphism of groups.

Definition 7.2. We write $K(\mathcal{L})$ for the kernel of $\phi_{\mathcal{L}}$ in $X(\bar{k})$, and $\operatorname{Pic}^0(X)$ the set of isomorphism classes of $\mathcal{L}$ with $\phi_{\mathcal{L}} = 0$. The Néron-Severi group of X is $\operatorname{NS}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$.

Remark. $x \in K(\mathcal{L})$ iff $T_x^*\mathcal{L} \cong \mathcal{L}$. By Theorem 6.1, this means that $K(\mathcal{L})$ consists of the $\bar{k}$-points of a closed subscheme of X. Indeed, we have the next proposition.

Definition 7.3. For a line bundle $\mathcal{L}$ on X, the associated Mumford line bundle on $X \times X$ is $\mathcal{M}(\mathcal{L}) = m^*\mathcal{L} \otimes \operatorname{pr}_1^* \mathcal{L}^\vee \otimes \operatorname{pr}_2^* \mathcal{L}^\vee$

Proposition 7.2. *$\mathcal{L} \in \operatorname{Pic}^0(X)$ iff $\mathcal{M}(\mathcal{L})$ is trivial.*

Proof. Assume $k = \bar{k}$ for simplicity.
For $x \in X(k)$, we have $m \circ (\mathrm{id}_X, x) = T_x, \operatorname{pr}_1 \circ (\mathrm{id}_X, x) = \mathrm{id}_X, \operatorname{pr}_2 \circ (\mathrm{id}_X, x) = x$. So $\mathcal{M}|_{X \times \{x\}} \cong T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ and $\mathcal{M}|_{\{e\} \times X}$ is trivial. By Theorem 6.1, $\mathcal{M}$ is trivial iff $T_x\mathcal{L} \cong \mathcal{L}$ for all x, i.e. $\mathcal{L} \in \operatorname{Pic}^0(X)$. □

Remark. This is one of the many, many different characterisations of Pic^0.

Let D be an effective divisor on X (since X is nonsingular, there is no difference between Weil and Cartier divisors).
Write $H(D) = \{x \in X(\bar{k}) : T_xD = D\}$. We certainly have $\mathcal{O}_X(T_x(D)) = T_{-x}^*\mathcal{O}_X(D)$ (indeed, if $D = \operatorname{div} f$, then $(T_x^* f)(y) = f(x + y)$ and so $\operatorname{div} T_x^* f = T_{-x}D$). And $H(D) \leq K(\mathcal{O}_X(D))$ because of this.

Remark. $H(D)$ is the $\bar{k}$-points of a closed subscheme of X, but for more obvious reasons than that for $K(\mathcal{L})$. Indeed, if $Y \subset X$ is a closed subset, then $T_xY = Y$ iff $\{x\} \times Y \subset m^{-1}(Y) \subset X \times X$, which happens iff $x \in \bigcap_{y \in Y}\{x \in X : (x, y) \in m^{-1}(Y)\} = \bigcap_{y \in Y} \operatorname{pr}_1((X \times \{y\}) \cap m^{-1}(Y))$, which is closed.

Theorem 7.3. *Suppose $\mathcal{L} = \mathcal{O}_X(D)$ for an effective divisor D, then the followings are equivalent:*
(i) $\mathcal{L}$ is ample.
(ii) $K(\mathcal{L})$ is finite.
(iii) $H(D)$ is finite.

Proof. Again assume $k = \bar{k}$.
(ii) $\implies$ (iii): Obvious.
(i) $\implies$ (ii): Assume for the sake of contradiction that $\mathcal{L}$ is ample but $K(\mathcal{L})$ is infinite. Then $K(\mathcal{L})$ consists of the k-points of some reduced closed subgroup

scheme of X, necessarily of positive dimension since $K(\mathcal{L})$ is infinite. The irreducible component containing e contains an abelian subvariety Y of positive dimension. The restriction $\mathcal{L}|_Y$ too is ample, so we might as well assume $X = Y$ (i.e. $K(\mathcal{L}) = X(k)$).
Then $m^*\mathcal{L} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}$ on $X \times X$. Let's pull it back via $d : X \to X \times X, d(x) = (x, -x)$. Then $m \circ d = e, \mathrm{pr}_1 \circ d = \mathrm{id}_X, \mathrm{pr}_2 \circ d = i = [-1]$, and hence $\mathcal{O}_X \cong \mathcal{L} \otimes i^*\mathcal{L}$.
Since i is an automorphism, $i^*\mathcal{L}$ too is ample, therefore $\mathcal{O}_X$ is ample. This however means that $\dim X = 0$ since X is complete, contradiction.
(iii) $\implies$ (i): Consider $\mathcal{O}_X(2D) = \mathcal{L}^{\otimes 2}$. By Theorem 6.8, this is isomorphic to $T_x^*\mathcal{L} \otimes T_{-x}^*\mathcal{L} = \mathcal{O}_X(T_x D + T_{-x} D)$ for any x. In other words, for any x, there is some $s_x \in H^0(X, \mathcal{O}_X(2D))$ whose divisor is $T_x D + T_{-x} D$.
Now, $y \in X(k)$ belongs to $T_x D \cup T_{-x} D$ iff one of $y \pm x$ is in D. So given any y, there is some x such that $y \notin T_x D \cup T_{-x} D$. Consequently, the map $f : X \to \mathbb{P}^N$ (where $N = h^0(X, \mathcal{O}_X(2D)) - 1$) induced by the sections of $\mathcal{O}_X(2D)$ is a morphism.
We claim that fibres of f are finite, which implies the ampleness of $\mathcal{O}_X(2D)$ since quasi-finite morphisms between complete varieties pulls ample invertible sheaves back to ample invertible sheaves.
If some fibre of f is infinite, then it must contain a curve C on X. Let $y \in C(k)$. Then there is some $x \in X(k)$ such that $s_x(y) \neq 0$ by what we've discussed. Consequently s_x is nonvanishing on C by definition of f, i.e. $C \cap (T_x D \cup T_{-x} D) = \varnothing$. We are then done by the next lemma. □

Lemma 7.4. *Suppose $k = \bar{k}$, $C \subset X$ is any curve and $Y \subset X$ an irreducible divisor with $C \cap Y = \varnothing$, then $T_{y_1 - y_2} Y = Y$ for any $y_1, y_2 \in C(k)$.*

Applying this to each irreducible component of $T_x D$ shows that $T_{y_1 - y_2}$ maps $T_x D$ to itself, hence D to itself, for all $y_1, y_2 \in C(k)$. But $C(k)$ is infinite, so $H(D)$ is infinite, contradicting (iii).

Proof. Let $U = \{x \in X(k) : T_x Y \not\supset C\}$ (i.e. collection of points x such that $T_x Y \cap C$ is finite). Since $C \cap Y = \varnothing$, we know that $T_{-x} Y \cap C = \varnothing = Y \cap T_x C$ for any $x \in U$ ("the degree of a divisor is constant in a connected family", see next section). For $y_1, y_2 \in C$ and $z \in Y(k)$, we have $z \in T_{z - y_2} C \cap Y$, in particular it is nonempty. Therefore $T_{z - y_2} C \subset Y$ and hence $z - y_2 + y_1 \in Y$, i.e. $T_{y_1 - y_2} Y = Y$. □

Corollary 7.5. *Abelian varieties are projective.*

Proof. Assume $k = \bar{k}$. We will find an ample sheaf on any abelian variety X. Let U be any nonempty open affine containing the identity e. Then $D = X \setminus U$ has codimension 1 since X is normal. Put the reduced induced closed subscheme structure on D.
Suppose $x \in H(D) = \{x \in X(k) : T_x D = D\}$. Then $T_x U = U$, so $x \in U(k)$, i.e. $H(D) \subset U(k)$. Now U is affine and $H(D)$ is the set of k-points of some closed subscheme of X. But closed subschemes of X are complete, hence $H(D)$ is finite and therefore $\mathcal{O}_X(D)$ is ample. □

Corollary 7.6. *For all $n \geq 1$, $\ker[n](\bar{k}) = \ker([n] : X(\bar{k}) \to X(\bar{k}))$ is finite, and $[n] : X \to X$ is surjective. In particular, $X(\bar{k})$ is divisible.*

Proof. Since X is complete, it suffices to show the finiteness of $\ker([n] : X(\bar{k}) \to X(\bar{k}))$. Assume WLOG that $k = \bar{k}$.
Suppose for the sake of contradiction that this kernel is infinite. Then $\ker[n]$ contains $V(k)$ for some subvariety $V \subset X$ of positive dimension. Let $\mathcal{L}$ be any ample invertible sheaf on X. Then $[n]^*\mathcal{L}$ is trivial on the fibres of $[n]$, hence $[n]^*\mathcal{L}|_V$ is trivial. But we know $[n]^*\mathcal{L} = \mathcal{L}^{\otimes n(n+1)/2} \otimes [-1]^*\mathcal{L}^{\otimes n(n-1)/2}$ is ample, so V admits a trivial ample invertible sheaf, which means that $\dim V = 0$. $\square$

Remark. One can show that if $\operatorname{char} k \nmid n$, then $\ker[n](\bar{k}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ where $g = \dim X$. If $\operatorname{char} k \mid n$, then $\# \ker[n](\bar{k}) < n^{2g}$.

Recall that for a complex torus X, $\operatorname{Pic}^0(X)$ is the dual complex torus of X. We can do something similar algebraically.

Theorem 7.7. *There is a dual abelian variety $\hat{X}$ to X with $\dim \hat{X} = \dim X$ together with an isomorphism $\psi : \hat{X}(\bar{k}) \to \operatorname{Pic}(X_{\bar{k}})$. Moreover, for all ample $\mathcal{L}$ on X, there is a unique surjective homomorphism $\lambda_{\mathcal{L}} : X \to \hat{X}$ with $\psi \circ \lambda_{\mathcal{L}}(\bar{k}) = \phi_{\mathcal{L}}$.*
In fact, $\hat{X}$ parameterises families of invertible sheaves. More precisely, there exists an "universal invertible sheaf" $\mathcal{P}$ on $X \times \hat{X}$ with the following property: Let S be any k-scheme and write $\operatorname{Pic}(X \times S)^0 = \{\mathcal{L} \in \operatorname{Pic}(X \times S) : \forall s \in S, \mathcal{L}|_{X \times \{s\}} \in \operatorname{Pic}^0(X \times \{s\})\}$. Then:
(i) For any $\mathcal{L} \in \operatorname{Pic}(X \times S)^0$, there is a unique $f : S \to \hat{X}$ such that $\mathcal{L} \cong (\mathrm{id}_X \times f)^*\mathcal{P} \otimes \operatorname{pr}_2^* \mathcal{M}$ for some $\mathcal{M} \in \operatorname{Pic}(S)$.
(ii) The identification in (i) gives a functorial bijection $\hat{X}(S) \to \operatorname{Pic}(X \times S)^0 / \operatorname{pr}_2^* \operatorname{Pic}(S) \cong \{\mathcal{L} \in \operatorname{Pic}(X \times S)^0 : \mathcal{L}|_{\{e\} \times S} \cong \mathcal{O}_S\}$.

Note that if we take $S = \operatorname{Spec} \bar{k}$, then the second bunch implies the first statement.

Idea of proof. First show the highly nontrivial statement that, if $\mathcal{L}$ is ample, then $\phi_{\mathcal{L}}$ surjects onto $\operatorname{Pic}^0(X_{\bar{k}})$ (we know that the image is contained in $\operatorname{Pic}^0(X_{\bar{k}})$ from example sheet).
Then we seek $\hat{X}$ to be the "quotient" of X by $\ker \phi_{\mathcal{L}}$. If $\operatorname{char} k = 0$, then this can be done by taking the quotient of X by the finite group $K(\mathcal{L})$ of automorphisms of X. We will soon see how this is done.
In positive characteristics, one have to know how to take quotient by the largest closed subgroup scheme $\underline{K}(\mathcal{L})$ such that $\mathcal{M}(\mathcal{L})$ is trivial. $\square$

Definition 7.4. A polarisation of an abelian variety X is an isogeny (i.e. surjective homomorphism) $\lambda : X \to \hat{X}$ such that, for some ample $\mathcal{L} \in \operatorname{Pic}(X_{\bar{k}})$, $\psi \circ \lambda = \phi_{\mathcal{L}}$.

8 Jacobians of Curves

Let X be a curve (nonsingular complete variety of dimension 1) over a field k. Write $g = g(X) = \dim H^0(X, \Omega_{X/k}) = \dim H^1(X, \mathcal{O}_X)$ for the genus of X. Let's review some facts about curves.

Definition 8.1. $\operatorname{Div}(X)$ is the free abelian group on the set of closed points of X. We have a degree homomorphism $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ given by $\sum_i n_i P_i \mapsto \sum_i n_i [\kappa(P_i) : k]$.

The divisor class group $\mathrm{Cl}(X)$ of X is the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors, i.e. divisors of the form $\mathrm{div}(f)$ for some $f \in k(X)^\times$. And we write $\mathrm{Cl}^0(X)$ for the kernel of deg in $\mathrm{Cl}(X)$.

We are going to prove:

Theorem 8.1. *There exists an abelian variety $J = \mathrm{Jac}(X)$ over k of dimension g together with an isomorphism $J(\bar{k}) \cong \mathrm{Cl}^0(X_{\bar{k}})$.*

Recall that there is a dictionary between divisors and invertible sheaves on X, where a divisor $D \in \mathrm{Div}(X)$ gives rise to the invertible sheaf $\mathcal{O}_X(D)$ whose sections on $U \subset X$ is $\{f \in k(X) : f = 0 \text{ or } (\mathrm{div}(f) + D)|_U \geq 0\}$. Every invertible sheaf arises this way. And $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ iff there is $f \in k(X)^\times$ with $\mathrm{div}(f) + D = D'$.
We write $L(D) = H^0(X, \mathcal{O}_X(D)) = \{f \in k(X) : \mathrm{div}(f) + D \geq 0\}$. We also write $\deg \mathcal{L}$ for $\deg D$ for any D with $\mathcal{L} \cong \mathcal{O}_X(D)$. And we write $\mathrm{Pic}^0(X)$ for the subgroup of $\mathrm{Pic}(X)$ consisting of those invertibel sheaves having degree 0. The canonical class $K_X \in \mathrm{Cl}(X)$ is the divisor class such that $\mathcal{O}_X(K_X) \cong \Omega_{X/k}$. Its degree is $2g - 2$, for example by the Riemann-Roch formula:

Theorem 8.2 (Riemann-Roch, sheaf version)**.** *For any invertible sheaf $\mathcal{L}$:*
(i) $h^0(\mathcal{L}) - h^1(\mathcal{L}) = 1 - g + \deg \mathcal{L}$.
(ii) (Serre duality) $H^1(X, \mathcal{L}) \cong H^0(X, \Omega_{X/k} \otimes \mathcal{L}^\vee)^\vee$.

Corollary 8.3 (Riemann-Roch, divisor version)**.** *Let $\ell(D) = \dim L(D)$. Then $\ell(D) - \ell(K_X - D) = 1 - g + \deg D$.*

One general thing that we need is to be able to take some kind of quotients.

Proposition 8.4. *Let V be a quasi-projective variety over k (not necessarily algebraically closed) and $G \subset \mathrm{Aut}(V)$ be a finite subgroup. Then there exists a unique variety $V' = V/G$ and a proper morphism $V \to V'$ with finite fibres such that:*
(i) $\forall \gamma \in G$, $\phi \circ \gamma = \phi$.
(ii) ϕ induces a bijection $V(\bar{k})/G \to V'(\bar{k})$ and an isomorphism $k(V') \to k(V)^G$.
(iii) For any morphism $\psi : V \to W$ of k-schemes such that $\psi \circ \gamma = \psi$ for all $\gamma \in G$, there is a unique $\theta : V' \to W$ such that $\theta \circ \phi = \psi$.

Sketch of proof. When $V = \mathrm{Spec}\, A$ is affine, then G can be viewed as a finite subgroup of $\mathrm{Aut}_k(A)$ and we can simply take $V' = \mathrm{Spec}\, A^G$. By some commutative algebra, A^G is a k-algebra of finite type and A is a finite B-module.
In general, the fact that V is quasi-projective shows that for any closed point $x \in V$ there is an open affine $U \subset V$ containing the orbit xG (since the orbit is finite). Now $\bigcap_{\gamma \in G} U\gamma$ is an open affine containing xG since V is separated, i.e. V can be covered by G-invariant open affines. So we use the affine case and glue. □

Remark. 1. The quasiprojective hypothesis can be weakened to the statement that any orbit is contained in an open affine. This is necessary: There exists a proper k-variety V and a free $\mathbb{Z}/2\mathbb{Z}$-action such that V/G does not exist as a scheme (Hironaka's example).
2. In fact, a morphism is proper and quasi-finite (i.e. has finite fibres) if and only if it is finite.

Back to curves.

Proposition 8.5. *Suppose S is any connected k-scheme and $[\mathcal{L}] \in \mathrm{Pic}(X \times S)$. Then:*
(i) $\deg \mathcal{L}|_{X\times\{s\}}$ *is independent of $s \in S$.*
(ii) For all $m \geq 0$, $\{s \in S : \dim_{\kappa(s)} H^0(X \times \{s\}, \mathcal{L}|_{X\times\{s\}}) \geq m\}$ is closed.

Proof. (ii) follows from Theorem 6.1. The proof of (i) is omitted, but it's basically because the Euler characteristic $h^0 - h^1 = 1 - g + \deg$ is constant in flat connected families. □

So if S is connected then $\mathrm{Pic}(X\times S) = \prod_{n\in\mathbb{Z}}$ where $\mathrm{Pic}^n(X\times S) = \{[\mathcal{L}] : \forall s \in S, \deg \mathcal{L}|_{X\times\{s\}} = n\}$. And for all $[\mathcal{G}] \in \mathrm{Pic}^n(X) = \{[\mathcal{L}] : \deg \mathcal{L} = n\}$, we have an isomorphism $\mathrm{Pic}^0(X \times S) \to \mathrm{Pic}^n(X \times S), [\mathcal{L}] \mapsto [\mathcal{L} \otimes \mathrm{pr}_1^* \mathcal{G}]$. In paticular, if $X(k) \neq \varnothing$, then $\mathrm{Pic}^0(X \times S) \cong \mathrm{Pic}^n(X \times S)$ for all n and $\mathrm{Pic}(X \times S) \cong \mathrm{Pic}^0(X \times S) \times \mathbb{Z}$.
From now on let's assume $k = \bar{k}$.

Proposition 8.6. *(i) If $\deg D = g$, then $\ell(D) \geq 1$.*
(ii) There exists an effective divisor D_0 of degree g with $\ell(D_0) = 1$.

In fact, most D_0 of degree g works for (ii).

Proof. (i) Riemann-Roch.
(ii) Let $\mathcal{L}$ be an invertible sheaf of degree $d \geq 2g + 1$. Then $h^1(\mathcal{L}) = h^0(\mathcal{L}^\vee \otimes \Omega_{X/k}) = 0$. So Riemann-Roch gives $h^0(\mathcal{L}) = d-g+1$. Moreover, the assumption that $d \geq 2g + 1$ implies that sections of $\mathcal{L}$ gives a closed immersion $X \hookrightarrow \mathbb{P}_k^{d-g}$ whose image is not contained in a hyperplane.
Since $k = \bar{k}$ is infinite, we can choose $P_1, \ldots, P_{d-g} \in X(k) \subset \mathbb{P}^{d-g}(k)$ not lying on any codimension 2 projective subspace. Then $H^0(X, \mathcal{L} \otimes \mathcal{O}(-\sum_i P_i)) = \{s \in H^0(X, \mathcal{L}) : s(P_1) = \cdots = s(P_{d-g}) = 0\}$ has dimension $h^0(\mathcal{L}) - (d - g) = 1$. So take $[D_0]$ to be the divisor class corresponding to $\mathcal{L} \otimes \mathcal{O}_X(-\sum_i P_i)$. Since $\ell(\mathcal{O}_X(D_0)) = 1$, we can choose D_0 to be effective. □

Now fix D_0 as in the proposition. For all $E \in \mathrm{Div}^0(X)$, there is some $D' = P_1 + \cdots + P_g$ such that $D' \sim D_0 + E$ by part (i) of the proposition. So the map $\pi_k : \{D' \geq 0 : \deg D' = g\} \to \mathrm{Cl}^0(X), D' \mapsto \mathcal{O}_X(D')$ is surjective.
But $\{D' \geq 0 : \deg D' = g\} = (X^g/S_g)(k)$ where S_g is the symmetric group on g letters, acting on X^g by permuting the coordinates! Write $X^{(g)} = X^g/S_g$, which turns out to be nonsingular (e.g. $\mathbb{A}^g/S_g = \mathrm{Spec}\, P \cong \mathbb{A}^g$ where P is the polynomial generated by the elementary symmetric polynomials).
$X^{(g)}$ is our first approximation to the Jacobian $J = \mathrm{Jac}(X)$. There will turn out to be a canonical morphism $\pi : X^{(g)} \to J$ inducing π_k. Note that most fibres of π would only have one element.
Fix $x_0 \in X(k)$.

Theorem 8.7 (Souped-up version of Theorem 8.1)**.** *There is an abelian variety J over k and $\mathcal{P} \in \mathrm{Pic}^0(X \times J)$ with $\mathcal{P}|_{\{x_0\}\times J} \cong \mathcal{O}_J$ such that, for any k-scheme S, $J(S)$ can be identified as the set of isomorphism classes of $\mathcal{L} \in \mathrm{Pic}^0(X \times S)$ with $\mathcal{L}|_{\{x_0\}\times S} \cong \mathcal{O}_S$, by sending $f \in J(S)$ to $(\mathrm{id}_X \times f)^*\mathcal{P}$. In particular, $J(k) \cong \mathrm{Pic}^0(X)$.*

Remark. We need to fix x_0 for the following reason: If one has some $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$, then for any $\mathcal{M} \in \operatorname{Pic}(S)$, $\mathcal{L}' = \mathcal{L} \otimes \operatorname{pr}_2^* \mathcal{M} \in \operatorname{Pic}^0(X \times S)$ and $\mathcal{L}|_{X \times \{s\}} \cong \mathcal{L}'|_{X \times \{s\}}$ for any $s \in S$. We want to ignore this variation, and asking things to be trivial on $\{x_0\} \times S$ for some fixed x_0 does just that, since $\mathcal{L}' \otimes \mathcal{L}^\vee|_{\{x_0\} \times S} = \mathcal{M}$ (one can also take a quotient but ehhh).

Lemma 8.8 (Version 0)**.** *There is a variety U_0 (which will be a dense open of J) and $\mathcal{P}_0 \in \operatorname{Pic}^0(X \times U_0)$ with $\mathcal{P}_0|_{\{x_0\} \times U_0} \cong \mathcal{O}_{U_0}$, such that for any variety S, $U_0(S)$ can be identified as the set of isomorphism classes of $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ with $\mathcal{L}|_{\{x_0\} \times S} \cong \mathcal{O}_S$ AND $h^0(\mathcal{L}|_{X \times \{s\}} \otimes \mathcal{O}_X(D_0)) = 1$ for any $s \in S$, by sending $f \in J(S)$ to $(\operatorname{id}_X \times f)^* \mathcal{P}_0$.*

Proof. We will construct U_0 as an open of $X^{(g)}$. There exists $\mathcal{M} \in \operatorname{Pic}(X \times X^{(g)})$ with $\mathcal{M}|_{X \times \{D'\}} \cong \mathcal{O}_X(D')$ for all $D' \in X^{(g)}(k)$ and $\mathcal{M}|_{\{x_0\} \times X^{(g)}} \cong \mathcal{O}_{X^{(g)}}$.
This is given as follows: $X \times X^g$ contains the diagonal $\Delta_X \times X^{g-1}$ as a closed subscheme. Let Y be the image of it under the quotient map.

$$\begin{array}{ccc} X \times X^g & \hookleftarrow & \Delta_X \times X^{g-1} \\ \downarrow & & \downarrow \\ X \times X^{(g)} & \hookleftarrow & Y \end{array}$$

Then for any $D' \in X^{(g)}(k)$ we have $Y|_{X \times \{D'\}} = D'$. Let $\mathcal{M}' = \mathcal{O}_{X \times X^{(g)}}(Y)$ and we can set $\mathcal{M} = \mathcal{M}' \otimes \operatorname{pr}_2^*(\mathcal{M}|^\vee_{\{x_0\} \times X^{(g)}})$.
Let $W = \{s \in X^{(g)} : h^0(\mathcal{M}|_{X \times \{s\}}) = 1\}$ which is open in $X^{(g)}$ by Theorem 6.4, and nonemtpy since $D_0 \in W(k)$. Take $(U_0, \mathcal{P}_0) = (W, \mathcal{M}|_W \otimes \operatorname{pr}_1^* \mathcal{O}_X(-D_0))$.
If $f : S \to W$ is any morphism, then $\mathcal{L} = (\operatorname{id}_X \times f)^* \mathcal{P}_0$ sure is in $\operatorname{Pic}^0(X \times S)$, is trivial on $\{x_0\} \times S$, and has $h^0(\mathcal{L}|_{X \times \{s\}} \otimes \mathcal{O}_X(D_0)) = 1$ by construction.
To see this procedure hits everything, we work backwards: For any $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ satisfying the conditions, consider $\mathcal{Q} = \mathcal{L} \otimes \operatorname{pr}_1^* \mathcal{O}_X(D_0)$, then $h^0(\mathcal{Q}_{X \times \{s\}}) = 1$ for all $s \in S$. We see from the proof of Theorem 6.1 that, locally on S, $\mathcal{L}$ has a section whose restriction to each fibre $X \times \{s\}$ is nonzero. One can check that such a section is unique up to units of $\mathcal{O}_S$, so we obtain, by gluing, a divisor on $X \times S$ with degree g on each fibre. This determines a morphism $S \to X^{(g)}$ whose image lies in W. □

Having constructed U_0 which we hope to be a dense open of J, which is supposedly a group variety, we are inspired to glue together some "translates" of it to cover J.
Let $(D_i)_i$ be effective divisors of degree g. We shall modify Lemma 8.8 by replacing the indices:

Lemma 8.9 (Version i)**.** *There is a variety U_i and $\mathcal{P}_i \in \operatorname{Pic}^0(X \times U_i)$ with $\mathcal{P}_i|_{\{x_0\} \times U_i} \cong \mathcal{O}_{U_i}$, such that for any variety S, $U_i(S)$ can be identified as the set of isomorphism classes of $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ with $\mathcal{L}|_{\{x_0\} \times S} \cong \mathcal{O}_S$ AND $h^0(\mathcal{L}|_{X \times \{s\}} \otimes \mathcal{O}_X(D_i)) = 1$ for any $s \in S$, by sending $f \in J(S)$ to $(\operatorname{id}_X \times f)^* \mathcal{P}_i$.*

Proof. $(U_i, \mathcal{P}_i) = (W, \mathcal{M}|_W \otimes \operatorname{pr}_1^* \mathcal{O}_X(-D_i))$. □

To glue, we identify a common open subscheme U_{ij} of U_i and U_j whose S-points are the set of $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ with $\mathcal{L}|_{\{x_0\} \times S} \cong \mathcal{O}_S$ and $h^0(\mathcal{L}|_{X \times \{s\}} \otimes$

$\mathcal{O}_X(D_i)) = 1 = h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_j))$. And for triple "intersections", U_{ijk} is just the open subscheme whose S-points are $\mathcal{L} \in \operatorname{Pic}^0(X \times S)$ with $\mathcal{L}|_{\{x_0\}\times S} \cong \mathcal{O}_S$ and $h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_i)) = h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_j)) = h^0(\mathcal{L}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_k)) = 1$.
So we get gluing data for $J = \bigcup_i U_i$ once we have chosen some collection of D_i.
Go back to this $X^{(g)}$ business. We set $W_0 = W \cong U_0$ and $W_i = \{s \in X^{(g)} : h^0(\mathcal{M}|_{X\times\{s\}} \otimes \mathcal{O}_X(D_1 - D_0)) = 1\}$ which is open in $X^{(g)}$ and contains a representative for the class of $2D_0 - D_i$ since $\ell(D_0) = 1$. We get maps $\pi_i : W_i \to U_i$ corresponding to $\mathcal{L}_i = \mathcal{M} \otimes \operatorname{pr}_1^* \mathcal{O}(-D_0)$.
Every $D \in X^{(g)}(k)$ lies in W_i for some D_i (e.g. $D_i \sim [2D_0 - D]$ would do). As $X^{(g)}$ is quasicompact, it is a finite union of W_i's. This gives a suitable family $(D_i)_i$ gluing to J, with a surjection $\pi : X^{(g)} \to J$ given by π_i.
The last thing to do is to show that J is an abelian variety. The multiplication map $m : J \times J \to J$ is locally defined as follows: $(x, y) \in U_i(k) \times U_j(k)$ correspond to some $\mathcal{P}_{i,x}, \mathcal{P}_{j,y} \in \operatorname{Pic}^0(X)$. We form $\mathcal{P}_{i,x} \otimes \mathcal{P}_{j,y}$ which corresponds to $z \in U_l(k)$ for some l. $\mathcal{P}_{i,x} \otimes \mathcal{P}_{j,y}$ is the fibre of $\mathcal{L} = \operatorname{pr}_1^* \mathcal{P}_i \otimes \operatorname{pr}_2^* \mathcal{P}_j$ on $U_i \times U_j$ above (x, y). Note that $h^0(\mathcal{L}|_{(x,y)} \otimes \mathcal{O}(D_l)) = 1$.
There is a neighbourhood V of $(x, y) \in U_i \times U_j$ for which the h^0 of $\mathcal{L} \otimes \mathcal{O}(D_l)$ is 1. Thus we get a morphism $V \to U_l$.
One can check that m extends to $J \times J \to J$, making it a group variety, and π shows the rest of the properties of an abelian variety.

A Proof of Theorem 6.6

Suppose k is a(n algebraically closed) field.

Theorem A.1 (Theorem 6.6). *Suppose X, Y, Z are k-varieties with X, Y complete. Suppose x, y, z are k-points of X, Y, Z and $\mathcal{L}$ is an invertible sheaf on $X \times Y \times Z$ which is trivial on $\{x\} \times Y \times Z$, $X \times \{y\} \times Z$ and $X \times Y \times \{z\}$. Then $\mathcal{L}$ is trivial.*

We have a map $\operatorname{Pic}(X \times Y) \oplus \operatorname{Pic}(X \times Z) \oplus \operatorname{Pic}(Y \times Z) \to \operatorname{Pic}(X \times Y \times Z)$ induced by projections. This theorem is essentially saying that this map is surjective.
We shall first prove the theorem for $Z = \operatorname{Spec} A$ for some finite local k-algebra A (e.g. $k[t]/(t^n)$). In particular, $Z = \{z\}$ and $A/\mathfrak{m}_A = \kappa(z) = k$.
Let's do induction on $\dim_k A$. If $\dim_k A = 1$, then $Z = \operatorname{Spec} k$, and so $X \times Y \times Z = X \times Y \times \{z\}$ and the theorem is obvious. For $\dim_k A > 1$, we choose an ideal $I \leq A$ with dimension 1. Let $Z_1 = \operatorname{Spec} A/I$ be the closed subscheme of Z defined by this ideal. And we know that the theorem is true for Z_1 by induction.

Lemma A.2. *Suppose V is a complete k-variety, then for any k-algebra B we have $H^0(V \times \operatorname{Spec} B, \mathcal{O}_{V\times \operatorname{Spec} B}) = B$.*

Proof. This is a special case of Corollary 6.3. □

Lemma A.3. *Suppose V is a complete k-variety, then there is an exact sequence*

$$0 \longrightarrow H^1(V, \mathcal{O}_V) \longrightarrow \operatorname{Pic}(V \times Z) \longrightarrow \operatorname{Pic}(V \times Z_1)$$

functorial in V.

Example A.1. When $A = k[t]/(t^2)$ and $I = (t)$, $\ker(\operatorname{Pic}(V \times \operatorname{Spec} A) \to \operatorname{Pic} V)$ may be viewed as the "tangent space" to Pic. And this tells us that this space is just $H^1(V, \mathcal{O}_V)$.

Proof. Note that $I^2 = 0$ and we have an exact sequence

$$0 \longrightarrow I \xrightarrow{a \mapsto 1+a} A^\times \longrightarrow (A/I)^\times \longrightarrow 0$$

and hence an exact sequence

$$0 \longrightarrow I\mathcal{O}_{V\times Z} \longrightarrow \mathcal{O}_{V\times Z}^\times \longrightarrow \mathcal{O}_{V\times Z_1}^\times \longrightarrow 0$$

(noting $V \times Z$ and $V \times Z_1$ have the same topological space). But now $I\mathcal{O}_{V\times Z} \cong \mathcal{O}_V$. The long exact sequence of cohomology then shows the result, noting $A^\times = H^0(X \times Z, \mathcal{O}_{V\times Z}^\times) \to (A/I)^\times = H^0(X \times Z_1, \mathcal{O}_{V\times Z_1}^\times)$ is surjective. □

Now, we know that $\mathcal{L}|_{X\times Y\times Z_1}$ is trivial by inductio hypothesis, so the preceding lemma gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(X\times Y, \mathcal{O}_{X\times Y}) \arrow[r] \arrow[d, "a"] & \operatorname{Pic}(X\times Y\times Z) \arrow[r, "c"] \arrow[d, "b"] & \operatorname{Pic}(X\times Y\times Z_1) \arrow[d] \\
0 \arrow[r] & H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y) \arrow[r] & \operatorname{Pic}(X\times Z) \oplus \operatorname{Pic}(Y\times Z) \arrow[r] & \operatorname{Pic}(X\times Z_1) \oplus \operatorname{Pic}(Y\times Z_1)
\end{tikzcd}$$

with exact rows, where the vertical maps are induced by (y^*, x^*) So $\mathcal{L} \in \ker b \cap \ker c \cong \ker a$, so we are done by the next lemma.

Lemma A.4. *a is an isomorphism.*

Proof. This follows from the Künneth formula: If X, Y are varieties over k, $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module, $\mathcal{G}$ a quasicoherent $\mathcal{O}_Y$-module, then

$$H^n(X \times Y, \operatorname{pr}_1^* \mathcal{F} \otimes \operatorname{pr}_2^* \mathcal{G}) = \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G})$$ □

Now let's extend to the case where $Z = \operatorname{Spec} A$ where A is a local Noetherian k-algebra. Let $z \in Z$ be the closed point, so $\kappa(z) = k$. Write $Z_n = \operatorname{Spec} A/\mathfrak{m}_A^n$. Then $\mathcal{L}|_{X\times Y\times Z_n}$ is trivial for all n, by what we have done.
Recall from the proof of Theorem 6.1 that there are finite cyclic A-modules M, M' such that for any homomorphism $A \to B$ of k-algebras, $H^0(X \times Y \times \operatorname{Spec} B, \mathcal{L}_B) = \operatorname{Hom}_A(M, B)$ and $H^0(X \times Y \times \operatorname{Spec} B, \mathcal{L}_B^\vee) = \operatorname{Hom}_A(M', B)$. By Lemma A.2, since $\mathcal{L}_{X\times Y\times Z_n} \cong \mathcal{O}_{X\times Y\times Z_n}$, we have $M \otimes A/\mathfrak{m}_A^n \cong A/\mathfrak{m}_A^n$. And therefore $\operatorname{ann}_A(M) \subset \bigcap_{n\geq 1} \mathfrak{m}^n = \{0\}$ which means that $M \cong M' \cong A$, so $\mathcal{L} \cong \mathcal{O}$.
Now for any k-variety Z, $\mathcal{L}|_{X\times Y\times \operatorname{Spec} \mathcal{O}_{Z,z}}$ is trivial. Take $F = \{z' \in Z : \mathcal{L}|_{X\times Y\times\{z'\}} \text{ trivial}\}$ which is closed in Z by Theorem 6.1, and contains the generic point of Z (which is also the generic point of $\operatorname{Spec} \mathcal{O}_{Z,z}$). So $F = Z$. Again by Theorem 6.1 we conclude that $\mathcal{L} = \operatorname{pr}_3^* \mathcal{M}$ for some $\mathcal{M}$ on Z. But then $\mathcal{O}_Z \cong \mathcal{L}_{\{x\}\times\{y\}\times Z} \cong \mathcal{M}$!