Representation Theory *

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This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part II course *Representation Theory* in Michaelmas 2021. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

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1 The Setup

Representation theory is the study of how symmetry (groups) occurs (acts) in nature (on finite dimensional vector spaces). This is useful not only in terms of geometrising the symmetries presented by (usually finite) groups, but also in terms of understanding the groups themselves from such actions of theirs.

1.1 Some Linear Algebra

The word "vector space" will always mean a finite-dimensional vector space throughout this course, unless stated otherwise. We usually also take the base field k to be an algebraically closed field of characteristic 0.

Definition 1.1. Given a vector space V, the general linear group $\mathrm{GL}(V)$ on V is the group (under composition, of course) of all invertible endomorphisms on V.

Since V is finite-dimensional (as we said so), there is an isomorphism $k^d \to V$ for some $d = \dim V$. A choice of such an isomorphism is essentially a choice of basis for V by taking the basis to be the image of the standard basis in k^d. We might write it as $V = ke_1 \oplus \cdots \oplus ke_d$ (where e_i are the images of the standard basis in k^d). Such a choice also induces a group isomorphism from $\mathrm{GL}(V)$ to the group of $d \times d$ matrices with nonzero determinant under matrix multiplication. As one can check, if α gets mapped to the matrix A here, then $\alpha(e_i) = \sum_j A_{ji} e_j$.

1.2 Group Representations

How did we explore symmetries of a set? Group actions.

Definition 1.2. An action of a group G on a set X is a map $G \times X \to X$ (written $(g, x) \mapsto g \cdot x$) with the conditions:
1. $\forall g \in G, x \in X, e \cdot x = x$.
2. $\forall g, h \in G, x \in X, g \cdot (h \cdot x) = (gh) \cdot x$.

Recall that such a group action of G on X is exactly a group homomorphism $G \to \operatorname{Sym} X = \{\text{bijections } X \to X\}$. Naturally, when X is in fact a vector space, we might want to replace Sym by GL.

Definition 1.3. A representation of a group G on a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

We usually call such a representation $\rho = (\rho, V) = (\rho, G, V)$. We sometimes just call it V when everything else is understood.
We can rephrase this definition by, essentially, unpacking the meaning of a group homomorphism to $\mathrm{GL}(V)$.

Definition 1.4. A representation ρ of G on V is the assignment of linear maps $\rho(g) : V \to V$ for each $g \in G$ such that:
1. $\rho(e) = \mathrm{id}_V$.
2. $\forall g, h \in G, \rho(gh) = \rho(g)\rho(h)$.

Remark. From this alternative definition, it immediately follows that $\rho(g)$ is invertible for every g and $\rho(g)^{-1} = \rho(g^{-1})$, so the two definitions are indeed the same.

Definition 1.5. The degree, or the dimension, of the representation ρ is just $\dim V$.

Definition 1.6. We say ρ is faithful if it has trivial kernel.

We are going to give some examples which are totally not just examples of group homomorphisms.

Example 1.1. 1. Let G be any group, then $\rho : G \to \mathrm{GL}(V)$ given by $\forall g \in G, \rho(g) = \mathrm{id}_V$ is the trivial representation (of G on V).
2. Let $G = C_2 = \{\pm 1\}$ and $V = \mathbb{R}^2$ (over $\mathbb{R}$). The map

$$\rho(\pm 1) = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a (faithful) representation. Geometrically, it realises the group as the reflectional symmetry of the plane across a line.
3. Let $G = (\mathbb{Z}, +)$ and let V be any vector space. Suppose ρ is a representation of G on V. Then of course $\rho(0) = \mathrm{id}_V$. Write $\alpha = \rho(1)$, then inductively $\rho(n) = \alpha^n$. Conversely, for any $\alpha \in \mathrm{GL}(V)$, $\rho(n) = \alpha^n$ is a representation of $\mathbb{Z}$ on V. So that's all of them (yay), i.e. the representations of $\mathbb{Z}$ are in bijection with the elements in $\mathrm{GL}(V)$.
4. Let $G = (\mathbb{Z}/N\mathbb{Z}, +)$, then by the same argument as in the above example, every representation ρ of G is in the form $\rho(n) = \alpha^n$ where the order of $\alpha \in \mathrm{GL}(V)$ is a factor of N.
5. Let $G = S_3$ be the symmetric group on three letters and $V = \mathbb{R}^2$. Consider an equilateral triangle in V centred at 0 with vertices labelled as $1, 2, 3$, the letters G permutes. Let G act on the triangle by permuting its vertices accordingly. Each of them extends to a linear transformation on V. Why? A transposition would induce a reflection in V across a line through the origin, which is linear. Since transpositions generate G, we get the required extensions and they clearly give a (faithful) representation of G on V. If you are bored you can try to write

down the matrices of the image of each $g \in G$ under a chosen basis.
6. Given a finite set X, we can form a vector space $kX = \{f : X \to k\}$ under pointwise addition and scalar multiplication. An action of G on X then induces a representation $\rho(g)(f)(x) = f(g^{-1} \cdot x)$. This is called the permutation representation of G on X.
7. In particular, if G is finite, then the action of G on itself by left multiplication induces a permutation representation of G on kG. This is called the regular representation of G, which is always faithful.
8. If ρ is a representation of G on V, the dual representation of G is a representation of it on $V^* = \mathrm{Hom}(V, k)$ given by $\rho^*(g)(f)(v) = f(\rho(g^{-1})(v))$.
9. In general, if $(\rho, V), (\rho', W)$ are representations of G, then we can construct the representation $(\alpha, \mathrm{Hom}_k(V, W))$ given by $\alpha(g)(f)(v) = \rho'(g)f(\rho(g)^{-1}v)$. When $W = k$ then this representation reduces to the one in the previous example. If we identify elements of $\mathrm{Hom}_k(V, W)$ as matrices (after choosing a basis), then we have $\alpha(g)(A) = \rho'(g)A\rho(g)^{-1}$.
10. If (ρ, V) is a representation of a group G and $\theta : H \to G$ is a group homomorphism, then $\rho \circ \theta$ is a representation of H on V. If $H \leq G$ and θ is the inclusion map, we call $\rho \circ \theta$ the restriction of ρ to H.

1.3 The Category of Representations

The natural action on S_3 on equilateral triangles centred at the origin, as we've seen, would induce representations of S_3 on $\mathbb{R}^2$. But it's clear that all of those representations should be "the same". This hints that we need a notion of isomorphism between representations in order to classify representations of a group G.

Definition 1.7. Let $(\rho, V), (\rho', V')$ be representations of the same group G. We say a linear isomorphism $\phi : V \to V'$ intertwines ρ and ρ' if $\rho'(g) \circ \phi = \phi \circ \rho(g)$ for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ V' & \xrightarrow[\rho'(g)]{} & V' \end{array}$$

We say $(\rho, V), (\rho', V')$ are isomorphic if such a ϕ exists.

Note that when ϕ intertwines ρ and ρ', then ϕ^{-1} (if exist) intertwines ρ' and ρ. Also, if ϕ intertwines ρ and ρ' and ϕ' intertwines ρ' and ρ'', then $\phi' \circ \phi$ intertwines ρ and ρ''. Consequently, isomorphism of representation is an equivalence relation.
Also, if (ρ, V) is a representation and $\phi : V \to V'$ is a linear isomorphism, then $\rho'(g) = \phi \circ \rho(g) \circ \phi^{-1}$ is a representation on V' and ϕ intertwines ρ, ρ'.
At the level of matrices, an intertwining map is just a change of basis if we identify the two vector spaces with each other via the map. From that point of view, an isomorphism of representations is just a choice of bases so that the two representations have coincidental matrices.

Example 1.2. 1. If G is trivial, then an isomorphism of G-representations is just an isomorphism of vector spaces.
2. If $G = (\mathbb{Z}, +)$, then $(\rho, V), (\rho', V')$ are isomorphic iff the $\rho(1)$ and $\rho'(1)$ have

the same matrix under some choice of bases of V, V'. Thus the degree d representations of G up to isomorphism are the conjugacy classes of invertible $d \times d$ matrices.
3. If $G = C_2 = \{\pm 1\}$, then the isomorphism classes of degree d representations correspond to conjugacy classes of $d \times d$ matrices of order 2. The minimal polynomial of an order 2 matrix divides $X^2 - 1 = (X+1)(X-1)$. So unless $\operatorname{char} k = 2$, the matrix has to be diagonalisable with diagonal entries ± 1. Consequently, there are exactly $d+1$ isomorphism classes of degree d representations of G.
4. If X, Y are finite sets acted on by G and suppose $f : X \to Y$ is a bijection such that $g \cdot f(x) = f(g \cdot x)$ for all $g \in G, x \in X$. Then $\phi : kX \to kY$ given by $\phi(\theta)(y) = \theta(f^{-1}(y))$ is an isomorphism between the corresponding permutation representations.

Definition 1.8. Suppose ρ is a representation of G on V and $W \leq V$. We say W is G-invariant if $\rho(g)(W) \subset W$ for all $g \in G$. If this were the case, then we say W is a subrepresentation of V which is a representation via $\rho_W = \rho$. It is said to be proper if $W \neq 0, V$.
We say V is simple (or irreducible) if V is nonzero and has no proper subrepresentation.

Example 1.3. 1. Any degree 1 representation is irreducible.
2. Suppose (ρ, k^2) is a representation of $G = C_2 = \{\pm 1\}$ given by $\rho(\pm 1) = \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then there are exactly two proper subrepresentations of ρ given by the axes. Indeed, since any proper subrepresentation has dimension 1, the spanning vector of it has to be an eigenvector of $\rho(-1)$, which are the standard basis vectors.
3. If (ρ, V) is any irreducible representation of C_2, then $\dim V = 1$. Indeed, if (ρ, V) is a representation, then $\rho(-1)^2 = \mathrm{id}_V$, so the minimal polynomial of $\rho(-1)$ divides $X^2 - 1 = (X-1)(X+1)$, so in particular it has an eigenvector whose span would be a subrepresentation which is proper when $\dim V \geq 2$. Furthermore, ρ has exactly 2 proper subrepresentations when $\operatorname{char} k \neq 2$ and exactly 1 when $\operatorname{char} k = 2$.
4. If $G = D_6$, then any complex representation of G (i.e. $k = \mathbb{C}$) has dimension at most 2. Identify $D_6 = \langle r, s | r^3, s^2, rsrs \rangle$, then $\rho(r)$ has an eigenvector v. Suppose $\rho(r)(v) = \lambda v$ (clearly $\lambda \neq 0$). We claim that $\operatorname{Span}\{v, \rho(s)v\} \leq V$ is G-invariant. Indeed, $\rho(s)\rho(s)v = \rho(s^2)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r^{-1})v = \lambda^{-1}\rho(s)v$ and r, s generate G.
It is then an easy exercise to further classify the irreducible representations of G up to isomorphism.
5. If $G = \mathbb{Z}$, we want to know when a complex representation (ρ, V) of it is irreducible. Choose basis $\{e_1, \ldots, e_n\}$ so that $\rho(1)$ is in Jordan normal form. If there is more than one Jordan block, then the subspace corresponding to any one of the Jordan blocks would be a subrepresentation. If $\rho(1) = J$ is a Jordan block, then it has subrepresentations given by the spans of $\{e_1, \ldots, e_k\}$ for $1 \leq k \leq n$. Hence all the irreducible representations of G have dimension 1.

Proposition 1.1. *Suppose (ρ, V) is a representation and $W \leq V$. The followings are equivalent:*
1. W is a subrepresentation.
2. There is a basis $\{e_1, \ldots, e_n\}$ of V extending a basis $\{e_1, \ldots, e_k\}$ of W under

which each $\rho(g)$ is k-block upper triangular.
3. Every basis $\{e_1, \dots, e_n\}$ of V extending a given basis $\{e_1, \dots, e_k\}$ of W makes each $\rho(g)$ k-block upper triangular.

Definition 1.9. If W is a subrepresentation of V, the quotient representation $\rho_{V/W}$ on V/W is given by $\rho_{V/W}(g)(v + W) = \rho(g)(v) + W$.

Proof. Consider the quotient representation. □

We are clearly not satisfied with merely the definition of isomorphisms – we also want the notion of a structure-compatible function between representations.

Definition 1.10. If (ρ, V) and (ρ', V) are representations of G, we say a linear map $\phi : V \to W$ is G-linear if $\rho'(g) \circ \phi = \phi \circ \rho(g)$ for all $g \in G$. We write $\text{Hom}_G(V, W)$ to denote all G-linear maps between (ρ, V) and (ρ', V).

Clearly G-linearity preserves under composition. From now on, we'll start to omit ρ and denote $\rho(g)v$ as $g \cdot v$ or simply gv.

Remark. 1. $\phi \in \text{Hom}_G(V, W)$ is an intertwining map iff it is bijective.
2. If $W \leq V$ is a subrepresentation, then the inclusion $W \hookrightarrow V$ is an element of $\text{Hom}_G(W, V)$ and the projection $V \to V/W$ is an element of $\text{Hom}_G(V, V/W)$.
3. Recall that $\text{Hom}_k(V, W) \supset \text{Hom}_G(V, W)$ is also a representation of G via $g\phi(v) = g(\phi(g^{-1}v))$. We can recover $\text{Hom}_G(V, W)$ from it by observing that $\phi \in \text{Hom}_G(V, W) \iff \forall g \in G, g\phi = \phi$.

Lemma 1.2 (First Isomorphism Theorem for Representations)**.** *Suppose (ρ, V) and (ρ', W) are representations of G and $\phi \in \text{Hom}_G(V, W)$, then $\ker \phi$ is a subrepresentation of V, $\text{Im}\,\phi$ is a subrepresentation of W and $V/\ker \phi \cong \text{Im}\,\phi$ as representations of G.*

Proof. The isomorphism is given by $\Phi(v + \ker \phi) = \phi(v)$ (duh!) which is already a linear isomorphism and for any $g \in G$ we have $g\Phi(v + \ker \phi) = g\phi(v) = \phi(gv) = \Phi(gv + \ker \phi) = \Phi(g(v + \ker \phi))$. □

2 Decomposition; Maschke's Theorem

Given a representation V of G and a G-invariant subspace $W \leq V$, when can we find a complement subspace of W that is also G-invariant?

Example 2.1. Suppose $G = C_2 = \{\pm 1\}$ and $V = \mathbb{R}^2$ is the representation that corresponds ± 1 to $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The x-axis is an invariant subspace of this representation, but only one of its complements is G-invariant, namely the y-axis.

Definition 2.1. We say a representation V of G is a direct sum of subspaces U and W, written $V = U \oplus W$, if both of them are G-invariant subspaces (i.e. subrepresentations) and V is the direct sum of U and W as vector spaces.
Conversely, given G-representations U and W, we can define their direct sum $U \oplus W$ to be a G-representation on the vector space direct sum of U and W given by $g(u, w) = (gu, gw)$,.

It's clear that these two contexts of direct sums are (isomorphism-wise) saying essentially the same thing.

Example 2.2. 1. If G acts on a finite set X and X decomposes to a disjoint union $X = X_1 \sqcup X_2$ of G-invariant subsets, then $kX = U_1 \oplus U_2$ where $U_i = \{f \in kX : \forall x \in X_i, f(x) = 0\}$. Clearly $U_1 \cong kX_2, U_2 \cong kX_1$, so $kX = kX_1 \oplus kX_2$. More generally, if the G-action on X decomposes into $X = O_1 \sqcup \cdots \sqcup O_r$, then $kX \cong \bigoplus_i kO_i$.
2. If G acts on a finite set X, then $U = \{f \in kX : \sum_x f(x) = 0\}$ and $W = \{f \in kX : f \text{ is constant}\}$ are G-invariant. If $\operatorname{char} k = 0$, then $U \cap W = 0$. It is also clear that $\dim_k U + \dim_k W = |X| - 1 + 1 = |X| = \dim_k kX$, so in that case $kX = U \oplus W$.
3. From our previous discussion, we have seen that every complex representations of C_2 is a direct sum of one-dimensional subspaces. Suppose now that G is a finite abelian group and V is a complex representation of G. Each $g \in G$ has finite order, hence has minimal polynomial dividing $X^{\operatorname{ord}(g)} - 1$ which has distinct roots, hence can be diagonalised. As G is abelian, all $g \in G$ can be simultaneously diagonalised. Indeed, if every $g \in G$ is a scalar matrix then it's trivial. Otherwise, there is some $g \in G$ that is not a scalar matrix and we decompose $V = \bigoplus_\lambda E(\lambda)$ where λ enumerates the eigenvalues of G (counting multiplicity) and $E(\lambda)$ is the eigenspace corresponding to λ. Each $E(\lambda)$ is G-invariant by commutativity. We can then show that V decomposes into a direct sum of one-dimensional subrepresentations by going over the elements of G and further decompose the space by diagonalising each of them in the subspaces. Hence all $g \in G$ can be simultaneously diagonalised, which mean that every representation of G is a direct sum of one-dimensional representations.
4. The representations of $\mathbb{Z}$ on $\mathbb{C}^n$ can be decomposed by Jordan blocks of 1. If there is only one Jordan block, then $\mathbb{C}^n$ has precisely $n - 1$ proper $\mathbb{Z}$-invariant subspaces $\operatorname{Span}\{e_1, \ldots, e_k\}$, none of which has any other as complement.

Proposition 2.1. *Suppose V is a representation of G decomposing into the direct sum of U and W as vector spaces, then the followings are equivalent:*
1. $V = U \oplus W$ as representations.
2. There are bases $\{v_1, \ldots, v_r\}$ of U and $\{v_{r+1}, \ldots, v_n\}$ of W (then $\{v_1, \ldots, v_n\}$ is a basis of V) under which every $g \in G$ is r-block diagonal.
3. Any bases $\{v_1, \ldots, v_r\}$ of U and $\{v_{r+1}, \ldots, v_n\}$ of W (ditto) would make every $g \in G$ r-block diagonal.

Proof. Obvious. □

Remark. This however doesn't mean that every basis of V would make every g block diagonal: The representation of C_2 in $\mathbb{R}^2$ given by $-1 \mapsto \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$ decomposes into $\operatorname{Span}\{e_1\} \oplus \operatorname{Span}\{e_1 - e_2\}$ but the matrix is not block diagonal.

In example sheet, we've seen that not every representation of $\mathbb{Z}$ has the property that every subrepresentation has a G-invariant complement. However,

Theorem 2.2 (Maschke)**.** *Suppose G be a finite group and V a representation of G over k with $\operatorname{char} k = 0$, then every G-invariant subspace of V has a G-invariant complement.*

Definition 2.2. A representation V of a group G is completely reducible if $V \cong W_1 \oplus \cdots \oplus W_r$ as representations for some irreducible subrepresentations W_i of G.

Corollary 2.3 (Complete Reducibility)**.** *Any representation of a finite group over a field with characteristic 0 is completely reducible.*

Proof. Clear from the preceding theorem (induction if you want). □

Before proving Theorem 2.2, let's first see an enlightening example.

Example 2.3. Let G act on a finite set X and consider $\mathbb{R}X$ equipped with a inner product $\langle f_1, f_2 \rangle = \sum_x f_1(x)f_2(x)$ preserved by all $g \in G$. It follows that if $W \leq \mathbb{R}X$ is a subrepresentation, then $W^\perp$ is G-invariant since whenever $g \in G, v \in W^\perp, w \in W$ we have $(gv, w) = (v, g^{-1}w) = 0$.

We will push this idea to establish the theorem over $\mathbb{C}$ before dealing with the general case. Recall that on a $\mathbb{C}$-vector space V, a Hemitian inner product (or simply an inner product) on V is a positive definite Hermitian sesquilinear form.

Definition 2.3. Suppose V is a representation of G. An inner product $(,)$ on V is G-invariant if $(gx, gy) = (x, y)$ for all $g \in G, x, y \in V$.

Equivalently (by the polarisation identity), $(gx, gx) = (x, x)$ for all $g \in G, x \in X$.

Lemma 2.4. *If $(,)$ is a G-invariant inner product on a representation V of G, then for any G-invariant subspace W, $W^\perp$ is a G-invariant complement.*

Proof. For any $g \in G, v \in W^\perp$ and $w \in W$, $(gv, w) = (v, g^{-1}w) = 0$. □

Theorem 2.2 will then follow from the existence of a G-invariant inner product on every complex representation of G. Equivalently, every $g \in G$ is unitary under some choice of basis for the representation.

Proposition 2.5 (Weyl's Unitary Trick)**.** *Any complex representation of a finite group G admits a G-invariant inner product.*

It then follows that studying the complex representations of a finite group is equivalent to studying its unitary representations.

Proof. Pick any Hermitian inner product $\langle , \rangle$ on V (say by identifying $V \cong \mathbb{C}^{\dim V}$ and pulling back the standard inner product) and define a new inner product on V by

$$(x, y) = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle$$

which, as one can check, is a G-invariant inner product. □

Corollary 2.6. *Theorem 2.2 holds for complex representations.*

Corollary 2.7. *Every finite subgroup G of $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a subgroup of $U(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A \text{ is unitary}\}$.*

Proof. Make $\mathbb{C}^n$ a representation of G via the inclusion $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$. Choose a G-invariant inner product on $\mathbb{C}^n$ and let P be the change-of-basis matrix from the standard basis on $\mathbb{C}^n$ to an orthonormal basis with respect to the new inner product. Then $P^{-1}GP \leq U(n)$. □

Now, how about the general case?

Proof of Theorem 2.2. The idea is that if $\pi : V \to V$ is a projection (i.e. $\pi^2 = \pi$), then $V = \operatorname{Im}\pi \oplus \ker\pi$. If it so happens that $\pi \in \operatorname{Hom}_G(V, V)$, then the two factors $\operatorname{Im}\pi, \ker\pi$ are also G-invariant, from which the theorem follows.
How do we do it? Pick any projection $\sigma : V \to V$ with $\operatorname{Im}\sigma = W$. Recall that $\operatorname{Hom}_k(V, V)$ is a G-representation via $g\phi(v) = g(\phi(g^{-1}v))$, so we define

$$\pi = \frac{1}{|G|} \sum_{g \in G} g\sigma$$

$\pi(W) \leq W$ since W is g-invariant; π also fixes every $w \in W$, so it follows that π is a projection onto W. It is also clear that π is G-linear, so we are done. □

Remark. 1. π (and hence $\ker\pi$) can be computed explicitly given V and W.
2. We don't exactly need $\operatorname{char} k = 0$: $\operatorname{char} k \nmid |G|$ suffices.
3. For any representation V of G with $\operatorname{char} k \nmid |G|$, we can write down the map

$$v \mapsto \frac{1}{|G|} \sum_{g \in V} gv$$

which is a k-linear projection onto $V^G = \{x \in V, \forall g \in G, gv = v\}$. We have $\dim V^G = |G|^{-1} \sum_{g \in G} \operatorname{tr} g$

3 Schur's Lemma

Recall that if V is a d-dimensional vector space, then $\operatorname{Aut}(V) \cong \mathrm{GL}_d(k)$ characterises the set of (oriented) basis for V. The decompositions $V = \bigoplus_i V_i$ of V into 1-dimensional subspaces V_i can in turn be parameterised by $\mathrm{GL}_d(k)/T$ where T is the subgroup of nonzero diagonal $d \times d$ matrices. If we want to remember the order of V_i, this parameterising space is $\mathrm{GL}_d(k)/N$ where N is the set of $d \times d$ matrices having exactly one nonzero entry on each row and column.
The moral of this story is that linear algebra in itself is quite "sloppy".
However, the structure of a simple representation makes things different.

Theorem 3.1 (Schur's Lemma)**.** *Suppose k is algebraically closed and V, W are irreducible representations of G over k, then:*
(i) Every element of $\operatorname{Hom}_G(V, W)$ is either zero or an isomorphism.
(ii) If k is algebraically closed, then

$$\dim_k \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

That is, simple representations are "rigid".

Proof. (i) If $\phi \in \operatorname{Hom}_G(V, W) \setminus \{0\}$, then $\ker\phi \lneq V$ is G-invariant. But V is simple, so $\ker\phi = 0$. Similarly, $\operatorname{Im}\phi \gneq \{0\}$ is also G-invariant, forcing it to equal W. Putting them together shows that ϕ is an isomorphism.
(ii) Suppose $\phi_1, \phi_2 \in \operatorname{Hom}_G(V, W)$ are nonzero. Then $\phi = \phi_1^{-1}\phi_2 \in \operatorname{Hom}_G(V, V)$ has an eigenvalue $\lambda \in k$ as k is algebraically closed. Now $\phi - \lambda \operatorname{id}_V$ is G-linear but not an isomorphism as its kernel contains the λ-eigenspace of ϕ. We then conclude by (i) that $\phi - \lambda \operatorname{id}_V = 0$, so $\lambda\phi_1 = \phi_2$. □

Proposition 3.2. *If V, V_1, V_2 are G-representations, then*

$$\operatorname{Hom}_G(V, V_1 \oplus V_2) \cong \operatorname{Hom}_G(V, V_1) \oplus \operatorname{Hom}_G(V, V_2)$$

$$\operatorname{Hom}_G(V_1 \oplus V_2, V) \cong \operatorname{Hom}_G(V_1, V) \oplus \operatorname{Hom}_G(V_2, V)$$

as vector spaces.

Proof. We've got the inclusions $\operatorname{Hom}_k(V, V_i) \hookrightarrow \operatorname{Hom}_k(V, V_1 \oplus V_2)$ which induces an isomorphism $\operatorname{Hom}_G(V, V_1) \oplus \operatorname{Hom}_G(V, V_2) \to \operatorname{Hom}_G(V, V_1 \oplus V_2)$ given by $(f_1, f_2) \mapsto f_1 + f_2$. This is G-linear. We have seen that $\operatorname{Hom}_G(V, W) = \operatorname{Hom}_k(V, W)^G = \{T \in \operatorname{Hom}_k(V, W) : \forall g \in G, gT = 0\}$ which implies the result. The rest is similar. □

So by induction,

Corollary 3.3. *If $V \cong \bigoplus_i V_i, W \cong \bigoplus_j W_j$ as representations, then we have $\operatorname{Hom}_G(V, W) \cong \bigoplus_{i,j} \operatorname{Hom}_G(V_i, W_j)$.*

Corollary 3.4. *Suppose k is algebraically closed and $V \cong \bigoplus_i V_i$ is a decomposition of the G-representation V into irreducible subrepresentations V_i, then for each irreducible G-representation W,*

$$|\{i : V_i \cong W\}| = \dim \operatorname{Hom}_G(W, V)$$

In particular, the left hand side is independent of the choice of decomposition.

Proof. It suffices to show that $\dim \operatorname{Hom}_G(W, V_i) = 1_{W \cong V_i}$ by the preceding corollary. But this is just the second part of Theorem 3.1. □

How would we, then, compute $\dim \operatorname{Hom}_G(W, V)$ from this?

Corollary 3.5. *If G is a finite group and V is a faithful simple $\mathbb{C}$-representation of it, then $Z(G) = \{g \in G : \forall h \in G, gh = hg\}$ is cyclic.*

Proof. Each $z \in Z(G)$ is an element of $\operatorname{Hom}_G(V, V)$. But $\operatorname{Hom}_G(V, V) = \mathbb{C}\,\mathrm{id}_V$ by Theorem 3.1, so $z = \lambda\,\mathrm{id}_V$ for some (necessarily unique) $\lambda \in \mathbb{C}^\times$. This gives an embedding $Z(G) \hookrightarrow \mathbb{C}^\times$, i.e. $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$, which has to be cyclic. □

Corollary 3.6. *Every irreducible $\mathbb{C}$-representation of an abelian group G has degree* 1.

Proof. Every $g \in G$ is an element of $\operatorname{Hom}_G(V, V)$, so $g = \lambda_g\,\mathrm{id}_V$ for some constants $\lambda_g \in \mathbb{C}^\times$ by Theorem 3.1. Then the 1-dimensional subspace spanned by any nonzero $v \in V$ would be G-invariant, so V cannot be irreducible unless it has degree 1. □

Example 3.1. 1. Let $G = C_4 = \langle x \rangle$, then the simple complex representations of G are

	1	x	x^2	x^3
ρ_1	1	1	1	1
ρ_2	1	i	-1	$-i$
ρ_3	1	-1	1	-1
ρ_4	1	$-i$	-1	i

2. Let $G = C_2 \times C_2 = \langle x, y \rangle$, then its simple complex representations are

	1	x	y	xy
ρ_1	1	1	1	1
ρ_2	1	-1	1	-1
ρ_3	1	1	-1	-1
ρ_4	1	-1	-1	1

In the examples, the number of irreducible representations equals the order of the group. This is not a coincidence.

Theorem 3.7. *Every finite abelian group G has precisely $|G|$ irreducible complex representations.*

Proof. We have already seen that every irreducible complex representations of G has order 1, so we are just classifying the homomorphisms $\mathbb{C} \to \mathbb{C}^\times$. By the classification of finite abelian groups, $G \cong C_{n_1} \times \cdots \times C_{n_k}$. Observe that if $G \cong G_1 \times G_2$, then the homomorphisms $G \to \mathbb{C}^\times$ are in bijection with the pairs of homomorphisms $G_1 \to \mathbb{C}^\times, G_2 \to \mathbb{C}^\times$ by commutativity. Inductively, it suffices to show the case when G is cyclic. In this case, the representation is determined by the image of the generator of G, which is necessarily a $|G|^{th}$ root of unity. Conversely, any choice of such a root of unity would yield such a representation, hence the total number of such representation is $|G|$. □

Lemma 3.8. *If $(\rho_1, V_1), (\rho_2, V_2)$ are non-isomorphic degree 1 complex representations of a finite group G, then*

$$\sum_{g \in G} \overline{\rho_1(g)} \rho_2(g) = 0$$

Proof. We've seen that $\mathrm{Hom}_k(V_1, V_2)$ is a G-representation via $g\phi = \rho_2(g) \circ \phi \circ \rho_1(g^{-1})$. Also, $\sum_{g \in G} gu \in \mathrm{Hom}_G(V_1, V_2)$, so $\sum_{g \in G} gu = 0$ by Theorem 3.1. As $\rho_1(g)$ is a root of unity, $\phi_1(g^{-1}) = \overline{\rho_1(g)}$. Take a bijection $\phi \in \mathrm{Hom}_k(V_1, V_2)$, then this gives

$$\sum_{g \in G} \overline{\rho_1(g)} \rho_2(g) \phi = 0 \implies \sum_{g \in G} \overline{\rho_1(g)} \rho_2(g)$$

which is what we wanted. □

Definition 3.1. Let V be a completely reducible representation of a group G and W any simple representation of G. The W-isotypic component of V is the smallest subrepresentation of V that contains every subrepresentation isomorphic to W.

Such a smallest subrepresentation exists since the structure of subrepresentation preserves under arbitrary intersections.
Being completely reducible, V is the sum of all its isotypic components by simply grouping the isomorphic components together. We might want to push this the condition to something stonger.

Definition 3.2. We say V has a unique isotypic decomposition if V is the direct sum of all its isotypic components.

This is analogous to the eigenspace decomposition of a diagonalisable matrix.

Corollary 3.9. *Suppose G is a finite abelian group, then any complex representation V of it has a unique isotypic decomposition.*

Proof. For each homomorphism $\theta_i : G \to \mathbb{C}^\times$ (indexed by $i \in \{1, \ldots, |G|\}$) as in Theorem 3.7, we define $V_i = \{v \in V : \forall g \in G, gv = \theta_i(g)v\}$ which an isotypic component. Now if $\sum_i v_i = 0$ for some $v_i \in V_i$, then for any $g \in G$, $\rho(g)\sum_i v_i = 0 \implies \sum_i \theta_i(g)v_i = 0$. Order the elements in G by $G = \{g_1, \ldots, g_{|G|}\}$, then $(\theta_i(g_j))$ is a $|G| \times |G|$ matrix that is invertible since its rows are nonzero and pairwise orthogonal by the preceding lemma. So necessarily $v_i = 0$ for all i, i.e. $V = \sum_i V_i = \bigoplus_i V_i$ is a direct sum. □

We'll generalise this result to any finite group in example sheet.

4 Characters

So far we've seen that if G is finite and $\operatorname{char} k = 0$, then any representation V of G decomposes into irreducible components. We can write it as $C = \bigoplus_i n_i V_i$ with pairwise non-isomorphic V_i, Moreover, when k is algebraically closed, we have $n_i = \dim \operatorname{Hom}_G(V_i, V)$. Next, we want to attempt to classify irreducible representations of a given finite group and understand how to compute these numbers n_i efficiently. Both tasks can be accomplished using character theory.

4.1 Definition

Definition 4.1. Given a representation (ρ, V) of G, the character of it is the function $\chi = \chi_\rho = \chi_V : G \to k, g \mapsto \operatorname{tr} \rho(g)$.

Recall that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, so the character is independent of the choice of basis and is preserved under isomorphisms.

Example 4.1. Let $G = D_6 = \langle s, t | s^2 = t^3, sts = t^{-1} \rangle$ act on $V = \mathbb{R}^2$ by symmetries of a triangle. Explicitly, we take t to be the anticlockwise rotation by $2\pi/3$ and s the reflection. Each $\rho(st^i)$ has eigenvalues ± 1, so $\chi(st^i) = 0$. t^r acts by anticlockwise rotation by $2\pi r/3$, hence has trace $\chi(t^r) = \cos(2\pi/3)$ which is 2 when $3 \mid r$ and -1 otherwise.

Proposition 4.1. *Let V be a representation of G with character $\chi = \chi_V$, then:*
(i) $\chi(e) = \dim V$.
(ii) $\chi(g) = \chi(hgh^{-1})$ for all $g, h \in G$.
(iii) If V' is another G-representation, then $\chi_{V \oplus V'} = \chi_V + \chi_{V'}$..
(iv) If $k = \mathbb{C}$ and $g \in G$ has finite order then $\chi(g^{-1}) = \overline{\chi(g)}$.

Proof. Trivial. □

In general, χ contains very little data about the representation – we just get an element of k for each conjugacy class of G. However, when G is finite and $k = \mathbb{C}$, we can show that it determines the representation up to isomorphism.

Definition 4.2. We say $f : G \to k$ is a class function if $f(ghg^{-1}) = f(g)$ for all $g, h \in G$. Write $\mathcal{C}_G$ to denote the k-vector space of class functions on G.

Notice that if $O_1, \dots, O_r$ are conjugacy classes in G, then the indicator functions $1_{O_i} : G \to \mathbb{C}$ form a basis for $\mathcal{C}_G$. In other words, $\dim \mathcal{C}_G = r$ is the number of conjugacy classes of G.
For the rest of this section, we'll deal with the case where G is finite and $k = \mathbb{C}$ unless specified otherwise.
$\mathcal{C}_G$ has the natural structure of a inner product space under

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

which makes the indicators 1_{O_i} pairwise orthogonal. Indeed,

$$\langle 1_{O_i}, 1_{O_j} \rangle = \frac{1}{|G|} \delta_{ij} |O_i| = \frac{\delta_{ij}}{|C_G(x_i)|}$$

for any $x_i \in O_i$ (with centraliser $C_G(x_i)$). Thus if $x_1, \dots, x_r$ are representatives of the orbits, then

$$\langle f_1, f_2 \rangle = \sum_{i=1}^{r} \frac{1}{|C_G(x_i)|} \overline{f_1(x_i)} f_2(x_i)$$

Example 4.2. If $G = D_6 = \langle s, t | s^2, t^3, stst \rangle$, then the conjugacy classes are $\{e\}, \{s, st, st^2\}, \{t, t^2\}$, so

$$\langle f_1, f_2 \rangle = \frac{1}{6} \overline{f_1(e)} f_2(e) + \frac{1}{2} \overline{f_1(s)} f_2(s) + \frac{1}{3} \overline{f_1(t)} f_2(t)$$

Theorem 4.2 (Orthonormality of Characters). *If V, V' are simple complex representations of a finite group G, then $\langle \chi_V, \chi_{V'} \rangle = 1_{V \cong V'}$.*

The theorem looks suspiciously like Theorem 3.1. In fact, that'll be how we prove it.

Lemma 4.3. *Suppose V, W are (complex) representations of (a finite group) G, then $\chi_{\mathrm{Hom}_k(V,W)}(g) = \overline{\chi_V(g)} \chi_W(g)$ for each $g \in G$.*

Proof. Fix $g \in G$, we choose bases $\{v_i\}$ for V and $\{w_j\}$ for W such that $g v_i = \lambda_i v_i, g w_j = \mu_j w_j$. This is possible since g has finite order, hence diagonalisable in V and W. Each $\alpha_{ij}(v_k) = \delta_{jk} w_i$ extends to a linear map $V \to W$, and they form a basis for $\mathrm{Hom}_k(V, W)$. Also, $(g\alpha_{ij})(v_k) = g(\alpha_{ij}(g^{-1} v_k)) = \lambda_k^{-1} \mu_i \delta_{jk} w_i = \lambda_j^{-1} \mu_i \delta_{jk} w_i$. So $g\alpha_{ij} = \lambda_j^{-1} \mu_i \alpha_{ij}$ which yields

$$\chi_{\mathrm{Hom}_k(V,W)}(g) = \sum_{i,j} \lambda_j^{-1} \mu_i = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$$

as desired. □

Lemma 4.4. *For any representation U of G, we have*

$$\dim U^G = \dim\{u \in U : \forall g \in G, gu = u\} = \langle 1, \chi_U \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_U(g)$$

where $1 : G \to \mathbb{C}, g \mapsto 1$ is the character of the trivial representation.

Proof. Let $\pi(u) = |G|^{-1} \sum_{g \in G} gu$ be the projection onto U^G, then $\dim U^G = \operatorname{tr} \pi = |G|^{-1} \sum_{g \in G} \chi_U(g)$. $\square$

Proposition 4.5. *Let V, W be representations of G, then* $\dim \operatorname{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle$.

Proof. By the precedinng lemmas, $\dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_k(V, W)^G = \langle 1, \chi_{\operatorname{Hom}_k(V,W)} \rangle = \langle 1, \overline{\chi_V} \chi_W \rangle = \langle \chi_V, \chi_W \rangle$. $\square$

Theorem 4.2 follows from this proposition and Theorem 3.1.
Hence simple representations are completely classified by their characters and the characters are linearly independent as they are orthogonal. Recall that $\dim \mathcal{C}_G$ is the number of conjugacy classes in G, so it follows that the number of simple representations of G is at most the number of conjugacy classes in G. In fact, we'll prove shortly that they are equal. But first, let's try to extend this classification to more general representations.

Corollary 4.6. *Two representations are isomorphic if and only if they have the same character.*

Proof. We've already seen that isomorphic representations have the same characters. By Theorem 2.2, every representation V of G is isomorphic to $\bigoplus_i n_i V_i$ for irreducible and pairwise non-isomorphic V_i. Note that if $V \cong \bigoplus_i n_i V_i$, then $\chi_V = \sum_i n_i \chi_{V_i}$. So it suffices to show that if $\sum_i n_i \chi_i = \sum_i m_i \chi_i$ then $n_i = m_i$ for all i. This is just saying that $\chi_1, \ldots, \chi_k$ are $\mathbb{Z}$-linearly independent. But we already know that they are $\mathbb{C}$-linearly independent by Theorem 4.2! $\square$

Consequently, the multiplicities in a complete reduction of a representation are completely determined by its character. Note also that it is necessary for the representation to be completely reducible (in the corollary this is given by Theorem 2.2). For example, the $\mathbb{Z}$-representations given by $1 \mapsto \mathrm{id}_{\mathbb{C}^2}$ and $1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are not isomorphic but possess the same character.
Our main goal now is to find ways to compute these characters of irreducible components $\chi_1, \ldots, \chi_k$. To start with, let's see how we might detect irreducibility of a representation using characters.

Corollary 4.7. *A representation of G with character χ is simple iff $\langle \chi, \chi \rangle = 1$.*

Proof. Theorem 4.2 gives the "only if" part. For the "if" direction, suppose the character decomposes into $\chi = \sum_i n_i \chi_i$ via the irreducible components, then $1 = \langle \chi, \chi \rangle = \sum_i n_i^2$ which can only mean that all n_i are 0 except for exactly one of them, which has to equal to 1. This gives irreducibility. $\square$

Example 4.3. Consider the action of D_6 on $\mathbb{C}^2$ by extending the equilateral triangle symmetry representation. We've know that $\chi(e) = 2, \chi(s) = 0, \chi(t) = -1$, so $\langle \chi, \chi \rangle = 1$, i.e. the representation is simple.

Theorem 4.8. *Irreducible characters of a finite group G form an orthonormal basis for $\mathcal{C}_G$.*

Proof. We've already shown that they form an orthonormal set. It remains to show that they span the whole space. Let I be the span of them. We will show that the orthogonal complement of I in $\mathcal{C}_G$ is zero. Suppose $f \in \mathcal{C}_G$, then for each representation (ρ, V) of G, we can extract a linear map $\phi_{f,V} \in \mathrm{Hom}_k(V,V)$ via

$$\phi_{f,V} = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho(g)$$

This is in fact G-linear:

$$\begin{aligned}\rho(h)\phi_{f,V}\rho(h^{-1}) &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho(hgh^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}\overline{f(h^{-1}gh)}\rho(g) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}\rho(g) = \phi_{f,V}\end{aligned}$$

Suppose now that $f \in I^\perp$ and V is irreducible, then by Theorem 3.1 we know that $\phi_{f,V} = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}$ and hence $\lambda \dim V = \mathrm{tr}\,\phi_{f,V} = |G|^{-1}\sum_{g\in G}\overline{f(g)}\chi_V(g) = \langle f, \chi_V\rangle = 0$ which can only mean that $\phi_{f,V} = 0$. By Theorem 2.2, we can drop the condition of V being irreducible and the statement remains true.
In particular, if $V = \mathbb{C}G$, then $0 = \phi_{f,\mathbb{C}G}1_e = |G|^{-1}\sum_{g\in G}\overline{f(g)}1_g$, so f is identically zero on G, i.e. $f = 0$. □

Corollary 4.9. *The number of irreducible characters of G equals the number of conjugacy classes of G.*

Corollary 4.10. *Fix $g \in G$, $\chi(g) \in \mathbb{R}$ for every character χ iff g, g^{-1} are conjugate.*

Proof. If g, g^{-1} are conjugate, then for any character χ we have $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$, so $\chi(g) \in \mathbb{R}$.
Conversely, if $\chi(g) \in \mathbb{R}$ for every character χ, then $\chi(g) = \chi(g^{-1})$ for every character χ Since the irreducible characters spans $\mathcal{C}_G$, $f(g) = f(g^{-1})$ for every class function f. In particular, by taking f to be the indicators for the conjugacy classes, we conclude that g, g^{-1} are conjugate. □

4.2 Character Table

Definition 4.3. The character table of a finite group G is constructed as follows: We list the conjugacy classes $O_1, \ldots, O_r$ of G (where usually we take $O_1 = \{e\}$ by convention). We also list the irreducible characters $\chi_1, \ldots, \chi_r$ (where again we usually take $\chi_1 = 1$ to be character of the trivial representation by convention). The character table is the matrix whose entries are $\chi_i(g_j)$ where $g_j \in O_j$.

Our previous discussion shows that character tables are always square.

Example 4.4. 1. The character table for $G = C_3 = \langle x\rangle$ is

	e	x	x^2
χ_1	1	1	1
χ_2	1	ω	$\bar\omega$
χ_3	1	$\bar\omega$	ω

where $\omega = e^{2\pi i/3}$.
2. $G = S_3$ has three conjugacy classes $\{e\}, \{(12),(13),(23)\}, \{(123),(321)\}$. We know it has the trivial character and the character given by the sign function. We can get the last one by orthonormality, which gives the character table

	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Indeed, χ_3 is the character of the representation on $\mathbb{C}^2$ extended from the symmetry of a triangle.

Note from the examples that the columns are orthogonal with respect to the standard inner product. Furthermore, the lengths of them are quite interesting, as they are the sizes of centralisers of the respective conjugacy classes. These are not accidents.

Proposition 4.11. *If G is a finite group with irreducible characters $\chi_1, \ldots, \chi_r$, then for any $g, h \in G$ we have*

$$\sum_{i=1}^r \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_G(g)| = |C_G(h)| & \text{if } g,h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $V_1, \ldots, V_r$ are the respective irreducible representations, then

$$\sum_{i=1}^r (\dim V_i)^2 = \sum_{i=1}^r \chi_i(1)^2 = |G|$$

Proof. Suppose X is the character table (viewed as a matrix) and let D be the diagonal matrix with diagonal entries $\delta_{ij}|C_G(g_i)|$ where g_i are the representatives of the conjugacy classes in G. We know that $\bar{X}D^{-1}X^\top = I$, so $\bar{X}^\top X = D$ since X is square and D is real. ☐

Example 4.5. 1. Take $G = S_4$, then it has character table

	e	$(12)(34)$	(123)	(12)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
χ_4	3	-1	0	-1	1
χ_5	2	2	-1	0	0

where we obtain χ_3 from decomposing the permutation representation of G derived from the its action on four letters, χ_4 by adjoining χ_2 to χ_3 and χ_5 using orthogonality of columns.
There is a surjective homomorphism from S_4 to S_3 with kernel given by $V_4 = \langle (12)(34), (13)(24) \rangle$, so after adjoining with χ_2, χ_1, χ_2, χ_5 are the irreducible characters of S_3.
2. Each irreducible representations of S_4 restrict to a representation of A_4. By looking at the restrictions of χ_1, χ_3, χ_5 we can compute the character table of A_4 (example sheet).

4.3 Permutation Representations

Recall that the group action of G on a finite set X would induce a representation on $\mathbb{C}X$ given by $gf(x) = f(g^{-1}x)$.

Lemma 4.12. *Suppose χ is the character of $\mathbb{C}X$, then $\chi(g)$ is the number of points in X fixed by g.*

Proof. Let $\{\delta_{x_1}, \ldots, \delta_{x_n}\}$ be the indicator functions of the elements in $X = \{x_1, \ldots, x_n\}$, then they form a basis for $\mathbb{C}X$ under which we have $g\delta_{x_i} = \delta_{gx_i}$. So the i^{th} diagonal element of the matrix of g (under this choice of basis) is 1 when $gx_i = x_i$ and 0 otherwise. □

Corollary 4.13. *If $V_1, \ldots, V_r$ are the irreducible representations of a finite group G, then the regular representation $\mathbb{C}G$ decomposes into $\mathbb{C}G = \bigoplus_i n_i V_i$ with $n_i = \dim V_i = \chi_i(e) > 0$. In particular, every irreducible representation is isomorphic to a subrepresentation of $\mathbb{C}G$ and $|G| = \sum_i (\dim V_i)^2$.*

Proof. $\chi_{\mathbb{C}G}(g) = |G| 1_{g=e}$ by the preceding lemma, so we have $n_i = \langle \chi_{\mathbb{C}G}, \chi_i \rangle = |G|^{-1} \overline{\chi_{\mathbb{C}G}(e)} \chi_i(e) = \chi_i(e)$. □

Proposition 4.14 (Burnside's Lemma)**.** *Let G be a finite group acting on a finite set of X, then $\langle 1, \chi_{\mathbb{C}X} \rangle$ equals the number of orbits of G in X.*

Proof. Suppose $X = \bigcup_i X_i$ is a decomposition into G-orbits, then $\mathbb{C}X \cong \bigoplus_i \mathbb{C}X_i$ and $\langle 1, \chi_{\mathbb{C}X} \rangle = \sum_i \langle 1, \chi_{\mathbb{C}X_i} \rangle$. So it suffices to show the case where G acts transitively on X. Now $|G| \langle 1, \chi_{\mathbb{C}X} \rangle = \sum_{g \in G} \chi(g) = \sum_{g \in G} |\{x \in X : gx = x\}| = \sum_{x \in X} |\{g \in G : gx = x\}| = \sum_{x \in X} |\operatorname{Stab}_G(x)| = \sum_{x \in X} |G|/|X| = |G|$. □

G acts on X, Y, then it also acts on $X \times Y$ via $g(x, y) = (gx, gy)$.

Corollary 4.15. *If G is a finite group acting on a finite set X, then $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle$ is the number of G-orbits in $X \times X$.*

Proof. $\chi_{\mathbb{C}(X \times X)}(g) = |\{x \in X : gx = x\}|^2 = \chi_{\mathbb{C}X}(g)^2$, so $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle = |G|^{-1} \sum_{g \in G} \chi_{\mathbb{C}X}(g)^2 = \langle 1, \chi_{\mathbb{C}(X \times X)} \rangle$. □

For any finite set X acted on by G with $|X| > 1$, then $\{(x, x) : x \in X\}$ and $\{(x, y) : x, y \in X, x \neq y\}$ are stable under the induced G-action on $X \times X$, so there are at least two G-orbits on $X \times X$.

Definition 4.4. We say G acts 2-transitively on X if G has exactly two orbits in $X \times X$.

Equivalently, for any $x, y, z, w \in X$ with $x \neq z, y \neq w$, there is some $g \in G$ taking x to y and z to w.
If G is 2-transitive on X, then $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}X} \rangle = 2$, so $\mathbb{C}X$ decomposes into $\mathbb{C} \oplus \{f \in \mathbb{C}X : \sum_{x \in X} f(x) = 0\}$ where the summands are irreducible and non-isomorphic.

Example 4.6. 1. S_n acts 2-transitively on $\{1, \ldots, n\}$ whenever $n \geq 2$.
2. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ act on $\mathbb{F}_p \cup \{\infty\}$ by Möbius transformations. Then the action is 2-transitive and it's an exercise to decompose the permutation representation of this action and calculate the characters of its components.

5 The Character Ring

Given a finite group G, $\mathcal{C}_G$ has the structure of a k-algebra with an inner product $\langle f_1, f_2 \rangle = |G|^{-1} \sum_{g \in G} f_1^*(g) f_2(g)$ where $f^*(g) = f(g^{-1})$ is an order 2 ring automorphism of $\mathcal{C}_G$. The structure of this is of considerable interest to representation theorists.
We've seen that some of the structure of $\mathcal{C}_G$ restricts to the characters, e.g. $\chi_{V \oplus V'} = \chi_V + \chi_{V'}, \chi_1 = 1, \langle \chi_V, \chi_{V'} \rangle = \dim \mathrm{Hom}_G(V_1, V_2)$. In fact, the product of characters is also a character. So we can obtain a subring of $\mathcal{C}_G$ from the characters of a group (counting the zero character $\chi_0 = 0$).

Definition 5.1. The character ring $R(G)$ of a group G is the ring $R(G) = \{\chi_1 - \chi_2 : \chi_1, \chi_2 \text{ characters}\}$.

To define multiplication, we need some preliminary theory on tensor products.

5.1 Tensor Product

We have seen that $\chi_{\mathbb{C}X}\chi_{\mathbb{C}Y} = \chi_{\mathbb{C}(X \times Y)}$. This phenomenon can, of course, be generalised. Suppose $v_1, \ldots, v_m$ is a basis for V and $w_1, \ldots, w_n$ a basis for W. Recall that the direct sum of vector spaces is constructed on $V \times W$ with $\mathrm{Span}\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ in mind. This results in a sum in dimension. What if we want a product?

Definition 5.2. The tensor product $V \otimes W$ is the k-vector space with basis given by the formal symbols $\{v_i \otimes w_j\}$.

This is an absolutely horrible definition, but it does give $kX \otimes kY \cong k(X \times Y)$ via $\delta_x \otimes \delta_y \mapsto \delta_{(x,y)}$.
For $v = \sum_i \lambda_i v_i, w = \sum_j \mu_j w_j$, we often write

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$$

One should, however, be aware that not every element of $V \otimes W$ can be written in this form.

Lemma 5.1. *The map $V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$ is bilinear.*

Proof. Check. □

One can also show that for any vector spaces U, V, W, there is a correspondence between linear maps $V \otimes W \to U$ and bilinear maps $V \times W \to U$.

Lemma 5.2. *If $x_1, \ldots, x_m$ is a basis for V and $y_1, \ldots, y_n$ is a basis for W, then $\{x_i \times y_j\}$ is a basis for $V \otimes W$.*

So the construction of $V \otimes W$ does not depend on the choice of basis.

Proof. Write $v_i = \sum_r A_{ri} x_r, w_j = \sum_s B_{sj} y_s$, then $v_i \otimes w_j = \sum_{r,s} A_{ri} B_{sj} x_r \otimes y_s$. But this is saying that $\{x_i \times y_j\}$ spans $V \otimes W$. □

Remark. Alternatively, we could've defined $V \otimes W$ in a coordinate-free fashion. Let F be the (infinite-dimensional) vector space with basis given by the formal symbols $\{v \otimes w : v \in V, w \in W\}$ and R a subspace of it generated by $x \otimes (\mu_1 y_1 + \mu_2 y_2) - \mu_1(x \otimes y_1) - \mu_2(x \otimes y_2), (\lambda x_1 + \lambda_2 x_2) \otimes y - \lambda_1(x_1 \otimes y) - \lambda_2(x_2 \otimes y)$ for all $x, x_1, x_2, y, y_1, y_2 \in V, \lambda_i, \mu_i \in k$. Then we can define $V \otimes W$ by F/R. It is isomorphic to the above construction via $(v \otimes w) + R \mapsto v \otimes w$.

Using this definition, it is however slightly easier to show that for vector spaces U, V, W, there is a natural (basis independent) isomorphism $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$.

Definition 5.3. Suppose V, W are as above and $\phi : V \to V, \psi : W \to W$ are k-linear, then we define their tensor product $\phi \otimes \psi : V \otimes W \to V \otimes W$ by extending $(\phi \otimes \psi)(v_i \otimes w_j) = \phi(v_i) \otimes \psi(w_j)$.

Example 5.1. If ϕ has matrix A_{ij} and ψ has matrix B_{rs}, then $\phi \otimes \psi$ would have matrix in the block form

$$(A \otimes B)_{ij} = \begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mm}B \end{pmatrix}$$

under the basis $\{v_i \otimes w_j\}$ ordered lexicographically.

Lemma 5.3. *$\phi \otimes \psi$ does not depend on the choice of basis.*

Proof. Check $(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$. □

Conceptually, this is just saying that $V \times W \to V \otimes W, (v, w) \mapsto \phi(v) \otimes \psi(w)$ is bilinear.

Remark. Of course, we could have the correspondence between linear maps $V \otimes W \to V \otimes W$ and bilinear maps $V \times W \to V \otimes W$.

Lemma 5.4. *Suppose $\phi, \phi_1, \phi_2 \in \mathrm{Hom}_k(V, V)$ and $\psi, \psi_1, \psi_2 \in \mathrm{Hom}_k(W, W)$, then*
(i) $(\phi_1 \phi_2) \otimes (\psi_1 \psi_2) = (\phi_1 \otimes \psi_1)(\phi_2 \otimes \psi_2)$.
(ii) $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$.
(iii) $\mathrm{tr}(\phi \otimes \psi) = \mathrm{tr}(\phi)\,\mathrm{tr}(\psi)$.

Proof. (i) and (ii) can be verified directly. (iii) can be seen from the matrix of $\phi \otimes \psi$. □

Definition 5.4. Given representations $(\rho, V), (\rho', V')$ of a group G, their tensor product $(\rho \otimes \rho', V \otimes V')$ is defined on $V \otimes V'$ by $(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$.

Proposition 5.5. *$(\rho \otimes \rho', V \otimes V')$ is indeed a representation and $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$.*

We sometimes write $\chi_\rho \otimes \chi_{\rho'}$ to denote $\chi_{\rho \otimes \rho'} = \chi_\rho \chi_{\rho'}$ for fun or otherwise.

Proof. Follows directly from the preceding lemma. □

Remark. 1. The construction means that $R(G)$ is indeed closed under multiplication.
2. If one of the representations is 1-dimensional, this construction reduces to a question on a previous example sheet.

We can extend this construction and define tensor products of representations of possibly different groups on their direct product. If (ρ, V) is a representation of G and (ρ', W) is a representations of H, then $V \otimes W$ can be made a representation of $G \times H$ with $(\rho \otimes \rho')(g, h) = \rho(g) \otimes \rho'(h)$. It is easy to see that this is indeed a representation and $\chi_{V \otimes W}(g, h) = (\chi_V \otimes \chi_W)(g, h) = \chi_V(g)\chi_W(h)$. Our previous construction can be recovered by taking $G = H$ and restricting the resulting representation to the diagonal subgroup in $G \times G$.

Example 5.2. If X, Y are finite sets acted on by G, then we have an isomorphism $kX \otimes kY \cong k(X \times Y)$ both as G-representations and $G \times G$-representations (as in the extended construction).

This extended definition has the advantage of granting prettier results.

Proposition 5.6. *Suppose G, H are finite groups, $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ all the simple $\mathbb{C}$-representations of G and $(\rho'_1, W_1), \ldots, (\rho'_s, W_s)$ that of H. For each i, j, the tensor product representation $(\rho_i \otimes \rho'_j, V_i \otimes W_j)$ of $G \times H$ is simple. Moreover, all simple $\mathbb{C}$-representations of $G \times H$ arise this way.*

Proof. Let χ_i be the character of V_i and ψ_j that of W_j, then by Corollary 4.7,

$$\begin{aligned}
\langle \chi_i \otimes \psi_j, \chi_k \otimes \psi_l \rangle &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \overline{\chi_i \otimes \psi_j(g, h)} \chi_k \otimes \psi_l(g, h) \\
&= \frac{1}{|G||H|} \sum_{g \in G, h \times H} \overline{\chi_i(g)\psi_j(h)} \chi_k(g)\psi_l(h) \\
&= \left(\frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_k(g) \right) \left(\frac{1}{|H|} \sum_{h \in H} \overline{\psi_j(h)} \psi_j(h) \right) = \delta_{ik}\delta_{jl}
\end{aligned}$$

So the representations are pairwise distinct by orthogonality and simple by Corollary 4.7. Now

$$\sum_{i,j} (\dim V_i \otimes W_j)^2 = \sum_i (\dim V_i)^2 \sum_j (\dim W_j)^2 = |G||H| = |G \times H|$$

Hence these are all of them, by Proposition 4.11. □

Note that the irreducibility part only works if we regard $V_i \times W_j$ as $G \times H$-representations. When V, W are irreducible ($\mathbb{C}$-)representations of G, $V \times W$ (over G instead of $G \times G$) is irreducible if one of V, W has dimension 1, but not in general.

Example 5.3. $G = S_3$ has character table

	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

But $\chi_3^2 = \chi_3 + \chi_2 + \chi_1$ is not simple.

In general, if $\chi_1, \ldots, \chi_r$ are simple complex characters of a finite group G, then we have $\chi_i \chi_j = \sum_k a_{ij}^k \chi_k$ with $a_{ij}^k \in \mathbb{N}$.

5.2 Symmetric and Exterior Powers

In fact, the tensor powers of a representation V with itself can never be simple unless $\dim V = 1$.
Given a vector space V, consider the linear map $\sigma = \sigma_V : V \otimes V \to V \otimes V$ extending from $\sigma(v \otimes w) = \sigma(w \otimes v)$. σ is an involution (i.e. $\sigma^2 = \mathrm{id}_V$), so whenever $\mathrm{char}\, k \neq 2$, σ decomposes $V \times V$ to two eigenspaces which is also the isotypical decomposition of σ as a representation of C_2. They have names.

Definition 5.5. The symmetric square of V is defined as $\mathfrak{S}^2 V = \{a \in V \otimes V : \sigma(a) = a\}$. The exterior square of V is defined as $\bigwedge^2 V = \{a \in V \otimes V : \sigma(a) = -a\}$.

Lemma 5.7. *Suppose $v_1, \dots, v_m$ is a basis for V, then $\mathfrak{S}^2 V$ has basis $\{v_i v_j = (1/2)(v_i \otimes v_j + v_j \otimes v_i)\}$ and $\bigwedge^2 V$ has basis $\{v_i \wedge v_j = (1/2)(v_i \otimes v_j - v_j \otimes v_i)\}$. In particular, $\dim \mathfrak{S}^2 V = m(m+1)/2$ and $\dim \bigwedge^2 V = m(m-1)/2$.*

Proof. Clear. □

Proposition 5.8. *Let (ρ, V) be a complex representation of a finite group G, then*
(i) $V \otimes V = \mathfrak{S}^2 V \oplus \bigwedge^2 V$ as representations of G. In particular, $V \otimes V$ can never be irreducible unless $\dim V = 1$.
(ii) $\chi_{\mathfrak{S}^2 V}(g) = (1/2)(\chi_V(g)^2 + \chi_V(g^2)), \chi_{\bigwedge^2 V}(g) = (1/2)(\chi_V(g)^2 - \chi_V(g^2))$

Proof. (i) is clear.
For (ii), it suffices (by (i)) to prove the first formula (recall that $\chi_V(g)^2 = \chi_{V \times V}(g)$). If $v_1, \dots, v_m$ is a basis consisting of eigenvectors of g in V with (distinct) eigenvalues $\lambda_1, \dots, \lambda_n$, then $g(v_i v_j) = \lambda_i \lambda_j v_i v_j$, thus $\chi_V(g)^2 + \chi_V(g^2) = (\sum_i \lambda_i)^2 + \sum_i \lambda_i^2 = 2(\sum_{i \leq j} \lambda_i \lambda_j) = 2\chi_{\mathfrak{S}^2 V}(g)$ as desired. □

One can also prove the formula for $\chi_{\bigwedge^2 V}$ directly, which of course will be left as exercise.

Example 5.4. S_4 has character table

	e	$(12)(34)$	(123)	(12)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
χ_4	3	-1	0	-1	1
χ_5	2	2	-1	0	0

We want to calculate the characters of symmetric and exterior squares of χ_3.

	e	$(12)(34)$	(123)	(12)	(1234)
χ_3^2	9	1	0	1	1
$\mathfrak{S}^2\chi_3$	6	2	0	2	0
$\bigwedge^2\chi_3$	3	-1	0	-1	1

We also have $\mathfrak{S}^2\chi_3 = \chi_5 + \chi_3 + 1, \bigwedge^2 \chi_3 = \chi_4 = \chi_2\chi_3$.

Of course, symmetric and exterior powers generalise. Let $V^{\otimes n}$ be the n-times tensor product of V with itself. For $w \in S_n$, we consider the linear map $\sigma(w) : V^{\otimes n} \to V^{\otimes n}$ extending $\sigma(w)(v_1 \otimes \cdots \otimes v_n) = v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(n)}$. $(\sigma, V^{\otimes n})$ is then a representation of S_n which commutes with the representation of any finite group G on $V^{\otimes n}$ induced from a G-representation on V. So when $V^{\otimes n}$ decomposes into S_n-isotypical components, each of them will be G-invariant.

Definition 5.6. Suppose V is a vector space, then the n^{th} symmetric power $\mathfrak{S}^n V$ of V is defined by $\{a \in V^{\otimes n} : \forall w \in S_n, \sigma(w)(a) = a\}$. The n^{th} alternating power $\bigwedge^n V$ of V is defined by $\{a \in V^{\otimes n} : \forall w \in S_n, \sigma(w)(a) = \operatorname{sgn}(w)(a)\}$.

Clearly $\mathfrak{S}^n V \oplus \bigwedge^n V = \{a \in V^{\otimes n} : \forall w \in A_n, \sigma(w)(a) = a\} \subsetneq V^{\otimes n}$ when $n > 2$. Assuming $\operatorname{char} k \nmid n!$, we define

$$v_1 \cdots v_n = \frac{1}{n!} \sum_{w \in S_n} v_{w(1)} \otimes \cdots \otimes v_{w(n)} \in \mathfrak{S}^n V$$

$$v_1 \wedge \cdots \wedge v_n = \frac{1}{n!} \sum_{w \in S_n} \operatorname{sgn}(w) v_{w(1)} \otimes \cdots \otimes v_{w(n)} \in \bigwedge^n V$$

One can show that if $v_1, \ldots, v_d$ is a basis for V, then $\{v_{i_1} \cdots v_{i_n} : 1 \le i_1 \le \cdots \le i_n \le d\}$ is a basis for $\mathfrak{S}^n V$ and $\{v_{i_1} \wedge \cdots \wedge v_{i_n} : 1 \le i_1 < \cdots < i_n \le d\}$ is a basis for $\bigwedge^n V$ (note that when $n > d$ then $\bigwedge^n V = \{0\}$). So for $g \in G$, we can compute $\chi_{\mathfrak{S}^n V}(g)$ and $\chi_{\bigwedge^n V}(g)$ in terms of the eigenvalues of g in V.
Notably, we have $\bigwedge^d V \cong \det \rho$ as representations.

Definition 5.7. Suppose $\operatorname{char} k = 0$. Given a vector space V, the tensor algebra of V is the infinite-dimensional vector space $TV = \bigoplus_{n \ge 0} V^{\otimes n}$ (with $V^{\otimes n} = k$ by convention) equipped with the structure of a noncommutative graded ring under the multiplication extending from $(v_1 \otimes \cdots \otimes v_r)(w_1 \otimes \cdots \otimes w_s) = v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_s$.
The symmetric algebra $\mathfrak{S}V$ of V is defined as the quotient $(TV)/(x \otimes y - y \otimes x : x, y \in V)$ which is a commutative graded ring with grading $\mathfrak{S}V \cong \bigoplus_{n \ge 0} \mathfrak{S}^n V$ (isomorphism extending from $x_1 \oplus \cdots \oplus x_n \mapsto x_1 \cdots x_n$).
The exterior algebra $\bigwedge V$ of V is defined as the quotient $(TV)/(x \otimes y + y \otimes x : x, y \in V)$ which is a graded-commutative ring with grading $\bigwedge V \cong \bigoplus_{n \ge 0} \bigwedge^n V$ (isomorphism extending from $x_1 \oplus \cdots \oplus x_n \mapsto x_1 \wedge \cdots \wedge x_n$).

The adjective graded-commutative means that if x is in the r^{th} graded piece and y is in the s^{th} graded piece then $xy = (-1)^{rs} yx$.

5.3 Duality

Recall that we have defined an operation $*$ on $\mathcal{C}_G$ given by $f^*(g) = f(g^{-1})$. This of course restricts to $R(G)$.

Definition 5.8. If G is a group and (ρ, V) a representation of G, then the dual representation (ρ^*, V^*) is given by $\rho^*(g)(\theta)(v) = \theta(g^{-1}v)$.

Lemma 5.9. $\chi_{V^*} = \chi_V^*$.

Proof. Choose a basis $v_1, \dots, v_d$ of V and let $\epsilon_1, \dots, \epsilon_d$ be the corresponding dual basis on V^*. Suppose A is the matrix of $\rho(g)$ under the basis $v_1, \dots, v_d$, then

$$\rho^*(g)(\epsilon_k)(v_i) = \epsilon_k(\rho(g)^{-1}v_i) = \epsilon_k\left(\sum_j (A^{-1})_{ji}v_j\right) = (A^{-1})_{ki}$$

So $\rho^*(g)$ has matrix $(A^{-1})^\top$ under the dual basis, hence $\chi_{V^*}(g) = \operatorname{tr}(A^{-1})^\top = \operatorname{tr}(A^{-1}) = \chi_V(g^{-1}) = \chi_V^*(g)$. $\square$

Definition 5.9. We say that V is self-dual if $V \cong V^*$ as representations of G.

Over $\mathbb{C}$, our established theory on $\mathbb{C}$-characters shows that V is self-dual iff χ_V takes real values.

Example 5.5. 1. Suppose $G = C_3 = \langle x\rangle$, $V = \mathbb{C}$, and G acts on V by $\rho(x) = e^{2\pi i/3}$, then V is not self-dual.
2. If $G = S_n$, every $g \in G$ is conjugate to g^{-1}, so every complex representation of G is self-dual.
3. Permutation representations (over $\mathbb{C}$) are always self-dual.

One can show (either directly or using characters) that if U, V, W are complex representations of G, then $V^* \otimes W \cong \operatorname{Hom}_k(V, W)$ and $\operatorname{Hom}_k(V \otimes W, U) \cong \operatorname{Hom}_k(V, \operatorname{Hom}_k(W, U))$ as G-representations. Then we know that if V is self-dual then $\langle 1, \chi_{\mathfrak{S}^2 V}\rangle, \langle 1, \chi_{\bigwedge^2 V}\rangle$ cannot both be zero.
We've now got a number of ways to build representations of a group G: Permutation representation from a G-action, representation induced from a group homomorphism $G \to H$ and a representation of H, tensor products, symmetric/exterior powers, decompositions of these into irreducible components, orthogonality of character table. Let's add one more to the list, then we'll stop.

6 Induction

6.1 Induced Representations

Given any group G and $g \in G$, we write $[g]_G$ to denote the conjugacy class of g in G. Its indicator function is of course denoted by $1_{[g]_G} \in \mathcal{C}_G$. It's worth noting that $[g^{-1}]_G = [g]_G^{-1}$ for any $g \in G$, so $1^*_{[g]_G} = 1_{[g^{-1}]_G}$.
When $H \le G$, $[g]_G \cap H$ is a union of conjugacy classes in H. Consequently, we the linear map $\Gamma : \mathcal{C}_G \to \mathcal{C}_H$ given by the restriction $f \mapsto f|_H$ is well-defined. We then have $\Gamma(1_{[g]_G}) = \sum_{[h]_H \subset [g]_G} 1_{[h]_H}$.
Recall that when G is finite we have a nondegenerate form on $\mathcal{C}_G$ given by

$$\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1^*(g) f_2(g)$$

In the case $k = \mathbb{C}$, this is a sesquilinear form, so Γ should have an adjoint Γ^* which by definition satisfies $\langle \Gamma(f_1), f_2\rangle_H = \langle f_1, \Gamma^*(f_2)\rangle_G$. In particular,

$$\begin{aligned}\langle 1_{[g^{-1}]_G}, \Gamma^*(f)\rangle_G &= \langle \Gamma(1_{[g^{-1}]_G}), f\rangle_H \\ &= \frac{1}{|H|}\sum_{[h]_H \subset [g]_G} |[h]_H| f(h) = \sum_{[h]_H \subset [g]_G} \frac{f(h)}{|C_H(h)|}\end{aligned}$$

On the other hand,

$$\langle 1_{[g^{-1}]_G}, \Gamma^*(f)\rangle = \frac{1}{|G|}\sum_{x\in[g]_G}\Gamma^*(f)(x) = \frac{\Gamma^*(f)(g)}{|C_G(g)|}$$

So we get an explicit formula for Γ^* given by

$$\Gamma^*(f)(g) = \sum_{[h]_H\subset[g]_G}\frac{|C_G(g)|}{|C_H(h)|}f(h)$$

that happens to satisfy the adjoint equation $\langle\Gamma(f_1), f_2\rangle_H = \langle f_1, \Gamma^*(f_2)\rangle_G$ for general fields k (with characteristic not dividing $|G|$), so we can indeed generalise this particuler occurence of adjointness to fields other than $\mathbb{C}$.
Does Γ^* map character ring to character ring? Suppose χ is a $\mathbb{C}$-character of H and ψ is an irreducible $\mathbb{C}$-character of G, then $\langle\psi, \Gamma^*(\chi)\rangle_G = \langle\Gamma(\psi), \chi\rangle_H$ is a nonnegative integer by Theorem 4.2 since both $\Gamma(\psi)$ and χ are characters. This means that

$$\Gamma^*(\chi) = \sum_{\psi \text{ irreducible character of } G}\langle\psi|_H, \chi\rangle_H\psi$$

is a character. So we can construct characters of G from characters of H via Γ^*. This method is known as induction.

Example 6.1. Let $G = S_3, H = A_3 = \{e, (123), (132)\}$, then $\Gamma^*(f)(e) = 2f(e), \Gamma^*(f)((12)) = 0, \Gamma^*(f)((123)) = f((123)) + f((132))$. Recall that A_3 has character table

A_3	e	(123)	(132)
χ_1	1	1	1
χ_2	1	ω	$\bar{\omega}$
χ_3	1	$\bar{\omega}$	ω

where $\omega = e^{2\pi i/3}$.
Under Γ^*, we obtain the characters

S_3	e	(12)	(123)
$\Gamma^*\chi_1$	2	0	2
$\Gamma^*\chi_2$	2	0	-1
$\Gamma^*\chi_3$	2	0	-1

We can of course decompose these to get a lot of irreducible characters of S_3: $\Gamma^*\chi_2$ is already irreducible and $\Gamma^*\chi_1$ decomposes into the sum of the trivial and sign characters of S_3.

Remark. As in the above example, the image of an irreducible character under Γ^* can be but need not be irreducible.

Let G be a finite group and W a k-vector space. The k-vector space $\text{Hom}(G, W) = \{f : G \to W\}$ has the structure of a G-representation under $(g \cdot f)(x) = f(g^{-1}x)$. If $w_1, \dots, w_n$ is a basis for W, then $\{\delta_g w_i : g \in G, 1 \le i \le n\}$ would be a basis for $\text{Hom}(G, W)$ where $\delta_g w_i(h) = 1_{g=h}w_i$. So $\dim\text{Hom}(G, W) = |G|\dim W$.

Lemma 6.1. $\text{Hom}(G, W) \cong (\dim W)(kG)$.

Proof. Fix a basis $\{w_i\}_{i=1}^n$ of W. Then $\Theta : (\dim W)(kG) \to \mathrm{Hom}(G, W)$ given by

$$\Theta((f_i)_{i=1}^n)(x) = \sum_{i=1}^n f_i(x) w_i$$

is an isomorphism. □

Remark. One can also prove this (for $k = \mathbb{C}$) using characters.

Suppose now that $H \leq G$ and W is a representation of H. $\mathrm{Hom}_H(G, W) = \{f \in \mathrm{Hom}(G, W) : \forall x \in G, h \in H, f(xh) = h^{-1} f(x)\}$ is a k-linear subspace of $\mathrm{Hom}(G, W)$.

Lemma 6.2. *$\mathrm{Hom}_H(G, W)$ is in fact a G-invariant subspace of $\mathrm{Hom}(G, W)$.*

Proof. Let $f \in \mathrm{Hom}_H(G, W)$ and $g, x \in G, h \in H$, then we have $(g \cdot f)(xh) = f(g^{-1}xh) = h^{-1} f(g^{-1}x) = h^{-1}((g \cdot f)(x))$. □

Example 6.2. If $W = 1_H$ and $f \in \mathrm{Hom}(G, W)$, then $f \in \mathrm{Hom}_H(G, W)$ iff f is constant on left cosets of H in G. So $\mathrm{Hom}_H(G, 1) \cong k(G/H)$ as G-representations.

Definition 6.1. If $H \leq G$ has finite index and W is a representation of H, its induced representation to G is $\mathrm{Ind}_H^G W = \mathrm{Hom}_H(G, W)$.

Of course, we don't just introduce two separate notions in the same section without somehow linking them together.

Proposition 6.3. *Suppose W is a representation of H, then*

$$\chi_{\mathrm{Ind}_H^G W}(g) = \Gamma^*(\chi_W)(g) = \sum_{[h]_H \subset [g]_G} \frac{|C_G(g)|}{|C_H(h)|} \chi_W(h)$$
$$= \frac{1}{|H|} \sum_{x \in G, x^{-1}gx \in H} \chi_W(x^{-1}gx) = \frac{1}{|H|} \sum_{x \in G} \chi_W^\circ(x^{-1}gx)$$

where $\chi_W^\circ \in \mathcal{C}_G$ is given by $\chi_W^\circ = 1_H \chi_W$

In particular, $\dim \mathrm{Ind}_H^G W = \chi_{\mathrm{Ind}_H^G W}(e) = [G : H] \dim W$.

Remark. $x^{-1}gx \in H \iff gxH = xH$, so if W is the trivial representation, then $\chi_{\mathrm{Ind}_H^G W}(g)$ is the number of cosets xH with $gxH = xH$, which in turn equals $\chi_{k(G/H)}(g)$.

Proof. For any $x, y, g \in G$, we have $x^{-1}gx = y^{-1}gy \iff xy^{-1} \in C_G(g)$. This gives

$$\sum_{[h]_H \subset [g]_G} \frac{|C_G(g)|}{|C_H(h)|} \chi_W(h) = \frac{1}{|H|} \sum_{x \in G} \chi_W^\circ(x^{-1}gx)$$

It remains to show that $\chi_{\mathrm{Ind}_H^G W}(g) = |G|^{-1} \sum_{x \in G} \chi_W^\circ(xyx^{-1})$.
Note that any $f \in \mathrm{Hom}_H(G, W)$ is determined by its values at (left) coset representatives of H. Conversely, given its values at coset representatives, we can also recover it using $f(xh) = h^{-1} f(x)$. We thus get an isomorphism of vector spaces $\Theta : \mathrm{Hom}_H(G, W) \to W^{\oplus r}, f \mapsto (f(x_i))_{i=1}^r$ where x_i are coset representatives of H. In particular, for $w \in W, j \in \{1, \ldots, r\}$, we can define

$\phi_{j,w}$ Hom via $\phi_{j,w}(x_k h) = \delta_{jk} h^{-1} \cdot w$ which has the property that Θ maps $\{\phi_{j,w} : w \in W\}$ to the j^{th} copy (indexed by the coset representatives) of W in the direct sum.
Any $g \in G$ would give a permutation $\sigma \in S_r$ and $h_1, \dots, h_r \in H$ such that $g^{-1} x_i = x_{\sigma(i)} h_i^{-1} \in x_{\sigma(i)} H$. Thus $g\phi_{i,w} = \phi_{\sigma^{-1}(i), h_{\sigma^{-1}(i)}}$, i.e. g acts on $W^{\oplus r}$ by the permutation matrix of σ. If $\sigma(i) = i$, then g acts on the i^{th} copy of W by $h_i = x_i^{-1} g x_i$, so $\operatorname{tr} g = \sum_{i=1}^r \chi_W^\circ(x_i^{-1} g x_i) = |H|^{-1} \sum_{x \in G} \chi_W^\circ(xgx^{-1})$. □

For a representation V of G and $H \leq G$, we write $\operatorname{Res}_H^G V$ to denote the H-representation on V obtained by restriction.

Corollary 6.4 (Frobenius Reciprocity)**.** *Let V be a representation of G and W a representation of $H \leq G$, then*
(i) $\langle \chi_V, \operatorname{Ind}_H^G \chi_W \rangle = \langle \operatorname{Res}_H^G \chi_V, \chi_W \rangle$.
(ii) $\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$ *as vector spaces.*

Proof. (i) is clear from $\operatorname{Ind}_H^G \chi_W = \Gamma^* \chi_W, \operatorname{Res}_H^G \chi_V = \Gamma \chi_V$. (ii) follows from (i) by the observing that $\langle \chi_V, \operatorname{Ind}_H^G \chi_W \rangle = \dim \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W)$ and that $\langle \operatorname{Res}_H^G \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$. □

Remark. One can prove (ii) directly by considering $\Theta : \operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \to \operatorname{Hom}_H(V, W)$ via $\Theta(\alpha)(v) = \alpha(v)(e)$ which has inverse $\Psi(\beta)(v)(g) = \beta(g^{-1}v)$.

6.2 Mackey Theory

We now want to study representations with the form $\operatorname{Res}_K^G \operatorname{Ind}_H^G W$ for subgroups $H, K \leq G$ and a representation W of H. One application of this is to characterise the situations when $\operatorname{Ind}_H^G W$ is irreducible (since Corollary 6.4 gives $\langle \operatorname{Ind}_H^G \chi_W, \operatorname{Ind}_H^G \chi_W \rangle_G = \langle \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi_W, \chi_W \rangle_H$).

Definition 6.2. Suppose $H, K \leq G$ are subgroups, the (K, H)-double cosets of G are defined as

$$KgH = \bigcup_{k \in K} kgH = \{kgh : k \in K, h \in H\}$$

It is easy to see that they partition G (e.g. by considering a double coset as the union of H-cosets in a K-orbit of G/H).

Definition 6.3. We write $K \backslash G / H$ to denote the set of (K, H)-double cosets in G.

Given any representation (ρ, W) of H and $g \in G$, we can define a representation $({}^g\rho, {}^gW)$ to be the representation of ${}^gH = gHg^{-1} \leq G$ on W given by $({}^g\rho)(ghg^{-1}) = \rho(h)$ for all $h \in H$.

Theorem 6.5 (Mackey's Restriction Formula)**.** *If G is a finite group, $H, K \leq G$, and W is a representation of H, then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{KgH \in K \backslash G / H} \operatorname{Ind}_{K \cap ({}^gH)}^K \operatorname{Res}_{K \cap ({}^gH)}^{({}^gH)} {}^gW$$

Proof. Recall that $\operatorname{Ind}_H^G W = \{f : G \to W : \forall g \in G, h \in H, f(gh) = h^{-1}f(g)\}$. For each double coset KgH, we can consider $V_{KgH} = \{f \in \operatorname{Ind}_H^G W : \forall x \notin KgH, f(x) = 0\} \cong \{f : KgH \to W : \forall x \in KgH, h \in H, f(xh) = h^{-1}f(x)\}$ which is a K-invariant subspace of $\operatorname{Ind}_H^G W$. Moreover, we have

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W = \bigoplus_{KgH \in K\backslash G/H} V_{KgH}$$

as K-representations.
It is then natural to seek a K-isomorphism

$$\Theta : V_{KgH} \to \operatorname{Hom}_{K\cap({}^gH)}(K, {}^gW) = \operatorname{Ind}_{K\cap({}^gH)}^K \operatorname{Res}_{K\cap({}^gH)}^{({}^gH)} {}^gW$$

Indeed, we can take $\Theta(f)(k) = f(kg)$. This is clearly well-defined, K-linear, and injective. Surjectivivity follows from a dimensional consideration:

$$\begin{aligned}\dim V_{KgH} &= |\operatorname{Orb}_K(gH)| \dim W = \frac{|K| \dim W}{|\operatorname{Stab}_K(gH)|}\\ &= \frac{|K| \dim W}{|K \cap {}^gH|} = \dim \operatorname{Hom}_{K\cap({}^gH)}(K, {}^gW)\end{aligned}$$

The theorem follows. □

Corollary 6.6. *Suppose $H, K \leq G$ and χ is a character of H, then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \chi = \sum_{KgH \in K\backslash G/H} \operatorname{Ind}_{K\cap({}^gH)}^K {}^g\chi$$

where ${}^g\chi$ is a class function on $K \cap ({}^gH)$ given by ${}^g\chi(ghg^{-1}) = \chi(h)$.

Corollary 6.7 (Mackey's Irreducibility Criterion). *If $H \leq G$ and W is a complex representation of H, then $\operatorname{Ind}_H^G W$ is irreducible iff W is irreducible and $\operatorname{Res}_{({}^gH)\cap H}^{{}^gH} {}^gW, \operatorname{Res}_{({}^gH)\cap H}^{H} W$ have no irreducible factors in common for any $g \notin H$.*

Proof. $\operatorname{Ind}_H^G W$ is irreducible iff $\langle \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi_W, \chi_W\rangle_H = 1$ by Corollary 6.4. The preceding corollary tells us how to decompose this thing

$$\begin{aligned}\langle \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi_W, \chi_W\rangle_H &= \sum_{HgH \in H\backslash G/H} \langle \operatorname{Ind}_{H\cap({}^gH)}^H \operatorname{Res}_{H\cap({}^gH)}^{{}^gH} {}^g\chi_W, \chi_W\rangle_H\\ &= \sum_{HgH \in H\backslash G/H} \langle \operatorname{Res}_{H\cap({}^gH)}^{{}^gH} {}^g\chi_W, \operatorname{Res}_{H\cap({}^gH)}^{H} \chi_W\rangle_{H\cap({}^gH)}\end{aligned}$$

From which the corollary is clear. □

Corollary 6.8. *If $H \lhd G$ and W is an irreducible complex representation of H, then $\operatorname{Ind}_H^G W$ is irreducible iff ${}^g\chi_W \neq \chi_W$ for all $g \notin H$.*

Proof. As $H \lhd G$, ${}^gH = H$ for all $g \in G$, so by the preceding corollary $\operatorname{Ind}_H^G W$ is irreducible iff ${}^gW$ is not isomorphic to W for any $g \notin H$, i.e. ${}^g\chi_W \neq \chi_W$ for all $g \notin H$. □

Example 6.3. (i) Take $C_n = \langle r \rangle \lhd D_{2n}$, then the irreducible representations of C_n have the form $\chi_k(r) = e^{2\pi i k/n}$. So $\operatorname{Ind}_{C_n}^{D_{2n}} \chi_k$ is irreducible iff χ_k is not real-valued (i.e. $k/n \notin (1/2)\mathbb{Z}$).
(ii) Take $G = S_n, H = A_n \lhd S_n$, then any simple character χ of A_n such that there is some $g \in S_n$ whose conjugacy class splits into two parts in A_n on which χ takes different values has $\operatorname{Ind}_H^G \chi$ simple.

6.3 Frobenius Groups

Definition 6.4. A finite group G is called a Frobenius group if it has a transitive action on a set X such that each nonidentity element in G fixes at most one element in X and there are no trivial stablisers.

Example 6.4. 1. $G = D_{2n}$ is a Frobenius group if we take X to be the vertices of a n-gon.
2.

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{F}_p, a \neq 0 \right\}$$

acts on $\{(x,1)^\top : x \in \mathbb{F}_p\}$ by matrix multiplication (in other words, it acts as the affine functions on $\mathbb{F}_p$) which makes it a Frobenius group.

Lemma 6.9. *G is a Frobenius group iff there is some nontrivial proper subgroup $H \lneq G$ such that $H \cap gHg^{-1} = \{e\}$ for all $g \notin H$.*

Proof. Suppose G acts on a set X that qualifies it as a Frobenius group, then we can pick $H = \operatorname{Stab}_G(x)$ for any $x \in X$. Conversely, suppose we have such an H, then we can take $X = G/H$ acted on by G via left multiplication. □

Theorem 6.10 (Frobenius)**.** *Let G be a finite group acting transitively on a set X such that each nonidentity $g \in G$ fixes at most one element of X, then $K = \{e\} \cup \{g \in G : \forall x \in X, gx \neq x\}$ is a normal subgroup of G with order $|X|$.*

Proof. Pick $x \in X$ and let $H = \operatorname{Stab}_G(x)$, then $|G| = |H||X|$. By hypothesis, $H \cap gHg^{-1} = \operatorname{Stab}_G(x) \cap \operatorname{Stab}_G(gx)$ for any $g \notin H$. This means that

$$\left| \bigcup_{x \in X} \operatorname{Stab}_G(x) \right| = \left| \bigcup_{g \in G} gHg^{-1} \right| = (|H| - 1)|X| + 1 \implies |K| = |X|$$

Moreover, $[h]_G \cap H = [h]_H$ and $C_G(h) = C_H(h)$ for any nonidentity $h \in H$.
We will show $K \trianglelefteq G$ by realising it as a kernel of a representation. If χ is a complex character of H, then

$$\operatorname{Ind}_H^G \chi(g) = \sum_{[h]_H \subset [g]_G} \frac{|C_G(g)|}{|C_H(h)|} \chi(h) = \begin{cases} |X|\chi(e) & \text{if } g = e \\ \chi(g) & \text{if } g \in H \setminus \{e\} \\ 0 & \text{if } g \in K \setminus \{e\} \end{cases}$$

Suppose $\chi_1, \ldots, \chi_r$ are the simple complex characters of H. Consider

$$\theta_i = \operatorname{Ind}_H^G \chi_i + \chi_i(e) 1_G - \chi_i(e) \operatorname{Ind}_H^G 1_H = \begin{cases} \chi_i(e) & \text{if } g \in K \\ \chi_i(g) & \text{if } g \in H \setminus \{e\} \end{cases}$$

So if θ_i is indeed a character then the kernel of the corresponding representation would contain K. Write $\theta_i = \sum_i n_i \chi_i$ for $n_i \in \mathbb{Z}$, then

$$\begin{aligned}\langle \theta_i, \theta_i \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 = \frac{1}{|G|} \left(\sum_{h \in H \setminus \{e\}} |X||\chi_i(h)|^2 + \sum_{k \in K} \chi_i(e)^2 \right) \\ &= \frac{|X|}{|G|} \sum_{h \in H} |\chi_i(h)|^2 = \langle \chi_i, \chi_i \rangle = 1\end{aligned}$$

So $\theta_i = \pm\chi$ for a irreducible character χ of G. But $\theta_i(e) = \chi_i(e) > 0$, so $\theta_i = \chi$ is a (simple) character. Consider now $\theta = \sum_i \chi_i(e)\theta_i$, then $\theta(h) = 0$ for any $h \in H \setminus \{e\}$ by Proposition 4.11 and we have $\theta(k) = |H| = \theta(e)$ for all $k \in K$, so $\ker\theta = K$ (noting that every element of G is either contained in a conjugate of H or K) as desired. □

Remark. 1. It follows that no Frobenius group is simple.
2. The normal subgroup K arising this way is called the Frobenius kernel of the G-action on X, and a subgroup of the form $H = \mathrm{Stab}_G(x)$ is called a Frobenius complement.
3. There is no known proof of the theorem without representation theory.

7 Arithmetic Properties of Characters

7.1 Algebraic Integers

Definition 7.1. A complex number $x \in \mathbb{C}$ is an algebraic integer if it is the root of a monic polynomial with integer coefficients.

Remark. 1. The set of algebraic integers forms a subring of $\mathbb{C}$.
2. If $A \subset \mathbb{C}$ is a subring whose additive group is a finitely-generated abelian group, then A consists exclusively of algebraic integers.
3. The only algebraic integers in $\mathbb{Q}$ are the integers.

Lemma 7.1. *If χ is a character of a finite group G, then all the values of χ are algebraic integers.*

Proof. For any $g \in G$, $\chi(g)$ is the sum of some $|G|^{th}$ roots of unity, each of which is an algebraic integer. □

7.2 The Group Algebra

There are two sensible ways to make kG a ring: With the pointwise multiplication, or with a multiplication defined by convolution

$$(f_1 f_2)(g) = \sum_{xy=g} f_1(x) f_2(y) = \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

We care about the latter structure more, since it does not entirely forget the group structure on G. For starters, we have $\delta_{y_1}\delta_{y_2} = \delta_{y_1 y_2}$, so the group itself does appear in the structure of kG. In particular,

$$\left(\sum_{g \in G} \lambda_g \delta_g \right) \left(\sum_{h \in G} \mu_h \delta_h \right) = \sum_{k \in G} \sum_{gh=k} \lambda_g \mu_h \delta_k$$

A finitely generated kG-module is then exactly the same as a representation of G: Any representation (ρ, V) of G makes V a kG-module via $f \cdot v = \sum_{g\in G} f(g)\rho(g)v$; Conversely, if M is a finitely-generated kG-module, then M is a k-vector space and a representation of G under $\rho(g)m = \delta_g \cdot m$. As an exercise, one can try to describe a permutation representation kX in terms of the language of kG-modules.

Definition 7.2. The centre $Z(R)$ of a ring R is the subring $Z(R) = \{f \in R : \forall g \in R, fg = gf\}$.

Lemma 7.2. *$f \in Z(kG)$ if and only if f is a class function on G.*

In particular, $\dim_k Z(kG)$ is the number of conjugacy classes in G.

Proof. $f \in Z(kG)$ iff $f\delta_g = \delta_g f$ for all $g \in G$. The latter however means exactly that f is constant on every conjugacy class of G. □

$Z(kG)$ is a commutative ring, but of course the multiplication on it is (in general) not the pointwise multiplication. To distinguish them, we introduce a different notation for the indicator functions

Definition 7.3. Suppose $O_1, \dots, O_r$ are conjugacy classes of G. The class sums $C_1, \dots, C_r$ are the class functions on G with

$$C_i(g) = \begin{cases} 1 & \text{if } g \in O_i \\ 0 & \text{otherwise} \end{cases}$$

Proposition 7.3. *There are some nonnegative integers a_{ijk} such that $C_iC_j = \sum_k a_{ijk}C_k$. Explicitly, we have $a_{ijk} = |\{(x,y) \in O_i \times O_j : xy = g_k\}|$ where $g_1, \dots, g_r$ is a collection of representatives for the orbits.*

Proof. Certainly $C_iC_j = \sum_k a_{ijk}C_k$ for some $a_{ijk} \in k$. We also have

$$a_{ijk} = (C_iC_j)(g_k) = \sum_{xy=g_k} C_i(x)C_j(y) = |\{(x,y) \in O_i \times O_j : xy = g_k\}|$$

which is exactly what we wanted. □

Definition 7.4. Such $\{a_{ijk}\}_{i,j,k}$ are called the structure constants of $Z(kG)$.

Suppose now that (ρ, V) is a representation of G and $z \in Z(kG)$. Then z can be realised as an element of $\mathrm{Hom}_G(V,V)$ via $zv = \sum_{g\in G} z(g)\rho(g)v$.
When $k = \mathbb{C}$ (which we will assume from this point onwards) and V is irreducible, we have Theorem 3.1 telling us that there are some $\lambda_z \in k$ with $zv = \lambda_z v$. So there is a homomorphism $w_\rho : Z(kG) \to k$ of k-algebras given by $w_\rho(z) = \lambda_z$. We then find, by taking traces, that $\lambda_z \dim V = \sum_{g\in G} z(g)\chi_V(g)$, so $w_\rho(C_i) = \chi_V(e)^{-1}|O_i|\chi_V(g_i)$ only depends on the character of (ρ, V). So we often use the notation $w_\chi = w_{\chi_\rho} = w_\rho$.

Lemma 7.4. *The values $w_\chi(C_i)$ are algebraic integers.*

Proof. We have

$$w_\chi(C_i)w_\chi(C_j) = \sum_{k=1}^{r} a_{ijk}w_\chi(C_k)$$

So $\mathbb{Z}[w_\chi(C_k) : 1 \le k \le r] \le \mathbb{C}$ is a finitely-generated $\mathbb{Z}$-module, hence consists entirely of algebraic integers. □

Lemma 7.5.

$$a_{ijk} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{\chi \text{ irreducible character of } G} \frac{\chi(g_i)\chi(g_j)\chi(g_k^{-1})}{\chi(e)}$$

Proof. We have

$$\frac{\chi(g_i)}{\chi(e)}|O_i|\frac{\chi(g_j)}{\chi(e)}|O_j| = w_\chi(C_i)w_\chi(C_j) = \sum_k a_{ijk}w_\chi(C_k) = \sum_k a_{ijk}\frac{\chi(g_k)|O_k|}{\chi(e)}$$

whenever χ is an irreducible character. So

$$\sum_\chi \frac{\chi(g_i)\chi(g_j)\chi(g_l^{-1})|G|}{\chi(e)|C_G(g_i)||C_G(g_j)|} = \sum_{k=1}^r \frac{1}{|C_G(g_k)|}\sum_\chi \chi(g_k)\chi(g_l^{-1}) = a_{ijl}$$

by Proposition 4.11. □

7.3 Degree of Irreducible Representations

Theorem 7.6. *If V is a simple representation of G, then* $\dim V \mid G$.

Proof. Let $\chi = \chi_V$ be the character of the representation. It suffices to show that $|G|/\chi(e)$ is an algebraic integer. Indeed, if $g_1, \dots, g_r$ are representatives of the conjugacy classes $O_1, \dots, O_r$ in G, then

$$\frac{|G|}{\chi(e)} = \frac{1}{\chi(e)}\sum_{g\in G}\chi(g)\chi(g^{-1}) = \frac{1}{\chi(e)}\sum_{i=1}^r |O_i|\chi(g_i)\chi(g_i^{-1}) = \sum_{i=1}^r w_\chi(C_i)\chi(g_i^{-1})$$

which (combined with Lemma 7.4) implies the claim. □

Example 7.1. 1. If G is a p-group, then the degree of every simple representation is a power of p. In particular, when $|G| = p^2$, then Proposition 4.11 shows that all simple representations of $|G|$ have degree 1, i.e. G is abelian.
2. If $G = A_n$ or S_n and $p > n$ is prime, then G has no simple representation whose degree is a multiple of p.

Corollary 7.7 (Burnside 1904)**.** *If (ρ, V) is a simple representation of G, then* $\dim V \mid |G/Z(G)|$.

Proof. If $z \in Z = Z(G)$, then Theorem 3.1 shows that g acts as a scalar λ_z on V. For each $n \geq 2$, $(\rho^{\otimes n}, V^{\otimes n})$ is an irreducible representation of G^n. $z = (z_1, \dots, z_n) \in Z^n$ would act on $V^{\otimes n}$ via a scalar $\lambda_{z_1}\cdots\lambda_{z_n}$. This means that $Z' = \{(z_1, \dots, z_n) \in Z^n : \lambda_{z_1}\cdots\lambda_{z_n} = e\}$ acts trivially on $V^{\otimes n}$, i.e. $V^{\otimes n}$ is in fact a simple representation of G^n/Z'. Now $|Z'| = |Z|^{n-1}$, so $|G^n/Z'| = |G/Z|^n|Z|$. By the preceding theorem, $(\dim V)^n \mid |G/Z|^n|Z|$. But this has to be true for every n, so necessarily $\dim V \mid |G/Z|$. □

Proposition 7.8. *If G is a simple group, then it has no irreducible representations of degree* 2.

Proof. Obvious when G is abelian or when $|G|$ is odd. If G is nonabelian and has even order, then G has an involution g by Cauchy's theorem. By a result from example sheet, every character χ of G has $\chi(g) \equiv \chi(e) \pmod 4$. If $\chi(e) = 2$, then $\chi(g) = \pm 2$, so $\rho(g) = \pm\,\mathrm{id}$ is in the center of the representation. So $\rho(h) = \pm\,\mathrm{id}$ for all $h \in G$, contradicting the simplicity of χ. □

Remark. It was proved in 1963 by Feit and Thompson (former lecturer of this course) that there is no nonabelain simple groups of odd order.

7.4 Burnside's $p^a q^b$ Theorem

Lemma 7.9. *Suppose α is a nonzero algebraic integer of the form $m^{-1}\sum_{i=1}^m \lambda_i$ with $\lambda_i^n = 1$ for all i, then $|\alpha| = 1$ (i.e. all λ_i are equal).*

Proof. By assumption, $\alpha \in \mathbb{Q}(\epsilon)$ with $\epsilon = e^{2\pi i/n}$. Let $G = \operatorname{Aut}(\mathbb{Q}(\epsilon)/\mathbb{Q})$, so $\{\beta \in \mathbb{Q}(\xi) : \forall \sigma \in G, \sigma\beta = \beta\} = \mathbb{Q}$ since this is a Galois extension. So $N(\alpha) = \prod_{\sigma \in G} \sigma\alpha$ is rational. G also fixes the set of algebraic integers, so $N(\alpha)$ is an algebraic integer too. But these just mean that $N(\alpha)$ is an integer. $0 < |\sigma\alpha| = |m^{-1}\sum_{i=1}^m \sigma\lambda_i| \leq 1$ for all $\sigma \in G$, so we can only have $|\sigma\alpha| = 1$ for all σ. In particular $|\alpha| = 1$. □

Lemma 7.10. *Suppose χ is a simple character of G and and O is a conjugacy class of G. If $\chi(e)$ and $|O|$ are coprime, then any $g \in O$ either has $|\chi(g)| = \chi(e)$ or $\chi(g) = 0$.*

Proof. Choose a, b such that $a\chi(e) + b|O| = 1$, then $\alpha = \chi(g)/\chi(e) = a\chi(g) + b|O|\chi(g)/\chi(e)$ is an algebraic integer by Lemma 7.4 and is the average of $\chi(e)$ many $|G|^{th}$ roots of unity, so either $\alpha = 0$ or $|\alpha| = 1$ by the preceding lemma. □

Proposition 7.11. *If G is a nonabelian group with a conjugacy class $O \neq \{e\}$ that has prime power order, then G is not simple.*

Proof. Suppose for the sake of contradiction that G is simple and $|O| = p^r$ and $g \in O$. If χ is a nontrivial simple character of G, then $|\chi(g)| < \chi(e)$. By the preceding lemma, we know that either $p \mid \chi(e)$ or $\chi(g) = 0$. Proposition 4.11 then gives

$$-\frac{1}{p} = \sum_{\chi \text{ nontrivial simple character of } G} \frac{\chi(e)}{p}\chi(g)$$

But the right-hand side is always an algebraic integer but the left-hand side is never, contradiciton. □

Theorem 7.12 (Burnside 1904)**.** *Suppose p, q are prime and G is a group with order $p^a q^b$ with $a + b \geq 2$, then G is not simple.*

Proof. The claim is clear if either $a = 0, b = 0, p = q$ or $G = Z(G)$. Assume otherwise. Let Q be a Sylow q-subgroup of G and $g \in Z(Q) \setminus \{e\}$. Then $q^b \mid |C_G(g)|$, so the conjugacy class of g has size equal to a power of p. The preceding proposition gives the result. □

Remark. 1. The same idea also shows that every group G of order $p^a q^b$ is soluble, i.e. there is a chain of subgroups

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$$

such that each G_{i+1}/G_i is cyclic.
2. The example A^5 shows that simple groups can occur when we allow more than two prime factors.
3. The first group-theoretic proof of the theorem was found in 1972.

8 Topological Groups

Got bored dealing with finite groups? Let's spice things up.
Some groups have a natural structure of a topological space compatible with its group structure.

Definition 8.1. A topological group G is a group with a topology such that the group multiplication $G \times G \to G$ and inversion $G \to G$ are continuous.

Example 8.1. 1. $\mathrm{GL}_n(\mathbb{C})$ is a topological group with topology inherited from $\mathbb{C}^{n\times n}$.
2. Any group can be made a topological group by giving it the discrete topology.
3. $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ are all topological groups with the topology inherited from $\mathrm{GL}_n(\mathbb{C})$.
4. Profinite groups (e.g. the p-adic numbers $\mathbb{Z}_p$) are topological groups.

$S^1 \cong \mathrm{U}(1)$ has uncountably many mutually non-isomorphic representations of degree 1, but not many of them is continuous. The aim is to study continuous $\mathbb{C}$-representations of topological groups.

Definition 8.2. A representation of a topological group G on a complex vector space V (with its natural topology) is a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$.

Unless stated otherwise, we will assume for the rest of this section that we only care about complex vector spaces.

Remark. 1. We can drop the continuity requirement if G is discrete.
2. For any topological space X, $\alpha : X \to \mathrm{GL}_n(\mathbb{C})$ is continuous iff all the components α_{ij} are continuous.

8.1 Compact Groups

So far, the fanciest trick we used to study representations is to take some sort of average over the group. This sure doesn't work directly for infinite groups, but if G is nice enough, we might just be able to replace the sum by an integral.

Definition 8.3. Let G be a topological group and let $C(G, \mathbb{R}) = \{f : G \to \mathbb{R} \text{ continuous}\}$. Then a linear operator $\mu_G : C(G, \mathbb{R}) \to \mathbb{R}$ is called a Haar integral if:
(i) $\mu_G(1) = 1$.
(ii) $\mu_G(g \mapsto f(xg)) = \mu_G(f) = \mu_G(g \mapsto f(gx))$.

We often use the notations

$$\mu_G(f) = \int_G f = \int_G f(g)\,\mathrm{d}g$$

Example 8.2. 1. If G is finite, then we can take

$$\int_G f = \frac{1}{|G|} \sum_{g\in G} f(g)$$

2. If $G = S^1$, we can take

$$\int_G f = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta$$

Note that if V is a $\mathbb{R}$-vector space, then a Haar integral induces a linear map $C(G,V) \to V$ by component-wise integral. Under this framework, whenever $\alpha : V \to W$ is linear, then we would have

$$\alpha\left(\int_G f\right) = \int_G \alpha f$$

If the structure on V in fact comes from its structure of a $\mathbb{C}$-vector space, then the integral is also $\mathbb{C}$-linear.

Theorem 8.1. *Any compact Hausdorff topological group admits a Haar integral.*

Proof. Omitted. □

Most topological groups we are interested (except $\mathrm{GL}_n(\mathbb{C})$) will be compact and Hausdorff, so we are not going to worry too much about the existence of a Haar integral. By confusion of terminology, compact Hausdorff topological groups are also known as compact groups.

Corollary 8.2 (Weyl's Unitary Trick)**.** *If G is a compact group, then every $\mathbb{C}$-representation of G has a G-invariant inner product.*

Proof. Choose any inner product $\langle,\rangle$ on V and refine it to

$$(v,w) = \int_G \langle gv, gw\rangle \,\mathrm{d}g$$

which works. □

Corollary 8.3 (Maschke's Theorem)**.** *If G is a compact group and V is a representation of G, then every subrepresentation has a G-invariant complement.*

Consequently, V has to be completely reducible.
We can also use Haar integral to define an inner product on $C(G,\mathbb{C})$ and hence on $\mathcal{C}_G \subset C(G,\mathbb{C})$ by

$$\langle f, f'\rangle = \int_G \overline{f(g)} f'(g)\,\mathrm{d}g$$

If (ρ, V) is a representation of G, then $\chi_\rho = \operatorname{tr}\rho$ is continuous and hence an element of $\mathcal{C}_G$.

Corollary 8.4 (Orthogonality of Characters)**.** *If G is a compact group and V, W are irreducible representations of G, then $\langle \chi_V, \chi_W\rangle = 1_{V\cong W}$.*

The isomorphism of representations for topological groups are defined exactly as before (recall that every linear isomorphism is a homeomorphism in finite-dimensional complex vector spaces).

Proof. Same as what we did for finite groups. □

For $U = \mathrm{Hom}_{\mathbb{C}}(V,W)$, we most certainly still have a G-linear projection $\pi : U \to U^G$ given by $\pi = \mu_G \circ \rho$.
It's also possible to show that the simple characters span $\mathcal{C}_G$, which requires some knowledge about Hilbert spaces.

8.2 Representations of S^1

Theorem 8.5. *Every simple representation of S^1 has degree 1 and has the form $z \mapsto z^n$ for $n \in \mathbb{Z}$.*

If you did Algebraic Topology then this can't be unexpected. If not, well...

Lemma 8.6. *If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continuous group homomorphism, then there's some $\lambda \in \mathbb{R}$ such that $\psi(x) = \lambda x$.*

Proof. Let $\lambda = \psi(1)$, then certainly $\psi(n) = \lambda n$ for all $n \in \mathbb{Z}$. Then $\psi(q) = \lambda q$ for all $q \in \mathbb{Q}$ since $m\psi(n/m) = \psi(n) = \lambda n$. The lemma then follows from continuity. □

Lemma 8.7. *If $\psi : (\mathbb{R}, +) \to S^1$ is a continuous group homomorphism, then there is some $\lambda \in \mathbb{R}$ such that $\psi(x) = e^{2\pi i \lambda x}$.*

Proof. Any continuous function $\psi : \mathbb{R} \to S^1$ with $\psi(0) = 1$, there is a unique continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\psi(x) = e^{2\pi i \alpha(x)}$. If you're doing Algebraic Topology, you can just refer to the path-lifting lemma. If not, the idea in this special case can be phrased as follows: α should be given locally by $(2\pi i)^{-1} \log \psi(x)$ for some branch of log. One should be able to glue together these choices to give a unique global α.
By the preceding lemma, it then suffices to show that any such α must be a group homomorphism. This follows from the continuity of $(a, b) \mapsto \alpha(a + b) - \alpha(a) - \alpha(b)$ which can only take integer values. □

Proof of Theorem 8.5. Since S^1 is abelian, every simple representation of S^1 has degree 1. Now let $\rho : S^1 \to \mathbb{C}^\times$ be a representation. $\rho(S^1)$ is compact, hence closed and bounded. But $\rho(z^n) = \rho(z)^n$, so $\rho(S^1) \subset S^1$. The map given by $x \mapsto \rho(e^{2\pi i x})$ is a continuous homomorphism $(\mathbb{R}, +) \to S^1$, hence $\rho(e^{2\pi i x}) = e^{2\pi i \lambda x}$. Also, $1 = \rho(1) = \rho(e^{2\pi i})$, so $\lambda \in \mathbb{Z}$. □

Write ρ_n to be the representation $z \mapsto z^n$ and χ_n (which is just ρ_n) its character. Which values can a character of S^1 take? Consider

$$\mathbb{N}[z, z^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} a_n z^n : a_n \in \mathbb{N}, \sum_{n \in \mathbb{Z}} a_n < \infty \right\}$$

where of course $0 \in \mathbb{N}$. Any representation V of S^1 decomposes into a direct sum of simple representations, so $\chi_V(z) = \sum_n a_n z^n \in \mathbb{N}[z, z^{-1}]$ with $\dim V = \sum_n a_n$, where a_n is the number of ρ_n in the decomposition.
On S^1, a Haar integral can be given by

$$\int_{S^1} f = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, \mathrm{d}\theta$$

So the coefficients (a_n) can be otherwise computed via

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\theta}) e^{-in\theta}\, \mathrm{d}\theta$$

So

$$\chi_V(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\phi}) e^{-in\phi}\, \mathrm{d}\phi \right) e^{in\theta}$$

which can certainly be interpreted as a Fourier series.

Remark. In fact, the theory of Fourier series shows that any continuous function on S^1 can be uniformly approximated by a linear combination of these χ_n's. Moreover, they actually form a complete orthonormal set in the Hilbert space

$$L^2(S^1) = \left\{ f : S^1 \to \mathbb{C} : \int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta < \infty \right\} / \{\text{a.e.}\}$$

That is, every function $f \in L^2(S^1)$ admits a series

$$\sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) e^{-in\phi} \, \mathrm{d}\phi \right) e^{in\theta}$$

converging to f under the norm

$$\|g\| = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta$$

So in some sense we can "decompose $L^2(S^1)$ as the direct sum of all ρ_n", whatever that's supposed to mean. This can be viewed as an analogue of decomposing $\mathbb{C}G$ into the direct sum of $\dim V$ copies of each irreducible representation V of G when G is finite.

8.3 Representations of SU(2)

Recall that $\mathrm{SU}(2) = \{A \in \mathrm{GL}_2(\mathbb{C}) : \bar{A}^\top A = I, \det A = 1\}$, i.e. matrices having the form $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ with $|a|^2 + |b|^2 = 1$. So SU(2) is homeomorphic to $S^3 \subset \mathbb{R}^4$. One can also construct the quaternions from SU(2) via

$$\mathbb{H} = \mathbb{R}\,\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : w, z \in \mathbb{C} \right\}$$

On $\mathbb{H}$, we can then define a norm via $\|A\| = \sqrt{\det A}$. Then the unit sphere in $\mathbb{H}$ under this norm would exactly be SU(2). Moreover, if $A \in \mathrm{SU}(2)$ and $X \in \mathbb{H}$ we have $\|AX\| = \|A\|\|X\| = \|X\|$.
A Haar integral on SU(2) can be given by

$$\int_{\mathrm{SU}(2)} f = \frac{1}{2\pi^2} \int_{S^3} f$$

Let's now try and understand conjugacy classes in SU(2). They are closely related to the subgroup $T = \{\mathrm{diag}(a, \bar{a}) : |a| = 1\} \leq \mathrm{SU}(2)$ which, in more advanced contexts, is known as a "maximal torus".

Lemma 8.8. *Let* $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
(i) $sts^{-1} = t^{-1}$ *for any* $t \in T$.
(ii) $s^2 = -I \in Z(\mathrm{SU}(2))$.
(iii) $N_{\mathrm{SU}(2)}(T) = T \cup (sT)$.

Proof. Direct computation. □

Proposition 8.9. *For every conjugacy class* O *in* SU(2), *there's some* $t \in T$ *such that* $O \cap T = \{t, t^{-1}\}$. *In particular, there is a bijection between conjugacy classes of* SU(2) *and* $[-1, 1]$ *given by* $A \leftrightarrow (1/2)\,\mathrm{tr}\,A$,

Proof. Every unitary matrix has has an orthonormal basis of eigenvectors. So for any $A \in O$, there is some $P \in \mathrm{U}(2)$ such that $P^{-1}AP$ is diagonal (i.e. in T). Replacing P by $\lambda^{-1}P$ where $\lambda^2 = \det P$ shows that A is in fact $\mathrm{SU}(2)$-conjugate to a diagonal matrix, i.e. $O \cap T \neq \varnothing$.
If $t \in O$ then $t^{-1} = sts^{-1} \in O \cap T$ by the preceding lemma. These are all the elements of $O \cap T$ by looking at the eigenvalues. □

Let O_x be the conjugacy class corresponding to $x \in [-1, 1]$, then it is just a real 2-sphere with radius $\sqrt{1-x^2}$.
We are now ready to discuss the representations of this group. Let V_n be the $\mathbb{C}$-vector space of homogeneous polynomials of degree n in two variables X, Y, i.e. $V_n = \mathrm{Span}_{\mathbb{C}}\{X^jY^{n-j} : 0 \leq j \leq n\}$. $\mathrm{GL}_2(\mathbb{C})$ acts on V_n via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(X,Y) = f(aX + cY, bX + dY)$$

Example 8.3. When $n = 0$, this is the trivial representation. When $n = 1$, this is the usual representation of $\mathrm{GL}_2(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^2)$ on $\mathbb{C}^2$. When $n = 2$, $V_2 = \mathbb{C}X^2 + \mathbb{C}XY + \mathbb{C}Y^2$ and

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

In general, $V_n \cong \mathfrak{S}^n V_1$ as representations of $\mathrm{GL}_2(\mathbb{C})$. We can of course restrict each of them to $\mathrm{SU}(2)$. In fact, these are precisely all the simple representations of $\mathrm{SU}(2)$.

Lemma 8.10. *A class function f on* $\mathrm{SU}(2)$ *is determined by its restriction to T. Furthermore, $f|_T$ is always even, i.e. $f(A) = f(A^{-1})$ for $A \in T$.*

Proof. Proposition 8.9. □

Let $\mathbb{N}[z, z^{-1}]^{\mathrm{ev}} = \{f \in \mathbb{N}(z, z^{-1}) : f(z) = f(z^{-1})\}$.

Lemma 8.11. *If χ is a character of* $\mathrm{SU}(2)$*, then $\chi|_T \in \mathbb{N}(z, z^{-1})^{\mathrm{ev}}$. More precisely, there always exists $f \in \mathbb{N}[z, z^{-1}]^{\mathrm{ev}}$ such that $\chi(\mathrm{diag}(z, \bar{z})) = f(z)$ for all $z \in S^1$.*

Proof. If V is a representation of $\mathrm{SU}(2)$, we restrict it to a representation $\mathrm{Res}_T^{\mathrm{SU}(2)} V$ of T. But T is just S^1 as a topological group, so $\chi_V|_T = \chi_{\mathrm{Res}_T^{\mathrm{SU}(2)} V} \in \mathbb{N}[z, z^{-1}]$. It is also even by the preceding lemma, so we are done. □

We have $\mathrm{diag}(z, \bar{z})X^jY^{n-j} = z^{n-2j}X^jY^{n-j}$, so the set $\{X^iY^{n-j} : 0 \leq j \leq n\}$ is an eigenbasis for T and $\chi_{V_n}(\mathrm{diag}(z, \bar{z})) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n}$. So we now know about $\chi_{V_n}|_T$, which is sufficient to recover χ_{V_n} in its entirety.

Theorem 8.12. *V_n are irreducible* $\mathrm{SU}(2)$*-representations.*

Proof. Let $0 \neq W \leq V_n$ be an $\mathrm{SU}(2)$-invariant subspace. Then W is T-invariant as well. $\mathrm{Res}_T^{\mathrm{SU}(2)} V_n = \bigoplus_{j=0}^n \mathbb{C}X^jY^{n-j}$ by the character calculation above. This is a direct sum of non-isomorphic representation of T, so W is spanned by a subset of $\{X^jY^{n-j} : 1 \leq j \leq n\}$. In particular, there is some $0 \leq j \leq n$ with $X^jY^{n-j} \in W$. As $2^{-1/2}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) \in \mathrm{SU}(2)$, we have $(X-Y)^j(X+Y)^{n-j} \in W$, so $X^n \in W$. Repeating the same argument shows that $(X+Y)^n \in W$. But then $X^jY^{n-j} \in W$ for all $0 \leq j \leq n$, i.e. $W = V_n$. □

Alternative proof. Suppose $f \in \mathrm{SU}(2)$ is a class function, then essentially

$$\int_{\mathrm{SU}(2)} f = \frac{1}{2\pi^2}\int_0^\pi f\left(\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix}\right) 4\pi \sin^2\theta \,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin^2\theta\,\mathrm{d}\theta$$

So it suffices to show that

$$\frac{1}{\pi}\int_0^{2\pi} |\chi_{V_n}(e^{i\theta})|^2 \sin^2\theta\,\mathrm{d}\theta = 1$$

which is trivial. □

Theorem 8.13. *Every simple representation of* SU(2) *is isomorphic to* V_n *for some* n.

Proof. Suppose V is a simple representation of SU(2), then $\chi_V|_T \in \mathbb{N}[z, z^{-1}]^{\mathrm{ev}}$. Observe that $\{\chi_{V_n}|_T\}_n$ is a ($\mathbb{Q}$-)basis of $\mathbb{Q}[z, z^{-1}]^{\mathrm{ev}}$, so $\chi_V = \sum_n a_n \chi_{V_n}$ for some $a_n \in \mathbb{Q}$ eventually zero. The procedure of clearing denominators and moving negative terms to the other side of the equation gives something of the form

$$m\chi_V + \sum_{i\in I} m_i \chi_{V_i} = \sum_{j\in J} m_j \chi_{V_j}$$

where $I \cap J = \varnothing$, $m, m_i, m_j \in \mathbb{N}, m \neq 0$. The result follows. □

What does tensor products in SU(2) look like?
Recall that if V, W are representations of a group G, $\chi_{V\otimes W} = \chi_V\chi_W$. Let's compute some examples of this for SU(2).

Example 8.4. 1. $\chi_{V_0\otimes V_n} = \chi_{V_n}$.
2. $\chi_{V_1\otimes V_1}(z) = (z + z^{-1})^2 = (\chi_{V_2} + \chi_{V_0})(z)$.
3. $\chi_{V_2\otimes V_1} = \chi_{V_3} + \chi_{V_1}$.

Proposition 8.14 (Clebsch-Gordon Rule)**.** *For every* $m, n \in \mathbb{N}$, $V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|n-m|}$.

Proof. WLOG $n \geq m$, then

$$\begin{aligned}\chi_{V_n\otimes V_m}(z) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}(z^m + z^{m-2} + \cdots + z^{2-m} + z^{-m}) \\ &= \sum_{j=0}^m \frac{z^{n+1+m-2j} - z^{-(n+1+m-2j)}}{z - z^{-1}} = \sum_{j=0}^m \chi_{n+m-2j}(z)\end{aligned}$$

which is what we wanted. □

One can also rephrase this combinatorially (exercise).
These discussions also lead to some properties of the representations of other topological groups closely related to SU(2).

Proposition 8.15. *There is an isomorphism* $\mathrm{SU}(2)/\{\pm I\} \to \mathrm{SO}(3)$ *of topological groups.*

Proof. Example sheet. □

Corollary 8.16. *Every irreducible representation of* SO(3) *has the form* V_{2n} *for some* $n \geq 0$.

Proof. $-I$ acts trivially on V_n iff n is even. □

9 Character Table of $\mathrm{GL}_2(\mathbb{F}_q)$

9.1 Finite Fields

Let $p > 2$ be prime and $q = p^a$ be some power of p. By techniques from Galois theory, there is a unique field $\mathbb{F}_q$ of order q and we have $\mathbb{F}_q^\times \cong C_{q-1}$.
The map $x \mapsto x^2$ is a group homomorphism with kernel $\{\pm 1\}$, hence exactly half of the elements in $\mathbb{F}_q^\times$ are squares. Let $\epsilon \in \mathbb{F}_q^\times$ be a non-square, then the field $\mathbb{F}_{q^2} \cong \{a + b\sqrt{\epsilon} : a, b \in \mathbb{F}_q\}$ with q^2 elements would contain a square root of ϵ. Consequently, every element of $\mathbb{F}_q$ has a square root in $\mathbb{F}_q^2$, i.e. every quadratic polynomial in $\mathbb{F}_q[X]$ factorises in $\mathbb{F}_{q^2}[X]$. Note that $(a + b\sqrt{\epsilon})^q = a^q + b^q(\sqrt{\epsilon})^q = a - b\sqrt{\epsilon}$, so we can write the roots of an irreducible quadratic in $\mathbb{F}_q$ as λ, λ^q for some $\lambda \in \mathbb{F}_{q^2}$.

9.2 Conjugacy Classes in $\mathrm{GL}_2(\mathbb{F}_q)$

The group we are interested in is of course

$$G = \mathrm{GL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q, ad - bc \neq 0 \right\}$$

which has order $(q^2 - 1)(q^2 - q) = q(q + 1)(q - 1)^2$.
Recall that conjugacy classes of 2×2 matrices are determined by their minimal polynomials. By Cayley-Hamilton, each $A \in G$ has a minimal polynomial m_A of degree at most 2 with nonzero constant coefficient.
If $m_A(X) = X - \lambda$ for some $\lambda \in \mathbb{F}_q^\times$, then essentially $A = \lambda I$ and $C_G(A) = G$, so they produce conjugacy classes of size 1.
If $m_A(X) = (X - \lambda)^2$ for some $\lambda \in \mathbb{F}_q^\times$, then A is conjugate to $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. We have

$$C_G\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = ZN = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{F}_q, a \neq 0 \right\}$$

So the size of the conjugacy class of A is $(q+1)(q-1)$ and there are $q - 1$ such classes.
If $m_A = (X - \mu)(X - \lambda)$ for some distinct $\mu, \lambda \in \mathbb{F}_q^\times$, then A is conjugate to $\mathrm{diag}(\lambda, \mu)$ and its centraliser T is the set of diagonal matrices in G. This gives $\binom{q-1}{2}$ conjugacy classes of size $q(q + 1)$.
There is a chance m_A is irreducible in $\mathbb{F}_q$, but then it can be factorised in $\mathbb{F}_{q^2}$ as $m_A(X) = (X - \alpha)(X - \alpha^q)$ where $\alpha = \lambda + \mu\sqrt{\epsilon}$. Then A is conjugate to $\left(\begin{smallmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{smallmatrix}\right)$ which is also conjugate to $\left(\begin{smallmatrix} \lambda & -\epsilon\mu \\ -\mu & \lambda \end{smallmatrix}\right)$. Then of course

$$C_G\left(\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} : a^2 - \epsilon b^2 \neq 0 \right\}$$

Call this group K. We have $|K| = (q - 1)(q + 1)$, so the conjugacy class of A has size $q(q - 1)$ and there are a total of $\binom{q}{2}$ such conjugacy classes.
A torus in G is a subgroup of it which is conjugate to a group of diagonal matrices after passing through a field extension of $\mathbb{F}_q$. A maximal torus is a torus that is not strictly contained in another torus. Under this terminology, both T, K are maximal tori. We call T a split torus (i.e. don't have to be passed through a field extension to be conjugate to the diagonal matrices) and

K a non-split torus.
G has many interesting subgroups: $Z = Z(G)$ which contains all the scalar matrices in G, $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{F}_q\}$ a Sylow p-subgroup of G and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : b \in \mathbb{F}_q, a, d \in \mathbb{F}_q^\times \right\}$$

which is called a Borel subgroup of G. Note that B is soluble as $N \lhd B$ and $B/N \cong (\mathbb{F}_q^\times)^2$. Under the transitive Möbius action of G on $\mathbb{F}_q \cup \{\infty\}$, we can identify $B = \mathrm{Stab}_G(\infty)$.
Write $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} s \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a + b\beta \\ d & \beta d \end{pmatrix}$$

Hence $|BsN| = q|B|$, so $BsN = G \backslash B$. Thus $BsN = BsB$ and $B\backslash G/B = \{B, BsB\}$ (the "Bruhat decomposition").
Clearly we are going to use Mackey theory, specifically Corollary 6.7. Note that ${}^sB \cap B = T$, so a good place to start is to do induction from the characters of B.

9.3 Character Table of the Borel subgroup

If $x, y \in B$ are conjugates in G, then they are either conjugates in B or there are $b_1, b_2 \in B$ such that $(b_1 s b_2) x (b_1 s b_2)^{-1} = y$. That is, either $[x]_B = [y]_B$ or $[sxs^{-1}]_B = [y]_B$. In other words, $[x]_G \cap B$ splits into at most 2 conjugacy classes in B.
Each conjugacy class of size 1 (the scalar matrices) clearly does not split. Each conjugacy class with representative $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ does not split either, except the sizes of each orbit now has $q-1$ elements. Each conjugacy class with representative $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ splits into two parts, giving $(q-1)(q-2)$ conjugacy classes each of size q. Notably, all of these keep their own centralisers.
Let $\Theta_q = \{$representations of $\mathbb{F}_q^\times\} \cong C_{q-1}$ with the group structure given by pointwise multiplication. As $B/N \cong (\mathbb{F}_q^\times)^2$, for each $\theta, \phi \in \Theta_q$ we can produce a degree 1 representation of B with kernel N given by

$$\chi_{\theta,\phi}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \theta(a)\phi(d)$$

This gives $(q-1)^2$ one-dimensional representations of B, so we've got $q-1$ more to find. We get them by induction from ZN. If $\gamma : (\mathbb{F}_q, +) \to \mathbb{C}^\times$ is a degree 1 representation and $\theta \in \Theta_q$, then

$$\rho_{\theta,\gamma} : ZN \to \mathbb{C}^\times, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \theta(a)\gamma(b)$$

is a representation of ZN. Calculation reveals that

$$\mathrm{Ind}_{ZN}^B \rho_{\theta,\gamma}\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \begin{cases} (q-1)\theta(\lambda) & \text{if } \gamma = 1 \\ -\theta(\gamma) & \text{otherwise} \end{cases}$$

and $\mathrm{Ind}_{ZN}^{B}\,\rho_{\theta,\gamma}(\mathrm{diag}(\lambda,\mu)) = 0$. So when $\gamma \neq 1$, we have

$$\langle \mathrm{Ind}_{ZN}^{B}\,\rho_{\theta,\gamma}, \mathrm{Ind}_{ZN}^{B}\,\rho_{\theta,\gamma}\rangle = \frac{(q-1)(q-1)^2 + (q-1)^2 + 0}{q(q-1)^2} = 1$$

$\mu_\theta = \mathrm{Ind}_{ZN}^{B}\,\rho_{\theta,\gamma}$ for $\gamma \neq 1$ then give the rest $q-1$ simple characters of B.

#Classes	$q-1$	$q-1$	$(q-1)(q-2)$
Size of Classes	1	$q-1$	q
Representative	$\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$
$\chi_{\theta,\phi}, \theta, \phi \in \Theta_q$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)^2$	$\theta(\lambda)\phi(\mu)$
$\mu_\theta, \theta \in \Theta_q$	$(q-1)\theta(\lambda)$	$-\theta(\lambda)$	0

9.4 Character Table of $\mathrm{GL}_2(\mathbb{F}_q)$

As $\det : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q^\times$ is surjective, each $\theta \in \Theta_q$ gets lifted to a degree 1 character χ_θ of $G = \mathrm{GL}_2(\mathbb{F}_q)$ by precomposing it with det. Let's first induce the 1-dimensional representations $\chi_{\theta,\phi}$ of B. Note that

$${}^{s}\chi_{\theta,\phi}\left(\begin{pmatrix} a & 0\\ c & d\end{pmatrix}\right) = \chi_{\theta,\phi}\left(\begin{pmatrix} d & c\\ 0 & a\end{pmatrix}\right) = \theta(d)\phi(a)$$

Also $B \cap {}^{s}B = T$, we have

$$\mathrm{Res}_{B\cap {}^{s}B}^{{}^{s}B}\,\chi_{\theta,\phi} = \mathrm{Res}_{B\cap {}^{s}B}^{B}\,\chi_{\theta,\phi} \iff \theta = \phi$$

So by Corollary 6.7, $W_{\theta,\phi} = \mathrm{Ind}_B^G\,\chi_{\theta,\phi}$ is irreducible iff $\theta \neq \phi$. We have

$$\chi_{W_{\theta,\phi}}\left(\begin{pmatrix} \lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = (q+1)\theta(\lambda)\phi(\lambda), \chi_{W_{\theta,\phi}}\left(\begin{pmatrix} \lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = \theta(\lambda)\phi(\lambda)$$

$$\chi_{W_{\theta,\phi}}\left(\begin{pmatrix} \lambda & 0\\ 0 & \mu\end{pmatrix}\right) = \theta(\lambda)\phi(\mu) + \theta(\mu)\phi(\lambda), \chi_{W_{\theta,\phi}}\left(\begin{pmatrix} \lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) = 0$$

It's then clear that $W_{\theta,\phi} \cong W_{\phi,\theta}$. So we get a total of $\binom{q-1}{2}$ irreduicble representations of degree $q-1$ (the "principal series representations").
Let's try and find the other representations by decomposing $W_{\theta,\theta}$. $W_{1,1} = \mathbb{C}(\mathbb{F}_q \cup \{\infty\})$ is secretly the Möbius representation, which decomposes into $W_{1,1} = 1 \oplus V_1$ with V_1 having degree q (the "Steinberg representation"). By looking at $\chi_\theta \otimes W_{1,1}$ we realise $W_{\theta,\theta} = \chi_\theta \oplus V_\theta$ where $\chi_\theta \otimes V_1$. Each V_θ is irreducible of degree q and we get $q-1$ of them.
So far, the incomplete character table looks like

#	$q-1$	$q-1$	$\binom{q-1}{2}$	$\binom{q}{2}$
	$\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{smallmatrix}\right)$
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\lambda^2 - \epsilon\mu^2)$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\lambda^2 - \epsilon\mu^2)$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu) + \phi(\lambda)\theta(\mu)$	0

We need $\binom{q}{2}$ more representations. Turns out, they are parameterised by representations φ of K such that $\varphi^q \neq \varphi$, but this time we won't try to construct the representations explicitly.

As $\operatorname{Ind}_B^G \mathbb{C}B = \mathbb{C}G$, the remaining characters must be summands of $\operatorname{Ind}_B^G \mu_\theta$. We have

$$\operatorname{Ind}_B^G \mu_\theta \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = (q+1)(q-1)\theta(\lambda), \operatorname{Ind}_B^G \mu_\theta \left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \right) = -\theta(\lambda)$$

$$\operatorname{Ind}_B^G \mu_\theta \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right) = \operatorname{Ind}_B^G \mu_\theta \left(\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix} \right) = 0$$

So

$$\langle \operatorname{Ind}_B^G \mu_\theta, \operatorname{Ind}_B^G \mu_\theta \rangle = \frac{1}{|G|}((q-1)(q+1)^2(q-1)^2 + (q-1)(q^2-1)) = q$$

Next, we attempt to induce from representations of K. $\mathbb{F}_{q^2} = \{\lambda + \mu\sqrt{\epsilon} : \lambda, \mu \in \mathbb{F}_q\}$ is in bijective correspondence with $K \cup \{0\}$ via

$$\lambda + \mu\sqrt{\epsilon} \leftrightarrow \begin{pmatrix} \lambda & \mu\epsilon \\ \mu & \lambda \end{pmatrix}$$

This gives an isomorphism of fields, i.e. one can identify K as the multiplicative group of $\mathbb{F}_{q^2}$. We also have $(\lambda+\mu\sqrt{\epsilon})^{q+1} = \det(\lambda+\mu\sqrt{\epsilon})$ by the correspondence above.
Suppose φ is a one-dimensional representation of K. Let $\Phi = \operatorname{Ind}_K^G \varphi$, then $\Phi(\lambda) = q(q-1)\varphi(\lambda)$ for $\lambda \in \mathbb{F}_q$, $\Phi(\alpha) = \varphi(\alpha) + \varphi(\alpha^q)$ for $\lambda \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, and $\Phi = 0$ at other conjugacy classes since they do not meet K. Now

$$\langle \Phi, \Phi \rangle = \frac{1}{|G|} \left((q-1)q^2(q-1)^2 + q(q-1) \sum_{\{\alpha,\alpha^q\} \subset K \setminus Z} |\varphi(\alpha) + \varphi(\alpha^q)|^2 \right)$$

The last sum looks scary, but it really isn't:

$$\begin{aligned} \sum_{\{\alpha,\alpha^q\} \subset K \setminus Z} |\varphi(\alpha) + \varphi(\alpha^q)|^2 &= \sum_{\{\alpha,\alpha^q\} \subset K \setminus Z} (\varphi(\alpha) + \varphi(\alpha^q))(\varphi(\alpha^{-1}) + \varphi(\alpha^{-q})) \\ &= \frac{q^2-1-(q-1)}{2} 2 + \sum_{\alpha \in K} \varphi^{q-1}(\alpha) - \sum_{\alpha \in Z} \varphi^{q-1}(\alpha) \\ &= (q^2-1) - (q-1) + |K| \langle \varphi^{q-1}, 1 \rangle_K \end{aligned}$$

So we have

$$\begin{aligned} \langle \operatorname{Ind}_K^G \varphi, \operatorname{Ind}_K^G \varphi \rangle &= \frac{q(q-1)}{q+1} + \frac{1}{(q-1)(q+1)}(q(q-1) + (q^2-1)\langle \varphi^{q-1}, 1 \rangle_K) \\ &= (q-1) + \langle \varphi^{q-1}, 1 \rangle_K \end{aligned}$$

and

$$\begin{aligned} \langle \Phi, \operatorname{Ind}_B^G \mu_\theta \rangle &= \frac{1}{|G|} \sum_{\lambda \in Z} (q^2-1)\theta(\lambda) q(q-1) \overline{\varphi(\lambda)} \\ &= |Z| \langle \operatorname{Res}_Z^K \varphi, \theta \rangle_Z = \begin{cases} |Z| & \text{if } \operatorname{Res}_Z^K \varphi = \theta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Given φ with $\varphi^{q-1} \neq 1$ (there are $q^2 - q$ such choices), there is some θ with $\operatorname{Res}_Z^K \varphi = \theta$. Then $\beta_\varphi = \operatorname{Ind}_B^G \mu_\theta - \operatorname{Ind}_K^G \varphi \in R(G)$ has $\langle \beta_\varphi, \beta_\varphi \rangle = 1$! This can only mean that one of $\pm\beta_\varphi$ is a character of G. We can take the positive sign since $\beta_\varphi(1) = (q^2 - 1) - q(q-1) = q - 1 > 0$.
The character table now looks like

#	$q-1$	$q-1$	$\binom{q-1}{2}$	$\binom{q}{2}$
	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	$\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$
χ_θ	$\theta(\lambda)^2$	$\theta(\lambda)^2$	$\theta(\lambda)\theta(\mu)$	$\theta(\lambda^2 - \epsilon\mu^2)$
V_θ	$q\theta(\lambda)^2$	0	$\theta(\lambda)\theta(\mu)$	$-\theta(\lambda^2 - \epsilon\mu^2)$
$W_{\theta,\phi}$	$(q+1)\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\lambda)$	$\theta(\lambda)\phi(\mu) + \phi(\lambda)\theta(\mu)$	0
β_φ	$(q-1)\varphi(\lambda)$	$-\varphi(\lambda)$	0	$-\varphi(\alpha) - \varphi(\alpha^q)$

The only way these $q^2 - q$ choices of β_φ would repeat itself is when $\beta_\varphi = \beta_{\varphi^q}$, so they give $\binom{q}{2}$ characters which is what's remaining.

Remark. 1. Here, these β_φ are not constructed explicitly as representations, only as characters. Drinfeld (1983) realised them as the ℓ-adic étale cohomology group of some algebraic curve X over $\mathbb{F}_q$. It is generalised later for all finite groups of Lie type, which is now known as the Deligne-Lusztig theory.
2. The irreducible representations of $\mathrm{PGL}_2(\mathbb{F}_q) = \mathrm{GL}_2(\mathbb{F}_q)/Z$ are exacly the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ where Z acts trivially. $\mathrm{PGL}_2(\mathbb{F}_q)$ has an order 2 subgroup $\mathrm{PSL}_2(\mathbb{F}_q) = \mathrm{SL}_2(\mathbb{F}_q)/Z(\mathrm{SL}_2(\mathbb{F}_q))$ whose character table can be constructed with techniques we've seen. Once we've done that, we'll see that $\mathrm{PSL}_2(\mathbb{F}_q)$ is simple for $q \geq 5$.