Modular Forms *

Zhiyuan Bai

Compiled on November 30, 2022

This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Modular Forms* in Michaelmas 2022. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

Contents

1 Introduction and Motivations **2**

2 Modular Forms of Level $\Gamma(1)$ **3**
2.1 Definition and Examples . . . 3
2.2 The Fundamental Domain . . . 5
2.3 Valence Formula . . . 6
2.4 Dimension and Structure of Spaces of Modular Forms . . . 8

3 Hecke Operators **11**
3.1 Lattices . . . 11
3.2 Hecke Operators . . . 12
3.3 Eigenforms . . . 14
3.4 The Petersson Inner Product . . . 16

4 L-Functions **20**
4.1 L-Functions associated to Modular Forms . . . 20
4.2 Applications of L-Functions . . . 24

5 Modular Forms in General **27**
5.1 Congruence Subgroups and their Cusps . . . 27
5.2 Jacobi's ϑ-Function . . . 29
5.3 ϑ-Functions and ζ-Functions of Lattices . . . 33

6 Non-Holomorphic Eisenstein Series **35**
6.1 Definition and Modularity . . . 35
6.2 Computing Fourier Coefficients . . . 37
6.3 The Prime Number Theorem . . . 40

*Based on the lectures under the same name taught by Prof. J. Thorne in Michaelmas 2022.

1 Introduction and Motivations

We denote the complex upper-half plane by $\mathfrak{h} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. We write $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) : \det g > 0\}$.

Lemma 1.1. $\mathrm{GL}_2(\mathbb{R})^+$ *acts transitively on* $\mathfrak{h}$ *by Möbius transformations.*

Proof. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, we have

$$\operatorname{Im}(g\tau) = \frac{1}{2}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{1}{2}\det(g)\frac{\operatorname{Im}\tau}{|c\tau+d|^2}$$

Hence $\mathrm{GL}_2(\mathbb{R})^+$ sends $\mathfrak{h}$ to itself. Transitivity follows from $x + iy = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) i$. □

Definition 1.1. Let $k \in \mathbb{Z}$. For $g \in \mathrm{GL}_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, we define $j(g,\tau) = c\tau + d$ (the "modular cocycle"). For a function $f : \mathfrak{h} \to \mathbb{C}$, we define $f|_k[g] : \mathfrak{h} \to \mathbb{C}$ via

$$f|_k[g](\tau) = f(g\tau)\det(g)^{k-1}j(g,\tau)^{-k}$$

Lemma 1.2. *This defines a right action of* $\mathrm{GL}_2(\mathbb{R})^+$ *on the set of functions* $f : \mathfrak{h} \to \mathbb{C}$.

Proof. We need to check that for any $g, h \in \mathrm{GL}_2(\mathbb{R})^+$ we have $f|_k[gh] = (f|_k[g])|_k[h]$. This again follows from calculations

$$\begin{aligned}(f|_k[g])|_k[h](\tau) &= (f|_k[g])(h\tau)\det(h)^{k-1}j(h,\tau)^{-k} \\ &= f(gh\tau)\det(g)^{k-1}j(g,h\tau)^{-k}\det(h)^{k-1}j(h,\tau)^{-k}\end{aligned}$$

So we would have our result if $j(gh,\tau) = j(g,h\tau)j(h,\tau)$ (the "cocycle relation"). This is true. A neat way to calculate this is by observing

$$j(h,\tau)j(g,h\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = g\left(j(h,\tau)\begin{pmatrix} h\tau \\ 1 \end{pmatrix}\right) = gh\begin{pmatrix} \tau \\ 1 \end{pmatrix} = j(gh,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix}$$

which gives what we wanted. □

Definition 1.2. For $k \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ a finite index subgroup. A weakly modular function f of level Γ and weight k is a meromorphic function on $\mathfrak{h}$ such that for all $\gamma \in \Gamma$, $f|_k[\gamma] = f$.

The goal of the course is to define and study complex vector spaces $M_k(\Gamma)$ of modular forms, which are nice modular functions. We'll show that these are finite dimensional, and we'll also study interesting family of endomorphisms on them, namely Hecke operators.
Why do we care about modular forms? The first motivation is the theory of elliptic functions. Suppose we have an elliptic curve E over $\mathbb{C}$ and ω a nonzero holomorphic differential on E. Then there is a unique lattice $\Lambda \leq \mathbb{C}$ such that we have an isomorphism $\phi : \mathbb{C}/\Lambda \to E$ (as Riemann surfaces and as groups) satisfying $\phi^*\omega = \mathrm{d}z$. One can show that E may be given by the equation $y^2 = x^3 - 60G_4(\Lambda) - 140G_6(\Lambda)$ where, for $k \geq 4$, we define

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$$

It turns out that $G_k(\mathbb{Z}+\mathbb{Z}\tau)$ are examples of modular forms, known as the holomorphic Eisenstein series.
Another, less classical, motivation is the connection with number theory. Modular forms have interesting "q-expansions": If f is a modular form, then it has a Fourier expansion $f(\tau)=\sum_n a_n q^n$ where $q=e^{2\pi i\tau/h}$ for some $h\in\mathbb{N}$ q-expansions usually have interesting arithmetic meanings, for example the Dedekind θ function $\theta(\tau)=\sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau}$. It happens that for even k, θ^k is always a modular form of weight $k/2$. This is important because essentially we have

$$\theta^k=\sum_m r_k(m)e^{i\pi m\tau}$$

where $r_k(m)=\#\{(n_1,\dots,n_k)\in\mathbb{Z}^k:\sum_i n_i^2=m\}$ is the number of ways to write m as the sum of k squares. So studying modular forms may be able to yield results about r_k's. For example, one of the simple formulas that one can prove by analysing θ^4 is $r_4(m)=8\sum_{d|m,4\nmid d} d$ – in particular, every positive integer is the sum of four squares.
A third motivation is its link to L-functions, like the Riemann ζ function $\zeta(s)=\sum_n n^{-s}$. We know that ζ has a meromorphic continuation to $\mathbb{C}$, has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$, and has an Euler product $\zeta(s)=\prod_p(1-p^{-s})^{-1}$. And we can use all of these together to prove the prime number theorem: $\pi(x)\sim x/\log x$.
These pleasant properties aren't always easy to prove (or even true) for general Dirichlet series $\sum_n a_n n^{-s}$. The ones with these properties are called L-functions. Modular forms can be used to construct L-functions with, provably, all these wonderful properties. So showing a Dirichlet series arises from a modular form could be an easy way to show that they are L-functions.
The last piece of motivation is its connection to the Langlands program. An example of this is the modularity theorem (also known as the Shimura-Taniyama-Weil conjecture) for elliptic curves, which implies Fermat's last theorem. This goes via Hecke operators and L-functions.

2 Modular Forms of Level $\Gamma(1)$

2.1 Definition and Examples

Recall that on an open $U\subset\mathbb{C}$, a meromorphic function on U is a holomorphic function $f:U-A\to\mathbb{C}$ where $A\subset U$ such that for any $a\in A$, there is some $\delta>0$, the punctured open disk $D^*(a,\delta)=D(a,\delta)-\{a\}$ is contained in $U-A$ and there is some k such that $(z-a)^k f(z)$ extends to a meromorphic function on $D(a,\delta)$. Two meromorphic functions are equivalent if they agree on some nonempty open subsets on U. Equivalently, a meromorphic function on U is simply a holomorphic function $U\to\mathbb{P}^1(\mathbb{C})$ of Riemann surfaces.

Lemma 2.1. *Let f be a weakly modular function of weight k and level $\Gamma(1)$, then there is a unique meromorphic function $\tilde{f}$ in $D^*(0,1)$ such that $f(\tau)=\tilde{f}(e^{2\pi i\tau})$.*

Proof. The map $\tau\mapsto e^{2\pi i\tau},\mathfrak{h}\mapsto D^*(0,1)$ is a holomorphic surjection and $\tau,\tau'\in\mathfrak{h}$ map to the same image iff $\tau'-\tau\in\mathbb{Z}$. For any $D(a,\delta)\subset D^*(0,1)$, there is a branch l of the logarithm defined in $D(a,\delta)$. We can then define $\tilde{f}$ in $D(a,\delta)$ via $\tilde{f}(q)=f((2\pi i)^{-1}l(q))$. This definition of $\tilde{f}$ is independent of the choice of the branch, since $f(\tau) = f(\tau+1)$ for all $\tau \in \mathfrak{h}$ (as f is fixed by the action of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$), hence we get a holomorphic $\tilde{f}$ on the whole $D^*(a, \delta)$. Uniqueness is clear. $\square$

Definition 2.1. Suppose we are in the situation of the previous lemma. We say f is meromorphic at infinity if $\tilde{f}$ is extends to a meromorphic function on $D(0, 1)$. We say it is holomorphic (resp. vanishing) at infinity if the extension is holomorphic (resp. vanishing) at 0.

If f is meromorphic at infinity, then $\tilde{f}$ is holomorphic on $D^*(0, \delta)$ for some $\delta > 0$ (equivalently, f is holomorphic on $\{\tau \in \mathfrak{h} : \operatorname{Im}\tau > -(2\pi)^{-1}\log\delta\}$) and has a Laurent expansion

$$\tilde{f}(q) = \sum_{n\in\mathbb{Z}} a_n q^n$$

where $a_n = 0$ for sufficiently negative n. So for τ with $\operatorname{Im}\tau > -(2\pi)^{-1}\log\delta$, we a Fourier expansion ("q-expansion") $f(\tau) = \sum_n a_n q^n, q = e^{2\pi i\tau}$. Therefore f is holomorphic (resp. vanishing) at infinity if $a_n = 0$ for all negative (resp. nonpositive) n.

Definition 2.2. A weakly modular function of weight k and level $\Gamma(1)$ is a modular function if it is meromorphic at infinity. It is a modular form if it is holomorphic in $\mathfrak{h}$ and at infinity. It is a cuspidal modular form (or cusp form) if it's a modular form vanishing at infinity.
We write $M_k(\Gamma(1))$ (resp. $S_k(\Gamma(1))$) to denote the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight k and level $\Gamma(1)$.

Example 2.1. For $k \in \mathbb{Z}$, we consider the series

$$G_k(\tau) = \sum_{\lambda\in\Lambda_\tau-\{0\}} \lambda^{-k} = \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} (m\tau+n)^{-k}$$

where $\Lambda_\tau = \mathbb{Z}\oplus\mathbb{Z}\tau$. Let's assume that this series converges absolutely. For $\gamma \in \Gamma(1)$, we have $\Lambda_{\gamma\tau} = j(\gamma,\tau)^{-1}\Lambda\tau$ by simple calculation, therefore

$$G_k(\gamma\tau) = \sum_{\lambda\in\Lambda_{\gamma\tau}-\{0\}} \lambda^{-k} = \sum_{\lambda\in\Lambda\tau-\{0\}} (j(\gamma,\tau)^{-1}\lambda)^{-k} = j(\gamma,\tau)^k G_k(\tau)$$

So it's not a stretch to guess that G_k is a modular form of weight k and level $\Gamma(1)$.

When does G_k converge absolutely, and is it actually a modular form?

Proposition 2.2. *For $k > 2$, G_k converges both absolutely and locally uniformly on $\mathfrak{h}$, hence defines a $\Gamma(1)$-invariant holomorphic function on $\mathfrak{h}$. Furthermore, it is holomorphic at infinity, i.e. a modular form.*

Proof. For $A \geq 2$, we define $\Omega_A = \{\tau \in \mathfrak{h} : \operatorname{Im}\tau \geq 1/A, |\operatorname{Re}\tau| \leq A\}$. Let $\tau \in \Omega_A$ and $x \in \mathbb{R}$. We have $|\tau + x| \geq 1/A$ and, when $x \geq 2A$, that $|\tau + x| \leq |x|/2$. Hence $|\tau + x| \geq (2A^2)^{-1}\max\{1, |x|\}$. Let $m, n \in \mathbb{Z}, m \neq 0$, then

$$\begin{aligned}|m\tau+n|^{-k} &= |m|^{-k}|\tau+n/m|^{-k} \leq |m|^{-k}(2A^2)^k \max\{1, |n/m|\}^{-k}\\ &= (2A^2)^k \max\{|m|, |n|\}^{-k}\end{aligned}$$

which, by the way, also holds (trivially) when $m = 0$. We can then estimate

$$\begin{aligned}\sum_{(m,n)\in\mathbb{Z}^2-\{0\}} |m\tau+n|^{-k} &\le (2A^2)^k \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} \max\{|m|,|n|\}^{-k} \\ &= (2A^2)^k \sum_{d\ge 1} d^{-k}\#\{(m,n)\in\mathbb{Z}^2 : \max\{|m|,|n|\}=d\} \\ &= (2A^2)^k \sum_{d\ge 1} 8d^{1-k} = 8(2A^2)^k\zeta(k-1) < \infty\end{aligned}$$

since $k > 2$. By Weierstrass M-test, $G_k(\tau)$ converges absolutely and uniformly on Ω_A, hence absolutely and locally uniformly on $\mathfrak{h}$.
To see that G_k is holomorphic at infinity, we only need to show that $\tilde{G}_k(q)$ has a limit as $q \to 0$. The limit equals to the limit of $G_k(\tau)$ as $\operatorname{Im}\tau \to \infty$ for $\tau \in \Omega_2$, say. We can simply calculate this:

$$\begin{aligned}\lim_{\tau\in\Omega_2, \operatorname{Im}\tau\to\infty} G_k(\tau) &= \lim_{\tau\in\Omega_2, \operatorname{Im}\tau\to\infty} \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} (m\tau+n)^{-k} \\ &= \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} \lim_{\tau\in\Omega_2, \operatorname{Im}\tau\to\infty} (m\tau+n)^{-k} \\ &= \sum_{n\in\mathbb{Z}-\{0\}} n^{-k} = \begin{cases} 2\zeta(k) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd}\end{cases}\end{aligned}$$

In any case, this is finite, so we are done. □

In fact, when k is odd, any weakly modular function of weight k and level $\Gamma(1)$ is 0, since it has to be fixed under the action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Definition 2.3. For $k \ge 4$ even, the normalised Eisenstein series is $E_k(\tau) = (2\zeta(k))^{-1}G_k(\tau)$.

We do this so that the constant term of the q-expansion of E_k is 1. We'll later show that the other coefficients of E_k are, in fact, also in $\mathbb{Q}$.
Certainly, sums of modular forms are modular forms of the same weight and level, and it's easy to check that product of elements of $M_k(\Gamma(1))$ and $M_l(\Gamma(1))$ is an element of $M_{k+l}(\Gamma(1))$.
In particular, E_4^3 and E_6^2 are both in $M_{12}(\Gamma(1))$. They both take value 1 at infinity, so $E_4^3 - E_6^2 \in S_{12}(\Gamma(1))$. We'll show that it is nonzero soon. To get there, we need to understand the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$.

2.2 The Fundamental Domain

Let's look at the set $\mathcal{F} = \{\tau \in \mathfrak{h} : \operatorname{Re}\tau \in [-1/2, 1/2], |\tau| \ge 1\}$ and $\mathcal{F}' = \mathcal{F} - (\{\tau : |\tau| = 1, \operatorname{Re}\tau > 0\} \cup \{\tau : \operatorname{Re}\tau = 1/2\})$. We'll also consider the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proposition 2.3. *(i) Any $\tau \in \mathfrak{h}$ is $\Gamma(1)$-conjugate to a unique element of $\mathcal{F}'$.*
(ii) If $\tau \in \mathcal{F}'$, $\mathrm{Stab}_{\Gamma(1)}(\tau) = \{\pm 1\}$ when $\tau \ne i, \rho$ where $\rho = e^{2\pi i/3}$, and $\mathrm{Stab}_{\Gamma(1)}(i) = \langle S\rangle, \mathrm{Stab}_{\Gamma(1)}(\rho) = \langle ST\rangle$.
(iii) $\Gamma(1)$ is generated by S and T.

Proof. Let $G = \Gamma(1)/\{\pm I\}$ and $H = \langle S, T\rangle \leq G$. We first show that any $\tau \in \mathfrak{h}$ is conjugate to something in $\mathcal{F}$. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, recall that we have $\operatorname{Im}(\gamma\tau)/|c\tau + d|^2$.
$\{1, \tau\}$ forms a basis for $\mathbb{C}$ over $\mathbb{R}$. So $\{(c, d) \in \mathbb{Z}^2 - \{0\} : |c\tau + d| < X\}$ is finite for all X. In particular, the set $\{|c\tau + d| : (c, d) \in \mathbb{Z}^2 - \{0\}\}$ has a minimum, and therefore $\{\operatorname{Im}(\gamma\tau) : \gamma \in H\}$ has a maximum. So by conjugating with H, we may assume that $\operatorname{Im}(\gamma\tau) \leq \operatorname{Im}\tau$ for all $\gamma \in H$. This is sufficient to imply $\tau \in \mathcal{F}$: Conjugating by T moves shows that τ is H-conjugate to something with real part in $[-1/2, 1/2)$. If $|\tau| < 1$, then $\operatorname{Im}(S\tau) = |\tau|^{-2}\operatorname{Im}\tau$, contradiction. Moving with S again shows that we can actually get to $\tau \in \mathcal{F}'$.
Suppose we now have $\tau, \tau' \in \mathcal{F}'$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $\tau' \in \gamma\tau$. We claim that $\tau = \tau'$, and $\gamma = \pm I$ except if $\tau = i, \rho$, in which case $\gamma \in \langle \pm S\rangle, \langle \pm ST\rangle$ respectively.
WLOG $\operatorname{Im}(\gamma\tau) > \operatorname{Im}\tau$, so $|c\tau + d| \leq 1$. If $\tau' \in \mathcal{F}'$, we have $\operatorname{Im}\tau' \geq \sqrt{3}/2$ with equality iff $\tau' = \rho$. So $1 \geq |c\tau + d| \geq |c|\sqrt{3}/2$, and therefore $|c| \leq 2/\sqrt{3}$. As $c \in \mathbb{Z}$, we conclude $|c| \leq 1$. WLOG $c \geq 0$, so either $c = 0$ or $c = 1$.
If $c = 0$, then $\gamma = \pm T^b$, so $b = 0, \gamma = \pm 1, \tau' = \tau$. If $c = 1$, then $|\tau + d| \leq 1$, so we must have either $d = 0, |\tau| = 1$ or $d = 1, \tau = \rho$.
In the former case, $\gamma = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\gamma\tau = a - 1/\tau$. Since $\operatorname{Re}\tau, \operatorname{Re}(\gamma\tau)$ are both in $[-1/2, 0]$ and $\operatorname{Re}(\gamma\tau) = a - \operatorname{Re}\tau$, either $a = 0, \operatorname{Re}\tau = \operatorname{Re}(\gamma\tau) = 0$ (in which case $\tau = i, \gamma = S$), or $a = -1, \operatorname{Re}(\gamma\tau) = \operatorname{Re}\tau = -1/2$ (in which case $\tau = \rho, \gamma = (ST)^2$).
In the latter case, we must have $\tau = \gamma\tau = \rho$ since $\operatorname{Im}\gamma\tau = \operatorname{Im}\tau/|\tau + 1|^2 = \operatorname{Im}\rho$. So $\rho = (a\rho + b)/(\rho + 1)$, hence $a\rho + b = \rho^2 + \rho = -1$, hence $a = 0, b = -1$ and $\gamma = ST$.
It remains to show that $\Gamma(1)$ is generated by S, T. We have $S^2 = -I$, so it suffices to show that $G = H$. Take $\tau = 2i$ (or anything in $\mathcal{F}'$ that's not i, ρ) and $\gamma \in G$, then $\gamma\tau \in \mathfrak{h}$, so by what we've shown earlier, there is some $\delta \in H$ such that $\delta\gamma\tau \in \mathcal{F}'$, therefore $\delta\gamma\tau = \tau$, hence $\delta\gamma \in \operatorname{Stab}_G(\tau) = \{I\}$, which means $\gamma = \delta^{-1} \in H$. ☐

Let f be a nonzero modular function of level $\Gamma(1)$ and weight k. For $\tau \in \mathfrak{h}$, we write $v_\tau(f)$ for the order of f as a meromorphic function. If $\gamma \in \Gamma(1)$, then $f(\gamma\tau) = f(\tau)j(\gamma, \tau)^k$, so $f_{\gamma\tau}(f) = f_\tau(f)$. We also define $v_\infty(f)$ to be the order of $\tilde{f}$ at 0.
If $\tau \in \mathfrak{h}$, we define

$$e_\tau = |\operatorname{Stab}_{\Gamma(1)/\{\pm 1\}}(\tau)| = \begin{cases} 1 & \text{if } \tau \neq i, \rho \\ 2 & \text{if } \tau = i \\ 3 & \text{if } \tau = \rho \end{cases}$$

2.3 Valence Formula

Proposition 2.4 (Valence formula)**.** *Let f be a nonzero modular function, then*

$$v_\infty(f) + \sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) = \frac{k}{12}$$

Proof. The sum is in fact finite, since f can only have finitely many zeros or poles in $\mathcal{F}$. Indeed, since $\tilde{f}$ is meromorphic, it is holomorphic and nonvanishing in $D^*(0,\delta)$ for some $\delta > 0$, which means that f is holomorphic and nonvanishing in $\{\tau \in \mathfrak{h} : \operatorname{Im}\tau \geq R\}$ for some $R > 0$. And there are only finitely many zeros and poles in the compact set $\mathcal{F} \cap \{\tau \in \mathfrak{h} : \operatorname{Im}\tau \leq R\}$ by the principle of isolated zeros.

We'll prove the formula by contour integration. Recall that the argument principle says that if $U \subset \mathbb{C}$ is open, g is meromorphic in U and γ is a simple closed positively-oriented contour missing every zeros and poles of g and bound a region D contained in U, then

$$\frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z = \sum_{a\in D} v_a(g)$$

We'll also use the pullback formula, whcih says that if $u : U \to V$ is a holomorphic function between open sets in $\mathbb{C}$, $f : V \to \mathbb{C}$ is holomorphic and γ a path in U, then

$$\int_{u\circ\gamma} f(z)\,\mathrm{d}z = \int_\gamma u^*(f(z)\,\mathrm{d}z) = \int_\gamma f(u(z))u'(z)\,\mathrm{d}z$$

Take $R > 2$ such that f has no zeros or poles in $\{\operatorname{Im}\tau \geq R\}$. We consider the contour γ that is the anticlockwise (i.e. positively oriented) boundary of $\mathcal{F} \cap \{\tau \in \mathfrak{h} : \operatorname{Im}\tau \leq R\}$. Starting with the top left, we name the vertices (anticlockwisely) $A, B = \rho, D = -\bar{\rho} = \rho + 1, E$ respectively. We also name $C = i$.

Suppose f has no zeros or poles on γ, then we have, by argument principle,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f)$$

On the other hand, we can calculate the contour integral one part at a time. If the integrand is omitted below, it's understood to be $f(z)^{-1}f'(z)\,\mathrm{d}z$. Note that if $u(\tau) = \tau + 1$, then $u(AB) = ED$ and $f \circ u = f$, therefore

$$\int_{ED} = \int_{u(AB)} = \int_{AB}, \text{ hence } \int_{AB\cup DE} = 0$$

Now take $u(\tau) = -1/\tau$, then $CD = u(CB)$ and $(f \circ u)(\tau) = f(\tau)\tau^k$. Therefore by pullback formula we have

$$\begin{aligned}\int_{CD} = \int_{u(CB)} &= \left(\int_{CB}\right) + k(\log B - \log C)\\ &= \left(\int_{CB}\right) + k(\log B - \log C) = \left(\int_{CB}\right) + 2\pi i\frac{k}{12}\end{aligned}$$

Now let $u(\tau) = e^{2\pi i\tau}$, then $u(AE)$ is a positively oriented circle in $D(0,1)$. So using the pullback formula again gives

$$v_\infty(f) = v_0(\tilde{f}) = \frac{1}{2\pi i}\int_{u(AE)} \frac{\tilde{f}'(z)}{\tilde{f}(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{AE}$$

Therefore we have

$$\sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau}v_\tau(f) = \int_\gamma = \frac{k}{12} - v_\infty(f)$$

which is the formula.
What if something on the contour is a zero or a pole? For zero or pole P on the interior of AB, we in fact get a pair of zero/pole $P, P+1$. Choose $\epsilon > 0$ small enough so that f has no zeros or poles in $D^*(P, 2\epsilon)$ and consider the same contour except half-circles of radius ϵ at $P, P+1$ are perturbed to the same direction. Then the contour integrals via AB and ED still cancel each other out, so nothing's wrong there. Similarly, we can deal with zeros or poles in the interior of BC and CD.
We need to do some work for possible zeros or poles at i, ρ. At ρ, this is done (for small $\epsilon > 0$) by perturbing the contour with γ_ϵ where the bit around $B = \rho$ is replaced with $t \mapsto \rho + \epsilon e^{it}$ where t ranges clockwise from $\pi/2$ to wherever it hits γ. And we also perturb it with the mirrored inward-bending contour around $D = -\bar{\rho}$, and by considering (exercise!) the integral with $\epsilon \to 0$ gives exactly the $v_\rho(f)/e_\rho$ part in the formula. A similar argument works for zeros and poles at $C = i$. □

Example 2.2. For $k = 4$, we have $v_\infty(E_4) + \sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} e_\tau^{-1} v_\tau(E_4) = 4/12 = 1/3$, hence $v_\infty(E_4) = v_\tau(E_4) = 0$ for $\tau \in \mathcal{F}', \tau \neq \rho$ and $v_\rho(E_4) = 1$.
For $k = 6$, we have $v_\infty(E_6) + \sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} e_\tau^{-1} v_\tau(E_6) = 6/12 = 1/2$, hence $v_\infty(E_6) = v_\tau(E_6) = 0$ for $\tau \in \mathcal{F}', \tau \neq i$ and $v_i(E_6) = 1$.

Example 2.3. Consider $\Delta = (E_4^3 - E_6^2)/1728 \in S_{12}(\Gamma(1))$. This is nonzero since it must be nonzero at i. The formula $v_\infty(\Delta) + \sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} e_\tau^{-1} v_\tau(\Delta) = 1$ then implies that Δ is nonvanishing on $\mathfrak{h}$ and has a simple zero at ∞.

2.4 Dimension and Structure of Spaces of Modular Forms

Theorem 2.5. *Let k be an even integer.*
(i) $M_k(\Gamma(1)) = 0$ if $k < 0$ or $k = 2$, and $M_0(\Gamma(1)) = \mathbb{C}$.
(ii) If $4 \leq k \leq 10$, then $M_k(\Gamma(1)) = \mathbb{C}E_k$.
(iii) Multiplication by Δ is an isomorphism $M_k(\Gamma(1)) \to S_{k+12}(\Gamma(1))$.

Proof. (i) This follows directly from Proposition 2.4.
(ii) By Proposition 2.4, for these values of k any nonzero $f \in M_k(\Gamma(1))$ has $v_\infty(f) = 0$. So $S_k(\Gamma(1)) = 0$ (this also follows from (i) and (iii), by the way). The fact that $M_k(\Gamma(1)) = \mathbb{C}E_k \oplus S_k(\Gamma(1))$ then gives the result.
(iii) For surjectivity, for any $f \in S_{k+12}(\Gamma(1))$, f/Δ is holomorphic in $\mathfrak{h}$ since Δ is nonvanishing in $\mathfrak{h}$, and at infinity since Δ has a simple zero there. Injectivity is clear. □

Corollary 2.6. *Let $k \geq 0$ be an even integer, then*

$$\dim M_k(\Gamma(1)) = \dim S_k(\Gamma(1)) + 1 = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{otherwise} \end{cases}$$

Proof. Follows from the preceding lemma using $M_k(\Gamma(1)) = \mathbb{C}E_k \oplus S_k(\Gamma(1))$. □

In particular, these spaces of modular forms are finite-dimensional.

Corollary 2.7. *Let $\mathcal{M}$ be the graded ring $\bigoplus_k M_k(\Gamma(1))$.. Then $\mathcal{M}$ is generated by E_4, E_6 as a $\mathbb{C}$-algebra.*

Proof. We want to show that $M_k(\Gamma(1))$ is spanned by $E_4^a E_6^b$ with $a, b \in \mathbb{Z}_{\geq 0}, 4a+6b = k$. This is certainly true for $k < 12$ by what we have already.
For $k \geq 12$ even, we can always find some $a, b \geq 0$ such that $4a + 6b = k$, so the assertion follows from Theorem 2.5(iii) and the fact that $E_4^a E_6^b$ cannot vanish at infinity (thus $M_k(\Gamma(1)) = \mathbb{C}E_4^a E_6^b \oplus S_k(\Gamma(1))$). □

Define now $j(\tau) = E_4^3/\Delta$, which is a modular function of weight 0 and level $\Gamma(1)$. This is holomorphic on $\mathfrak{h}$ with a simple pole at infinity.

Theorem 2.8. *The map $j : \mathfrak{h} \to \mathbb{C}$ is surjective and τ, τ' have the same image iff they are $\Gamma(1)$-conjugate.*
Moreover, any modular functionof weight 0 and level $\Gamma(1)$ is a rational function in j.

Proof. Let $z \in \mathbb{C}$. We want to show that there is a unique $\Gamma(1)$-orbit in $\mathfrak{h}$ such that $j(\tau) = z$, i.e. such that $v_\tau(j - z) > 0$. But $j - z$ is a modular function of weight 0 and level $\Gamma(1)$. So what we want follows simply from using Proposition 2.4 on $j - z$ and observing that it is holomorphic except at infinity, where it has a simple pole.
Now suppose f is a nonzero modular function of weight 0. Multiplying by a polynomial in j, we can assume WLOG that f is holomorphic in $\mathfrak{h}$. Multiplying by Δ^N then makes it an element of $M_{12N}(\Gamma(1))$, which is generated by $E_4^a E_6^b, 4a + 6b = 12N$. So it suffices to show that for every $a, b, 4a + 6b = 12N$, $E_4^a E_6^b/\Delta^N$ is a rational function in j. But we must have $3 \mid a$ and $2 \mid b$ by solving $4a + 6b \equiv 0 \pmod{12}$, so the result follows from the fact that $j - 1728 = E_6^2/\Delta$. □

Remark. One can show that $j(\tau)$ is the j-invariant of the elliptic curve $\mathbb{C}/\Lambda_\tau$.

Proposition 2.9. *Let $k \geq 4$ be even, then*

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 0} \sigma_{k-1}(n) q^n, \sigma_a(n) = \sum_{d|n} d^a$$

Proof. We start with the formula

$$-i\pi \left(1 + 2 \sum_{n \geq 1} q^n \right) = i\pi \frac{q+1}{q-1} = \pi \cot(\pi\tau) = \frac{1}{\tau} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right)$$

with the series converging absolutely and locally uniformly in $\tau \in \mathfrak{h}$. By local uniform convergence, we are allowed term-by-term differentiation. Doing this $k - 1$ times yields

$$-2 \sum_{n=1}^{\infty} (2\pi i n)^{k-1} q^n = (-1)^{k-1}(k-1)! \tau^{-k} \sum_{n \in \mathbb{Z}}^{\infty} \frac{1}{(\tau+n)^k}$$

noting $k > 1$. Therefore

$$\begin{aligned} G_k(\tau) &= \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} \frac{1}{(m\tau+n)^k} = \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n^k} + \sum_{m\in\mathbb{Z}-\{0\},n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} \\ &= 2\sum_{n\geq 1}^{\infty} \frac{1}{n^k} + 2\sum_{m\geq 1,n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{m\geq 1}\sum_{n\geq 1} n^{k-1}q^{nm} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 0}\sigma_{k-1}(n)q^n \end{aligned}$$

as desired. □

Corollary 2.10. *$E_k(\tau) = 1 + \sum_{n\geq 1} a_n q^n$ with $a_n \in \mathbb{Q}$ for all $n \geq 1$. Moreover, if $k = 4, 6$, then we in fact have $a_n \in \mathbb{Z}$.*

Proof. Famously (or indeed, from example sheet), $\pi^k/\zeta(k) \in \mathbb{Q}$ for any even $k \geq 4$. Therefore $a_n \in \mathbb{Q}$.
As for $k = 4, 6$: We have $\zeta(4) = \pi^4/90$, so $(2\pi i)^4/(\zeta(4)(4-1)!) = 240 \in \mathbb{Z}$; $\zeta(6) = \pi^6/945$, so $(2\pi i)^6/(\zeta(6)(6-1)!) = -504 \in \mathbb{Z}$. □

From the proof, we then know that

$$E_4(\tau) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n, E_6(\tau) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$$

Corollary 2.11. *$\Delta = q + \sum_{n\geq 2} a_n q^n, j = q^{-1} + \sum_{n\geq 0} b_n q^n$ with $a_n, b_n \in \mathbb{Z}$.*

Proof. Since $j = E_4^3/\Delta$, it suffcies to show the result for Δ. Write $E_4 = 1 + 240U(q)$ and $E_6 = 1 - 504V(q)$, then

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = \frac{3(240)U + 3(240)^2U^2 + (240)^3U^3 + 2(504)V - (504)^2V^2}{1728}$$

To show Δ has integer coefficients, it suffices to prove that the coefficients of $(2(240)U + 2(504)V)/1728 = (5U + 7V)/12$ in q are integers. That is, $5\sigma_3(n) + 7\sigma_5(n) \equiv 0 \pmod{12}$ for any $n \geq 1$. Equivalently, $\sigma_3(n) \equiv \sigma_5(n) \pmod{12}$. But $12 \mid d^5 - d^3$ for all $d \in \mathbb{Z}$, so this must be true.
And the leading coefficient of Δ is $(3(240) + 2(504))/1728 = 1$. □

For a modular form f of level $\Gamma(1)$, we write $a_n(f)$ to denote the n^{th} coefficient of the q-expansion of f, i.e. $f = \sum_n a_n(f)q^n$.

Theorem 2.12. *Let $k \geq 4$ be even. Then $M_k(\Gamma(1))$ possesses a unique basis $f_0, \ldots, f_N$ satisfying:*
(i) For all $0 \leq i, j \leq N$, we have $a_i(f_j) = \delta_{ij}$.
(ii) For all $0 \leq i \leq N, n \in \mathbb{Z}_{\geq 0}$, we have $a_n(f_i) \in \mathbb{Z}$.

Proof. Let $N = \dim S_k(\Gamma(1))$. Write $k = 12a + d$ where $a, d \in \mathbb{Z}_{\geq 0}$ and $d \in \{0, 4, 6, 8, 10, 14\}$. Then $N + 1 = \dim M_k(\Gamma(1)) = a + 1$. Write $k = 4A + 6B$ for some $A, B \in \mathbb{Z}_{\geq 0}$. Define, for each $i = 0, \ldots, N$, $g_i = E_4^A E_6^B \Delta^i E_6^{2(N-i)}$ whicih is an element of $M_k(\Gamma(1))$. Then $\forall n \geq 0$, $a_n(g_i) \in \mathbb{Z}$. The leading term of g_i of q^i, so performing row reductions gives $f_0, \ldots, f_N$ satisfying the conditions.
They form a basis, for the linear forms $a_0, \ldots, a_N \in M_k(\Gamma(1))^*$ are linear independent due to the existence of $f_0, \ldots, f_N$. And indeed $f_0, \ldots, f_N$ is the dual basis of $a_0, \ldots, a_N$, which also gives their uniqueness. □

3 Hecke Operators

3.1 Lattices

The vector spaces of modular forms and cusp forms have additional symmetries. They have very interesting endomorphisms (the Hecke operators) which have important arithmetic meaning. They can be constructed group theoretically by using the action of $\mathrm{GL}_2(\mathbb{Q})^+$, or geometrically. We're going to pursue the geometric approach (thank god).
Recall that in a finite-dimensional real vector space V, a lattice $\Lambda \leq V$ is any discrete cocompact (i.e. with compact quotient) subgroup.

Lemma 3.1. *Let $\Lambda \leq V$ be a subgroup, then Λ is a lattice iff there is a basis $e_1, \ldots, e_n$ for V such that $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$.*

So every lattice arises as a change of basis of $\mathbb{Z}^n \leq \mathbb{R}^n$.

Proof. Example sheet. □

Let $\mathcal{L}$ be the collection of lattices in $\mathbb{C}$. $\mathbb{C}^\times$ acts on $\mathcal{L}$ by $z\Lambda = \{z\lambda : \lambda \in \Lambda\}$. How do lattices relate to the modular action on $\mathfrak{h}$?

Proposition 3.2. *The map $\mathfrak{h} \to \mathcal{L}, \tau \mapsto \Lambda_\tau$ descends to a bijection $\Gamma(1)\backslash\mathfrak{h} \to \mathbb{C}^\times\backslash\mathcal{L}$.*

Proof. We first show that $\mathfrak{h} \to \mathbb{C}^\times\backslash\mathcal{L}$ is surjective. Suppose $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, then $\mathrm{Im}(e_1/e_2) \neq 0$. After possibly reversing e_1 and e_2, let's assume that $\mathrm{Im}(e_1/e_2) > 0$. Then $\Lambda = e_2\Lambda_{e_1/e_2}$, hence the surjectivity.
Suppose $\tau \in \mathfrak{h}, \gamma \in \Gamma(1)$ then $\Lambda_{\gamma\tau} = j(\gamma, \tau)^{-1}\Lambda_\tau$, so we do have a well-defined (surjective) map $\Gamma(1)\backslash\mathfrak{h} \to \mathbb{C}^\times\backslash\mathcal{L}$.
Finally, we show that this map is injective. Suppose $\Lambda_\tau = z\Lambda_{\tau'}$ for some $z \in \mathbb{C}^\times$, then $z\tau' = a\tau + b, z = c\tau + d$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$. But then we simply have $\tau' = (a\tau + b)/(c\tau + d)$, which also means that $\det\gamma = \operatorname{sgn}\mathrm{Im}\,\tau'/\operatorname{sgn}\mathrm{Im}\,\tau = 1$, therefore $\gamma \in \Gamma(1)$. □

This shows that functions $f : \langle \to \mathbb{C}$ invariant under the weight 0 action of $\Gamma(1)$ are the same as functions $F : \mathcal{L} \to \mathbb{C}$ invariant under multiplication by $\mathbb{C}^\times$, and they are related by $f(\tau) = F(\Lambda_\tau)$. We say a function $F : \mathcal{L} \to \mathbb{C}$ is of weight $k \in \mathbb{Z}$ if for any $\Lambda \in \mathcal{L}, z \in \mathbb{C}^\times$ we have $F(z\Lambda) = z^{-k}F(\Lambda)$.

Proposition 3.3. *The map $F \mapsto (f(\tau) = F(\Lambda_\tau))$ defined a bijection between the space V_k of functions $F : \mathcal{L} \to \mathbb{C}$ of weight k and the set W_k of functions $f : \mathfrak{h} \to \mathbb{C}$ with $f|_k[\gamma] = f$.*

Proof. We'll first show that if F is of weight k, then $f(\tau) = F(\Lambda_\tau)$ invariant under the weight k-action of $\Gamma(1)$. Indeed, $f|_k[\gamma](\tau) = f(\gamma\tau)j(\gamma,\tau)^{-k} = F(\Lambda_{\gamma\tau})j(\gamma,\tau)^{-k} = F(j(\gamma,\tau)\Lambda_{\gamma\tau}) = F(\Lambda_\tau) = f(\tau)$. So the map is well-defined. We'll show that it's bijective by exhibiting an inverse. For $f : \mathfrak{h} \to \mathbb{C}$ invariant under the weight k action of $\Gamma(1)$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \in \mathcal{L}$ with $\mathrm{Im}(e_1/e_2) >$, we define $F(\Lambda) = e_2^{-k}f(e_1/e_2)$. This is independent of the choice of basis, for if $\Lambda = \mathbb{Z}e_1' \oplus \mathbb{Z}e_2'$ and $\mathrm{Im}(e_1'/e_2') > 0$, then there is some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $e_1' = ae_1 + be_2, e_2' = ce_1 + de_2$, then

$$\begin{aligned}(e_2')^{-k} f(e_1'/e_2') &= (ce_1 + de_2)^{-k} f\left(\frac{ae_1 + be_2}{ce_1 + de_2}\right) \\ &= e_2^{-k} j(\gamma, e_1/e_2)^{-k} f(\gamma(e_1/e_2)) = e_2^{-k} f(e_1/e_2)\end{aligned}$$

We also have $F(z\Lambda) = (ze_2)^{-1} f(e_1/e_2) = z^{-k} F(\Lambda)$, so indeed the proposed inverse too is well-defined. It's easy to check that these two identifications are inverse to each other. □

Consequently $M_k(\Gamma(1))$ embeds into V_k, and we'll define Hecke operators on V_k using the geometry of lattices, and show that $M_k(\Gamma(1))$ is invariant under it.

3.2 Hecke Operators

Definition 3.1. For each $n \in \mathbb{N}$, then n-th Hecke operator $T_n : V_k \to V_k$ via

$$(T_n F)(\Lambda) = n^{k-1} \sum_{\Lambda' \leq \Lambda, [\Lambda : \Lambda'] = n} F(\Lambda')$$

We define $T_n : W_k \to W_k$ by the identification $V_k \cong W_k$.

Note that there are only finitely many $\Lambda' \leq \Lambda$ such that $[\Lambda : \Lambda'] = n$. Indeed, if this were the case then $n(\Lambda/\Lambda') = 0$, so $n\Lambda \subset \Lambda'$. Recall that the set of subgroups of $\Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^2$ is in bijection with the set subgroups of Λ containing $n\Lambda$, hence the latter must be finite and hence the set of subgroups $\Lambda' \leq \Lambda$ with index n is also finite.
Let's also check that $T_n F$ is indeed in V_k. We have

$$\begin{aligned}(T_n F)(z\Lambda) &= n^{k-1} \sum_{\Lambda' \leq z\Lambda, [z\Lambda : \Lambda'] = n} F(\Lambda') = n^{k-1} \sum_{\Lambda' \leq \Lambda, [\Lambda : \Lambda'] = n} F(z\Lambda') \\ &= n^{k-1} \sum_{\Lambda' \leq \Lambda, [\Lambda : \Lambda'] = n} z^{-k} F(\Lambda') = z^{-k} (T_n F)(\Lambda)\end{aligned}$$

Proposition 3.4. *Fix a weight k and consider the Hecke operators (T_n) on V_k.*
(i) If $n, m \in \mathbb{N}$ are coprime, then $T_n T_m = T_{nm} = T_m T_n$.
(ii) If p is prime, then $T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$.

Proof. Let $n, m \in \mathbb{N}$, not necessarily coprime. then

$$\begin{aligned}(T_n T_m F)(\Lambda) &= n^{k-1} \sum_{\Lambda' \leq \Lambda, [\Lambda : \Lambda'] = n} (T_m F)(\Lambda') \\ &= n^{k-1} m^{k-1} \sum_{\Lambda' \leq \Lambda, [\Lambda : \Lambda'] = n} \left(\sum_{\Lambda'' \leq \Lambda', [\Lambda' : \Lambda''] = m} F(\Lambda'') \right) \\ &= (nm)^{k-1} \sum_{\Lambda'' \leq \Lambda, [\Lambda : \Lambda''] = mn} a(\Lambda, \Lambda'') F(\Lambda'')\end{aligned}$$

where

$$\begin{aligned}a(\Lambda, \Lambda'') &= \#\{\Lambda' \leq \Lambda : [\Lambda : \Lambda'] = n, [\Lambda' : \Lambda''] = m\} \\ &= \#\{A \leq \Lambda/\Lambda'' : [\Lambda/\Lambda'' : A] = n\}\end{aligned}$$

Let's calculate this in a slightly more general context. If B is an abelian group and $l \in \mathbb{Z}$, we write $B[l] = \{b \in B : lb = 0\}$. If $n, m \in \mathbb{N}$ are coprime and $|B| = mn$, then $B = B[n] \times B[m]$ and $B[n]$ is the unique subgroup of B of index n. Hence if n, m are coprime, then $a(\Lambda, \Lambda'') = 1$ for any $\Lambda'' \leq \Lambda, [\Lambda : \Lambda''] = mn$. This immediately implies (i).
Now let p be a prime and $n \geq 1$, then for $\Lambda'' \leq \Lambda, [\Lambda : \Lambda''] = p^{n+1}$, we have $a(\Lambda, \Lambda'') = \#\{A \leq \Lambda/\Lambda'' : \#A = p\}$. Choose a $\mathbb{Z}$-basis e_1, e_2 for Λ and $a \geq b \geq 0$ such that $a + b = n + 1$ such that $p^a e_1, p^b e_2$ is a basis for Λ'' (we can do this by the theory of Smith normal form).
If $b = 0$, then $\Lambda'' = \mathbb{Z}p^{n+1}e_1 \oplus \mathbb{Z}e_2$, so Λ/Λ'' is cyclic of order p^{n+1}. Therefore $a(\Lambda, \Lambda'') = 1$.
If $b \geq 1$, then $\Lambda/\Lambda'' \cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$. But subgroups of $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ of order p corresponds to subgroups of $(\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z})[p] \cong (\mathbb{Z}/p\mathbb{Z})^2$ of order p. So $a(\Lambda, \Lambda'') = p + 1$.
Observe that in the case $b = 0$ we have $\Lambda'' \nleq p\Lambda$ but in the case $b \geq 0$ we always have $\Lambda'' \leq p\Lambda$. So (sorry for the bad typesetting)

$$\begin{aligned}
&(T_{p^n}T_pF)(\Lambda) \\
&= p^{(n+1)(k-1)} \sum_{\Lambda'' \leq \Lambda, [\Lambda:\Lambda''] = p^{n+1}} a(\Lambda, \Lambda'')F(\Lambda'') \\
&= p^{(n+1)(k-1)} \left(\sum_{\Lambda'' \leq \Lambda, [\Lambda:\Lambda''] = p^{n+1}} F(\Lambda'') + \sum_{\Lambda'' \leq p\Lambda, [p\Lambda:\Lambda''] = p^{n-1}} pF(\Lambda'') \right) \\
&= T_{p^{n+1}}F(\Lambda) + p^{(n+1)(k-1)} \cdot p \sum_{\Lambda'' \leq \Lambda, [\Lambda:\Lambda''] = p^{n-1}} F(p\Lambda'') \\
&= T_{p^{n+1}}F(\Lambda) + p^{(n+1)(k-1)} \cdot p \cdot p^{-k} \sum_{\Lambda'' \leq \Lambda, [\Lambda:\Lambda''] = p^{n-1}} F(\Lambda'') \\
&= (T_{p^{n+1}}F)(\Lambda) + p^{k-1}(T_{p^{n-1}}F)(\Lambda)
\end{aligned}$$

as desired (phew?). □

Corollary 3.5. *For any $n, m \in \mathbb{N}$, we have $T_nT_m = T_mT_n$.*

Proof. By (ii) in the preceding proposition, T_{p^n} is a polynomial in T_p with integer coefficients for all prime p and integer n. So the statement holds for prime powers. Hence it holds for arbitrary n, m due to part (i) of the proposition. □

Proposition 3.6. *Let $n \in \mathbb{N}$ and a $\mathbb{Z}$-basis e_1, e_2 for $\Lambda \in \mathcal{L}$. Then any $\Lambda' \leq \Lambda$ with index n are precisely those of the form $\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$, where $a, d \in \mathbb{N}, ad = n, b \in \mathbb{Z}, 0 \leq b < d$.*

Proof. Recall that if $M \in M_{m\times m}(\mathbb{Z})$ has $\det M \neq 0$ and N is a finite free $\mathbb{Z}$-module with basis $\omega_1, \ldots, \omega_m$, then $\bigoplus_i \mathbb{Z} \sum_j M_{ij}\omega_j \leq M$ has index $|\det M|$. This is done by looking at the Smith normal form of M. We then immediately see that any lattice of the said form has index n.
Now for any $\Lambda' \leq \Lambda$ with index n, we can consider the short exact sequence

$$0 \rightarrowtail \mathbb{Z}e_2/(\Lambda \cap \mathbb{Z}e_2) \cong (\Lambda' + \mathbb{Z}e_2)/\Lambda' \rightarrowtail \Lambda/\Lambda' \rightarrowtail \Lambda/(\Lambda' + \mathbb{Z}e_2) \rightarrowtail 0$$

Let $a = |\Lambda/(\Lambda' + \mathbb{Z}e_2)|, d = |\mathbb{Z}e_2/(\Lambda \cap \mathbb{Z}e_2)|$, then $n = ad$. We have $d = \inf\{d \geq 1 : de_2 \in \Lambda'\}$ and $a = \inf\{a \geq 1 : \exists b \in \mathbb{Z}, ae_1 + be_2 \in \Lambda'\}$. If $b, b' \in \mathbb{Z}$ are such that $ae_1 + be_2, ae_1 + b'e_2 \in \Lambda'$, then $(b - b')e_2 \in \Lambda'$ and therefore $b_1 \equiv b_2 \pmod{d}$. So we see that there is some unique $0 \leq b < d$ such that $ae_1 + be_2 \in \Lambda'$. Then $\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2 \leq \Lambda'$ and both and them has index n in Λ, therefore $\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2 = \Lambda'$. □

Proposition 3.7. *Let $f \in W_k$, then*

$$(T_n f)(\tau) = n^{k-1} \sum_{a,b,d \in \mathbb{Z}_{>0}, ad=n, 0 \leq b < d} d^{-k} f\left(\frac{a\tau + b}{d}\right)$$
$$= \sum_{a,b,d \in \mathbb{Z}_{>0}, ad=n, 0 \leq b < d} f|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]$$

Proof. $F(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}d) = d^{-k} F(\mathbb{Z}((a\tau + b)/d) \oplus \mathbb{Z}) = d^{-k} f((a\tau + b)/d)$. □

Corollary 3.8. *If $f \in W_k$ is holomorphic on $\mathfrak{h}$, then $T_n f$ is also holomorphic on $\mathfrak{h}$.*

Corollary 3.9. *If $f \in M_k(\Gamma(1))$ has q-expansion $\sum_{m \geq 0} a_m q^m$, then $T_n f \in M_k(\Gamma(1))$ and it has q-expansion $\sum_{m \geq 0} c_m q^m$ where*

$$c_m = \sum_{l \in \mathbb{N}, l | \gcd(m,n)} l^{k-1} a_{mn/l^2}$$

Proof.

$$\begin{aligned}
T_n f(\tau) &= n^{k-1} \sum_{a,b,d \in \mathbb{Z}_{>0}, ad=n, 0 \leq b < d} d^{-k} f\left(\frac{a\tau + b}{d}\right) \\
&= n^{k-1} \sum_{a,b,d \in \mathbb{Z}_{>0}, ad=n, 0 \leq b < d} d^{-k} \sum_{m \geq 0} a_m e^{2\pi i m (a\tau + b)/d} \\
&= n^{k-1} \sum_{a,d \in \mathbb{N}, ad=n} d^{-k} \sum_{m \geq 0} a_m e^{2\pi i m a\tau/d} \sum_{0 \leq b < d} e^{2\pi i m b/d} \\
&= n^{k-1} \sum_{a,d \in \mathbb{N}, ad=n} d^{-k} \sum_{m \geq 0} a_m e^{2\pi i m a\tau/d} (d 1_{d|m}) \\
&= \sum_{a,d \in \mathbb{N}, ad=n} (n/d)^{k-1} \sum_{m \geq 0} a_{dm} q^{am} = \sum_{l \in \mathbb{N}, l|n} l^{k-1} \sum_{m \geq 0} a_{mn/l} q^{lm}
\end{aligned}$$

By the uniqueness of Laurent expansions, we conclude $T_n f \in M_k(\Gamma(1))$. Collecting the relevant terms gives the formula for c_m. □

Corollary 3.10. *(i) $a_1(T_n f) = a_n(f)$.*
(ii) $a_0(T_n f) = a_0(f)\sigma_{k-1}(n)$. In particular, $S_k(\Gamma(1))$ is also invariant under T_n.

3.3 Eigenforms

Next, we'll try to understand the spectral decomposition of $M_k(\Gamma(1))$ under T_n's. That is, we want to decompose $M_k(\Gamma(1))$ into simultaneous eigenspaces of T_n's.

Example 3.1. $M_4(\Gamma(1))$ is one-dimensional, so E_4 is an eigenvector of everything.
$S_{12}(\Gamma(1))$ too is one-dimensional, so $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is an eigenvector of everything. Let's actually calculate the eigenvalue. Suppose $T_n\Delta = \alpha_n\Delta$, then $\tau(n) = a_n(\Delta) = a_1(T_n\Delta) = \alpha_n$. So $T_n\Delta = \tau(n)\Delta$. Ramanujan had conjectures in 1916 that if p is prime then $\tau(p^n)\tau(p) = \tau(p^{n+1}) + p^n\tau(p^{n-1})$ and $\tau(mn) = \tau(m)\tau(n)$ if m, n are coprime. But these follows immediately from Proposition 3.4.

In general,

Definition 3.2. If $f \in M_k(\Gamma(1))$ is a simultaneous eigenvector of all T_n, we say it is an eigenform. If furthermore that $a_1(f) = 1$, then we say it is a normalised eigenform.

Lemma 3.11. *Let $k > 0$ and $f \in M_k(\Gamma(1))$ is an eigenform, then:*
(i) There is a nonzero scalar multiple of f which is normalised.
(ii) If f is normalised, then $T_nf = a_n(f)f$ for every n.

Proof. Let $\alpha_n \in \mathbb{C}$ be the eigenvalue of T_n on f. Corollary 3.10 shows that $a_n(f) = a_1(T_nf) = \alpha_n a_1(f)$. This implies both assertions. □

Proposition 3.12. *Let $k \geq 4$ be even. Then $G_k(\tau)$ is an eigenform on which T_n has eigenvalue $\sigma_{k-1}(n)$.*

Proof. Each T_n is a polynomial in operators T_p for prime numbers p. So to show that T_k is an eigenvector for all T_n's, it is enough to show that it is an eigenvector for all T_p's for p prime.
Recall that G_k is associated to the funtion on lattices $G_k(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-k}$, so

$$T_pG_k(\Lambda) = p^{k-1} \sum_{\Lambda'\leq\Lambda,[\Lambda:\Lambda']=p} \sum_{\lambda\in\Lambda'-\{0\}} \lambda^{-k} = p^{k-1} \sum_{\lambda\in\Lambda-\{0\}} a(\Lambda,\lambda)\lambda^{-k}$$

where $a(\Lambda, \lambda)$ is the number of subgroups $\Lambda' \leq \Lambda$ with index p whih contains λ. Note that if $\Lambda' \leq \Lambda$ has index p, then $p\Lambda \leq \Lambda'$. So if $\lambda \in p\Lambda$ then $\lambda \in \Lambda'$. This means that $a(\Lambda, \lambda)$ equals to the number of subgroups $\Lambda' \leq \Lambda$ wiht index p, which is $p - 1$.
If $\lambda \notin p\Lambda$, then the image of λ in $\Lambda/p\Lambda \cong (\mathbb{Z}/p\mathbb{Z})^2$ is nonzero, therefore has order p. In particular, $\mathbb{Z}\lambda + p\Lambda$ has index p in Λ, so this is the only $\Lambda' \leq \Lambda$ with index p that can possibly contain λ, so $a(\Lambda, \lambda) = 1$. Therefore

$$\begin{aligned} T_pG_k(\Lambda) &= p^{k-1} \sum_{\lambda\in p\Lambda-\{0\}} (p+1)\lambda^{-k} + \sum_{\lambda\in\Lambda-p\Lambda} \lambda^{-k} \\ &= p^{k-1} \sum_{\lambda\in\Lambda-\{0\}} \lambda^{-k} + p^k \sum_{\lambda\in\Lambda-\{0\}} (p\lambda)^{-k} \\ &= (1+p^{k-1}) \sum_{\lambda\in\Lambda-\{0\}} \lambda^{-k} = \sigma_{k-1}(p)G_k(\Lambda) \end{aligned}$$

So G_k is indeed an eigenform.
As for the eigenvalues, if $T_nG_k = \alpha_nG_k$, then $\sigma_{k-1}(n)a_0(G_k) = a_0(T_nG_k) = a_0(\alpha_nG_k) = \alpha_na_n(G_k)$ and $a_n(G_k) = 2\zeta(k) \neq 0$, so $\alpha_n = \sigma_{k-1}(n)$ (alternatively, we could use the preceding lemma). □

Remark. 1. The decomposition $M_k(\Gamma(1)) = \mathbb{C}G_k \oplus S_k(\Gamma(1))$ is therefore compatible with the spectral decomposition of the T_n's. So we've reduced the spectral problem for the Hecke operators on $M_k(\Gamma(1))$ to that on $S_k(\Gamma(1))$.
2. We have $E_4E_6 = E_{10}$, but it is usually not the case that a product of eigenforms is an eigenform, e.g. $E_4^3 \neq E_{12}$.
3. E_k is not a normalised eigenform, which in fact should be

$$\begin{aligned} F_k = a_1(G_k)^{-1}G_k &= \frac{\zeta(k)(k-1)!}{(2\pi i)^k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n \\ &= \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n \end{aligned}$$

where B_k is the k-th Bernoulli number.

Proposition 3.13. *For all $k \geq 4$ even, the eigenvalues of T_n are algebraic numbers living in a fixed number field (i.e. a finite extension of $\mathbb{Q}$).*

To emphasise, the number field can depend on k.

Proof. We'll show that the characteristic polynomial $\det(X \operatorname{id}_{S_k(\Gamma(1))} - T_n)$ has integer coefficients. To see this, let $f_1, \ldots, f_N$ be a basis for $S_k(\Gamma(1))$ such that $a_i(f_j) = \delta_{ij}$ and $a_n(f_j) \in \mathbb{Z}$ for all n, j. This is possible due to Theorem 2.12. Then $f = \sum_{j=1}^N a_j(f)f_j$ for every $f \in S_k(\Gamma(1))$. The matrix of T_n with respect to this basis has integer coefficients due to Corollary 3.9, so the eigenvalues can only be algebraic integers.
Note also that $a_n(f) = \sum_{i=1}^N a_i(f)a_n(f_i) \in \mathbb{Q}(a_1(f), \ldots, a_N(f))$, hence all these eigenvalues live in the same number field. ☐

The proof gives an algorithm to compute the actions of T_n on $S_k(\Gamma(1))$.

Example 3.2. Take $k = 24$, then we know that $\dim S_{24}(\Gamma(1)) = 2$. Let's compute the eigenvalues of T_2. We can take $f_2 = \Delta^2 = q^2 - 48q^3 + 1080q^4 + \cdots$ and $f_1 = \Delta E_6^2 + 1032\Delta^2 = q + 195660q^3 + 12080128q^4 + \cdots$ (use a computer I guess). So T_2 has matrix

$$\begin{pmatrix} a_2(f_1) & a_2(f_2) \\ a_4(f_1) + 2^{23}a_1(f_1) & a_4(f_2) + 2^{23}a_1(f_2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 20468736 & 1080 \end{pmatrix}$$

which has eigenvalues $12(45 \pm \sqrt{144169})$. So all $a_n(f)$ for normalised eigenforms $f \in S_{24}(\Gamma(1))$ lie in $\mathbb{Q}(\sqrt{144169})$. Don't know if this is worth noting, but 144169 happens to be prime.

Enough waffling, let's diagonalise.

3.4 The Petersson Inner Product

Definition 3.3. Let $f : \mathfrak{h} \to \mathbb{C}$ be continuous and invariant under the weight 0 action of $\Gamma(1)$. Then we define

$$\int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{\mathcal{F}'} f(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{\mathcal{F}} f(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

provided the latter converges absolutely (here and onwards we write $\tau = x + iy$ as convention).

The idea here is that the area form $y^{-2}\,\mathrm{d}x\,\mathrm{d}y$ is invariant under the action of $\mathrm{GL}_2(\mathbb{R})^+$. It in fact descends to an area form on the surface $\Gamma(1)\backslash\mathfrak{h}$, and the integration we're defining here is the same as the one given by the usual integration theory on manifolds. Not having differential geometry as a prerequisite, we are going to do this in an elementary way.

Lemma 3.14. *Let $f, g \in S_k(\Gamma(1))$, then $f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k$ is invariant under the weight 0 action of $\Gamma(1)$. Furthermore,*

$$\int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

is absolutely convergent.

Proof. For $\gamma \in \Gamma(1)$, we have

$$\begin{aligned} f(\gamma\tau)\overline{g(\gamma\tau)}\,\mathrm{Im}(\gamma\tau)^k &= f(\tau)j(\gamma,\tau)^k\overline{g(\tau)j(\gamma,\tau)}^k\,\mathrm{Im}(\tau)^k|j(\gamma,\tau)|^{-2k} \\ &= f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \end{aligned}$$

Since f is a cusp form, $\tilde{f}(q) = qf_0(q)$ with $f_0 : D(0,1) \to \mathbb{C}$ holomorphic. Hence for all $\delta \in (0,1)$, there is some $C_{f,\delta} = \sup_{q\in\overline{B(0,\delta)}} |f_0(q)| \geq 0$ such that $|\tilde{f}(q)| \leq |q|C_{f,\delta}$.
Note that we have $|q| = |e^{2\pi i x}e^{-2\pi y}| = e^{-2\pi y}$, so there is some $C_f > 0$ such that for all $\tau \in \mathfrak{h}$ with $\mathrm{Im}\,\tau \geq 1/2$, we have $|f(\tau)| \leq C_f e^{-2\pi y}$. Then

$$\begin{aligned} \int_{\mathfrak{F}} |f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k|\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &\leq \int_{x=-1/2}^{1/2}\int_{y=\sqrt{3}/2}^{\infty} C_f C_g e^{-4\pi y}y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{y=\sqrt{3}/2}^{\infty} C_f C_g e^{-4\pi y}y^{k-2}\,\mathrm{d}y < \infty \end{aligned}$$

as desired. ☐

Definition 3.4. The Petersson inner product on $S_k(\Gamma(1))$ is defined by

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

This is an inner product by the preceding lemma.

Theorem 3.15. *For every $n \in \mathbb{N}$, T_n is self-adjoint with respect to the Petersson inner product, i.e. $\langle T_n f, g\rangle = \langle f, T_n g\rangle$ for any $f, g \in S_k(\Gamma(1))$.*

Definition 3.5. For a lattice $\Lambda \leq \mathbb{C}$, its covolume is

$$\mathrm{covol}(\Lambda) = \int_{\mathbb{C}/\Lambda} \mathrm{d}x\,\mathrm{d}y = \int_{D_\Lambda} \mathrm{d}x\,\mathrm{d}y = \det\begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix}$$

where D_Λ is a fundamental parallelogram of $\mathbb{C}/\Lambda$ and $e_j = x_j + iy_j$ with $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \mathrm{Im}(e_1/e_2) > 0$.

Sketch of Proof. We know that each T_n is an integer polynomial in T_p for p prime. So it suffices to show the self-adjointness of T_p.
If $f, g \in S_k(\Gamma(1))$, then $f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k \in W_0$. Thus it corresponds to an element of V_0. We claim that if f corresponds to $F \in V_k$ and g corresponds to $G \in V_k$, then $f(\tau)\overline{g(\tau)}\,\mathrm{Im}(\tau)^k$ corresponds to $F(\Lambda)\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k \in V_0$.
Indeed, this is in V_0 because $\mathrm{covol}(z\Lambda) = |z|^2\,\mathrm{covol}(\Lambda)$ for any $z \in \mathbb{C}^\times$, and we have the correspondence as $\mathrm{covol}(\Lambda_\tau) = \mathrm{Im}\,\tau$ for any $\tau \in \mathfrak{h}$.
Now suppose $A : \mathbb{C}^\times\backslash\mathcal{L} \to \mathbb{C}$ is a function corresponding to a continuous function $a \in W_0$, then we write

$$\int_{\mathbb{C}^\times\backslash\mathcal{L}} A(\Lambda)\,\mathrm{d}\Lambda = \int_{\Gamma(1)\backslash\mathfrak{h}} a(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

$$\begin{aligned}\langle T_pf, g\rangle &= \int_{\Gamma(1)\backslash\mathfrak{h}} T_pf(\tau)g(\tau)\,\mathrm{Im}(\tau)^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= \int_{\mathbb{C}^\times\backslash\mathcal{L}} (T_pF)(\Lambda)\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k\,\mathrm{d}\Lambda\\ &= p^{k-1}\int_{\mathbb{C}^\times\backslash\mathcal{L}}\sum_{\Lambda'\le\Lambda,[\Lambda:\Lambda']=p} F(\Lambda')\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k\,\mathrm{d}\Lambda\end{aligned}$$

Write $\mathcal{L}_p = \{(\Lambda', \Lambda) : \Lambda' \le \Lambda, [\Lambda : \Lambda'] = p, \Lambda \in \mathcal{L}\}$. There is a bijection betwee $\Gamma_0(p)\backslash\mathfrak{h} \to \mathbb{C}^\times\backslash\mathcal{L}_p$, where $\Gamma_0(p) = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma(1) : c \equiv 0 \pmod p\}$ and the bijection sends τ to $(\mathbb{Z}p\tau \oplus \mathbb{Z}, \mathbb{Z}\tau \oplus \mathbb{Z})$. If $A : \mathbb{C}^\times\backslash\mathcal{L}_p \to \mathbb{C}$ is a function which corresponds to a continuous function $a : \Gamma_0(p)\backslash\mathfrak{h} \to \mathbb{C}$, we similarly define

$$\int_{\mathbb{C}^\times\backslash\mathcal{L}_p} A(\Lambda)\,\mathrm{d}(\Lambda, \Lambda') = \int_{\Gamma_0(p)\backslash\mathfrak{h}} a(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

Recall that if $\Lambda' \le \Lambda$ has index p, then $p\Lambda \le \Lambda'$. So we can define a map $\iota : \mathcal{L}_p \to \mathcal{L}_p, (\Lambda', \Lambda) \to (p\Lambda, \Lambda')$. This descends to a map $\mathbb{C}^\times\backslash\mathcal{L}_p \to \mathbb{C}^\times\backslash\mathcal{L}_p$ which is a measure-preserving involution. Indeed, under the identification of $\mathbb{C}^\times\backslash\mathcal{L}_p$ with $\Gamma_0(p)\backslash\mathfrak{h}$, ι corresponds to the action of $\eta_p = \left(\begin{smallmatrix} 0 & -1\\ p & 0\end{smallmatrix}\right)$ and $\eta_p^*(y^{-2}\,\mathrm{d}x\,\mathrm{d}y) = y^{-2}\,\mathrm{d}x\,\mathrm{d}y$. So let's change variables using i. This gives

$$\begin{aligned}\langle T_pf, g\rangle &= p^{k-1}\int_{\mathbb{C}^\times\backslash\mathcal{L}_p} F(\Lambda')\overline{G(\Lambda)}\,\mathrm{covol}(\Lambda)^k\,\mathrm{d}(\Lambda, \Lambda')\\ &= p^{k-1}\int_{\mathbb{C}^\times\backslash\mathcal{L}_p} F(p\Lambda)\overline{G(\Lambda')}\,\mathrm{covol}(\Lambda')^k\,\mathrm{d}(\Lambda, \Lambda')\\ &= p^{k-1}\int_{\mathbb{C}^\times\backslash\mathcal{L}_p} F(\Lambda)\overline{G(\Lambda')}\,\mathrm{covol}(\Lambda)^k\,\mathrm{d}(\Lambda, \Lambda') = \langle f, T_pg\rangle\end{aligned}$$

as desired. □

Consequently,

Theorem 3.16. *For all $k \ge 0$, $S_k(\Gamma(1))$ has a basis $f_1, \ldots, f_N$ of normalised eigenforms, unique up to reordering, such that:*
(i) For all $n \in \mathbb{N}$, $T_n(f_i) = a_n(f_i)f_i$.
(ii) For every i, there is a number field $k_{f_i} \subset \mathbb{R}$ such that $\forall n, a_n(f_i) \subset \mathcal{O}_{k_{f_i}}$.

Proof. General spectral theory of self-adjoint endomorphisms (which the T_n's are, by the preceding theorem) and proposition 3.13 imply the existence.
As for uniqueness, suppose they are not unique, then there must be a simultaneous eigenspace of the T_n's with dimension at least 2. If f_1, f_2 are normalised eigenforms in this eigenspace, then $a_n(f_1) = a_n(f_2)$ for all n since they are both just eigenvalues of T_n in this eigenspace. This then means that $f_1 = f_2$, contradiction. □

The sequence of eigenvalues $(a_1(f), a_2(f), \ldots)$ has great arithmetic significance. Recall Ramanujan's conjectures on the τ function (Example 3.1).

Lemma 3.17. *If p is a prime, then*

$$\sum_{n=0}^{\infty} \tau(p^n)X^n = (1 - \tau(p)X + p^{11}X^2)^{-1}$$

Proof. We calculate

$$\begin{aligned}&(1 - \tau(p)X + p^{11}X^2) \sum_{n=0}^{\infty} \tau(p^n)X^n \\ &= 1 + \sum_{n\geq 2}(\tau(p^n) - \tau(p^{n-1})\tau(p) + p^{11}\tau(p^{n-2}))X^n = 1\end{aligned}$$

by Ramanujan's conjecture on the τ function. □

Let's factorise $1 - \tau(p) + p^{11}X^2 = (1 - \alpha_p X)(1 - \beta_p X), \alpha_p, \beta_p \in \mathbb{C}$. If $\tau(p)^2 - 4p^{11} \leq 0$, then α_p, β_p are conjugate complex numbers with $|\alpha_p| = |\beta_p| = p^{11/2}$. When $\tau(p)^2 - 4p^{11} > 0$, α_p, β_p are distinct real numbers, which has less symmetry. Ramanujan conjectured that the first case always happens.

Theorem 3.18 (Ramanujan-Petersson Conjecture)**.** *If $f \in S_k(\Gamma(1))$ is a normalised eigenform, then for any prime p, $|a_p(f)| \leq 2p^{(k-1)/2}$.*

Remark. This is proved by Deligne in 1973.

Why is this important? Many applications of modular forms to number theory use either this conjecture or generalisations of it. Ramanujan's proved the formula

$$r_{24}(p) = \frac{16}{691}(1 + p^{11}) + \frac{33152}{691}\tau(p)$$

for any odd prime p. Recall that $r_{24}(n)$ is the number of ways to express n as the sum of 24 squares. How does Theorem 3.18 come in? It gives rise to the estimate $\tau(p) = O(p^{11/2})$, so we have $r_{24}(p) = (16/691)(1 + p^{11}) + O(p^{11/2})$.

Proposition 3.19. *Let*

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z} \right\} \leq \Gamma(1)$$

Suppose $f : \mathfrak{h} \to \mathbb{C}$ is invariant under the (weight 0) action of Γ_∞. Suppose in addition that for any $\tau \in \mathfrak{h}$,

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} |f(\gamma\tau)| < \infty$$

and that

$$\int_{x=-1/2}^{1/2}\int_0^\infty |f(x+iy)|\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} < \infty$$

Then $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} f(\gamma\tau)$ *is measurable, invariant under the (weight* 0*) action of* $\Gamma(1)$*, and satisfies*

$$\int_{\Gamma(1)\backslash\mathfrak{h}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} f(\gamma\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{x=-1/2}^{1/2}\int_{y=0}^\infty f(x+iy)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

This could be viewed as "unfolding" $\Gamma(1)\backslash\mathfrak{h}$ into $[-1/2,1/2]+i[0,\infty)$, which is a fundamental domain of the $\Gamma_\infty/\{\pm1\}$-action on $\mathfrak{h}$.

Proof. By Fubini's theorem, if we have the absolute convergence

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\int_{\mathcal{F}} |f(\gamma\tau)|\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} < \infty$$

Then $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} f(\gamma\tau)$ is measurable, and there is an equality

$$\int_{\mathcal{F}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} f(\gamma\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\int_{\mathcal{F}} f(\gamma\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

How does this help? If $(\gamma_i)_{i\in I}$ is a set of representatives for $\Gamma_\infty\backslash\Gamma(1)$, then

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\int_{\mathcal{F}} f(\gamma\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \sum_{i\in I}\int_{\gamma_i\mathcal{F}^\circ} f(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

Let $S=\{\tau\in\mathfrak{h} : \operatorname{Re}\tau\in(-1/2,1/2)\}$. From example sheet, we know that $\gamma\mathcal{F}^\circ$ doesn't meet $1/2+\mathbb{Z}+i\mathbb{R}$ for any $\gamma\in\Gamma(1)$. And by definition of S, there is a unique $\delta\in\Gamma_\infty/\{\pm1\}$ such that $\delta\gamma\mathcal{F}^\circ\subset S$.
This means that we can choose our coset representative γ_i to be the unique (up to ±1) choice such that $\gamma_i\mathcal{F}^\circ\subset S$. Then

$$S = \left(\coprod_{i\in I}\gamma_i\mathcal{F}^\circ\right)\cup(S\cap W)$$

where $W\subset\mathfrak{h}$ is a measure zero subset. We therefore conclude

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)}\int_{\mathcal{F}} f(\gamma\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{x=-1/2}^{1/2}\int_0^\infty f(x+iy)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

which does everything. □

4 L-Functions

4.1 L-Functions associated to Modular Forms

The primary motivating example of an L-function is the Riemann ζ function $\zeta(s)=\sum_{n\geq1}n^{-s}$, which is absolute convergent and holomorphic in $\{\operatorname{Re}s>1\}$

with the property that:
(i) ζ has a meromorphic continuation to $\mathbb{C}$, with a simple pole at $s = 1$.
(ii) We have a functional equation $\xi(s) = \xi(1-s)$ where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the "completed ζ function".
(iii) There is an Euler product $\zeta(s) = \prod_{p \text{ prime}}(1-p^{-s})^{-1}$.
In general, when we say a Dirichlet series $\sum_{n\geq 0} a_n n^{-s}$ is an L-function if it has analogous properties (or in some cases, has some of the conjectured properties, or has been conjectured to have analogous properties).

Example 4.1. For $N \geq 2$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a group homomorphism, the associated Dirichlet L-function is given by $L(\chi, s) = \sum_{n\geq 1, \gcd(n,N)=1} \chi(n) n^{-s}$.

Modular forms give rise to a huge class of L-functions in a reasonably easy way.

Definition 4.1. If f is a modular form of weight k and level $\Gamma(1)$, then its associated Dirichlet series is $L(f, s) = \sum_{n\geq 1} a_n(f) n^{-s}$.

Example 4.2. Consider $F_k = -B_k/(2k) + \sum_{n=1} \sigma_{k-1} q^n$, the normalised eigenform associated with G_k. Then

$$\begin{aligned}L(F_k, s) &= \sum_{n\geq 1} \sigma_{k-1}(n) n^{-s} = \sum_{n\geq 1, m|n} m^{k-1} n^{-s} = \sum_{a\geq 1}\sum_{d\geq 1} d^{k-1}(ad)^{-s} \\ &= \left(\sum_{a\geq 1} a^{-s}\right)\left(\sum_{d\geq 1} d^{-(s+1-k)}\right) = \zeta(s)\zeta(s+1-k)\end{aligned}$$

Boring. So let's look at cusp forms instead.

Proof. For any $f \in S_k(\Gamma(1))$, $L(f, s)$ converges absolutely on $\{s \in \mathbb{C} : \operatorname{Re} s > 1 + k/2\}$ and defines a holomorphic function there. □

Proof. We'll use the Weierstrass M-test to show that $L(f, s)$ converges absolutely and uniformly in $\{\operatorname{Re} s > 1 + k/2 + \delta\}$ for every $\delta > 0$. Notationally, we'll always write $s = \sigma + it$. Clearly $|n^{-s}| = n^{-\sigma}$.
In example sheet, you'll prove that (since f is cuspidal) there is some $C_f > 0$ such that $|a_n(f)| \leq C_f n^{k/2}$. So in the given region we have

$$\sum_{n\geq 1} |a_n(f) n^{-s}| \leq \sum_{n\geq 1} |a_n(f)| n^{-\sigma} \leq C_f \sum_{n\geq 1} n^{-(\sigma - k/2)} \leq C_f \sum_{n\geq 1} n^{-(1+\delta)}$$

as desired. □

If we assume Theorem 3.18, then we can actually get absolute convergence in the much bigger region $\{\operatorname{Re} s > (1+k)/2\}$.

Theorem 4.1. *Let $f \in S_k(\Gamma(1))$, then:*
(i) $L(f, s)$ has an analytic continuation to $\mathbb{C}$.
(ii) The function $\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s)$ has an analytic continuation to $\mathbb{C}$ and satisfies the functional equation $\Lambda(f, k-s) = i^k \Lambda(f, s)$.

Before proving the theorem, let's just discuss the function $\Gamma(s)$. This is defined as

$$\Gamma(s) = \int_{y=0}^{\infty} e^{-y} y^s \frac{\mathrm{d}y}{y}$$

Proposition 4.2. *(i) $\Gamma(s)$ converges absolutely in $\{\mathrm{Re}\, s > 0\}$ and defines a holomorphic function there.*
(ii) $\Gamma(s)$ admits a meromorphic continuation to $\mathbb{C}$ with simple poles at the non-positive integers.
(iii) The continuation of $\Gamma(s)$ is nonvanishing.

Proof. (i) We need to show that

$$\int_{y=0}^{\infty} |e^{-y} y^s| \frac{\mathrm{d}y}{y} = \int_{y=0}^{\infty} e^{-y} y^{\sigma} \frac{\mathrm{d}y}{y}$$

is finite for $\sigma > 0$, which is an easy exercise. To show that it defines a holomorphic function, we consider for $N > 0$ the function

$$\Gamma_N(s) = \int_{y=1/N}^{N} e^{-y} y^s \frac{\mathrm{d}y}{y}$$

This is continuous and holomorphic. Indeed, for $\mathrm{Re}\, s > 0$ and $\epsilon > 0$. Since $[1/N, N]$ is compact, we can find $\delta > 0$ such that for any $s' \in \mathbb{C}$ and $y \in [1/N, N]$ we have $|y^{s-1} - y^{s'-1}| < \epsilon$ whenever $|s - s'| < \delta$. Then

$$|\Gamma_N(s) - \Gamma_N(s')| \leq \int_{y=1/N}^{N} e^{-y} |y^{s-1} - y^{s'-1}| \,\mathrm{d}y < \epsilon \int_{y=1/N}^{N} e^{-y} \,\mathrm{d}y$$

which shows continuity. By Morera's theorem, to check that $\Gamma_N(s)$ is holomorphic it suffices to check

$$\oint_{\gamma} \Gamma_N(s) \,\mathrm{d}s = 0$$

for all closed contour γ in $\{\mathrm{Re}\, s > 0\}$. But by Fubini's theorem,

$$\oint_{\gamma} \Gamma_N(s) \,\mathrm{d}s = \int_{y=1/N}^{N} e^{-y} \left(\oint_{\gamma} y^s \,\mathrm{d}s \right) \frac{\mathrm{d}y}{y} = \int_{y=1/N}^{N} e^{-y} \cdot 0 \frac{\mathrm{d}y}{y} = 0$$

To show that $\Gamma(s)$ is continuuos and holomorphic, it suffices to show that $\Gamma_N \to \Gamma$ locally uniformly in $\{\mathrm{Re}\, s > 0\}$. For $\sigma_1 > \sigma_0 > 0$, let's show uniform convergence in $\{\sigma_0 \leq \sigma \leq \sigma_1\}$. This follows from the estimate, for s in this region, that

$$\begin{aligned} |\Gamma(s) - \Gamma_N(s)| &\leq \int_{y=0}^{1/N} e^{-y} y^{\sigma} \frac{\mathrm{d}y}{y} + \int_{y=N}^{\infty} e^{-y} y^{\sigma} \frac{\mathrm{d}y}{y} \\ &\leq \int_{y=0}^{1/N} e^{-y} y^{\sigma_0} \frac{\mathrm{d}y}{y} + \int_{y=N}^{\infty} e^{-y} y^{\sigma_1} \frac{\mathrm{d}y}{y} \end{aligned}$$

(ii) (iii) Integration by parts shows that $s\Gamma(s) = \Gamma(s+1)$ for $\mathrm{Re}\, s > 0$. This allows us to extend the definition of Γ as a meromorphic function to the whole of $\mathbb{C}$. □

Proof of Theorem 4.1. Part (iii) of the preceding proposition means that it suffices to show (ii). Define

$$F(s) = \int_{y=0}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y}$$

We claim that F converges absolutely in $\mathbb{C}$ and defines a holomorphic function. We have $f(-1/\tau) = f(\tau)\tau^k$, so $f(1/y) = f(iy)(iy)^k$. This change of variables gives

$$F(s) = \int_{y=0}^{1} f(iy)y^s \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} = \int_{y=1}^{\infty} f(iy)(i^k y^{k-s} + y^s)\frac{\mathrm{d}y}{y}$$

Since f is cuspidal, there is some $C_f > 0$ such that for all $y \geq 1$ we have $|f(iy)| \leq C_f e^{-2\pi y}$. So $|f(iy)y^s| \leq C_{f,s}e^{-\pi y}$ for some $C_{f,s} > 0$. Therefore the integral converges absolutely. Using the same agrument we used for $\Gamma(s)$ shows that $F(s)$ is holomorphic in $\mathbb{C}$.
Now what is F? We have the estimate

$$\sum_{n=1}^{\infty} |a_n| \int_{y=0}^{\infty} e^{-2\pi n y} y^\sigma \frac{\mathrm{d}y}{y} = \sum_{n=1}^{\infty} |a_n| n^{-\sigma} (2\pi)^{-\sigma} \Gamma(\sigma)$$

This is finite since $L(f,s)$ is absolutely convergent when $\sigma > 1 + k/2$. We can therefore safely calculate

$$\begin{aligned} F(s) &= \int_{y=0}^{\infty} \sum_{n=1}^{\infty} a_n e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} = \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} \\ &= (2\pi)^{-s}\Gamma(s)L(f,s) = \Lambda(f,s) \end{aligned}$$

For the functional equation, recall that

$$\Lambda(f,s) = \int_{y=1}^{\infty} f(iy)(i^k y^{k-s} + y^s)\frac{\mathrm{d}y}{y}$$

from where it's clear. □

Remark. The poles of Γ then gives us a bunch of zeros of $L(f,s)$. They are called the trivial zeros.

Theorem 4.3. *Let $f \in S_k(\Gamma(1))$ be a normalised eigenform. Then we have an Euler product*

$$L(f,s) = \sum_{n=1}^{\infty} a_n(f) n^{-s} = \prod_{p \text{ prime}} (1 - a_p(f)p^{-s} + p^{k-1-2s})^{-1}$$

This can either be taken as an identity of formal Dirichlet series, or as limits convergent for $\operatorname{Re} s > 1 + k/2$.

Proof. Let's prove the formal version first. If $n \in \mathbb{N}$ can be factored as $n = \prod_j p_j^{e_j}$, then $a_n(f) = \prod_j a_{p_j^{e_j}}(f)$, since $a_n(f)$ is also the eigenvalue of T_n on f and T_n factors as such. So

$$L(f,s) = \prod_{p \text{ prime}} \sum_{j=0}^{\infty} a_{p^j}(f) p^{-js} = \prod_{p \text{ prime}} (1 - a_p(f)p^{-s} + p^{k-1-2s})^{-1}$$

due to the relation $a_{p^n}(f)a_p(f) = a_{p^{n+1}}(f) + p^{k-1}a_{p^{n-1}}(f)$. The limit version is the consequence of something you'll prove in example sheet. □

4.2 Applications of L-Functions

We'll take the following theorem as a blackbox, which is always a good start.

Theorem 4.4 (Wiener-Ikehara Tauberian Theorem)**.** *Suppose $(a_n)_{n\geq 1}$ is a sequence of complex numbers with $f(s) = \sum_{n\geq 1} a_n n^{-s}$ absolutely convergent in $\{\operatorname{Re} s > 1\}$. Suppose also that f has a meromorphic continuation to an open domain containing $\{\operatorname{Re} s \geq 1\}$, holomorphic on $\{\operatorname{Re} s = 1\}$ with the possible exception of a simple pole at $s = 1$. Set α to be the residue of f at $s = 1$. Then*

$$\sum_{1\leq n\leq x} a_n = \alpha x + o(x)$$

Proposition 4.5. *Suppose $\zeta(s) = \sum_{n\geq 1} n^{-s}$ has a meromorphic continuation to $\mathbb{C}$ which is holomorphic and nonvanishing on $\{\operatorname{Re} s = 1, s \neq 1\}$ and has a simple pole at $s = 1$, then the prime number theorem*

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

holds.

Proof. There's a branch of $\log\zeta(s)$ in $\{\operatorname{Re} s > 1\}$ given by

$$\log\zeta(s) = \sum_{p \text{ prime}} -\log(1 - p^{-s}) = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}$$

Therefore

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \sum_{k=1}^{\infty} -(\log p)p^{-ks} = -\sum_{p \text{ prime}} (\log p)p^{-s} - \sum_{p \text{ prime}} \sum_{k=2}^{\infty} (\log p)p^{-ks}$$

Since ζ is meromorphic in $\mathbb{C}$, so is ζ'/ζ. As ζ is holomorphic and nonvanishing on $\{\operatorname{Re} s = 1, s \neq 1\}$ and has a simple pole at $s = 1$, ζ'/ζ must also be holomorphic on $\{\operatorname{Re} s = 1, s \neq 1\}$ has has a simple pole at $s = 1$ of residue -1.
Now the tail sum $\sum_{p \text{ prime}} \sum_{k\geq 2} (\log p)p^{-ks}$ is converges absolutely and locally uniformly in $\{\operatorname{Re} s > 1/2\}$. Therefore $\sum_{p \text{ prime}} (\log p)p^{-s}$ has a meromorphic continuation to $\{\operatorname{Re} s > 1/2\}$, holomorphic to $\{\operatorname{Re} s = 1, s \neq 1\}$ with a simple pole at $s = 1$ with residue 1.
Theorem 4.4 then shows that $\sum_{p\leq x \text{ prime}} \log p = x + o(x)$. The next lemma shows that this implies the prime number theorem. □

Lemma 4.6. *Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers with $1 < x < y$. Suppose $f : [x, y] \to \mathbb{C}$ is a continuously differentiable function. Write $A(t) = \sum_{1\leq n\leq t} a_n$, then*

$$\sum_{x<n\leq y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_{t=x}^{y} A(t)f'(t)\,\mathrm{d}t$$

Proof. Exercise. □

If we take $a_n = 1_{n=p \text{ prime}} \log p$, then $A(t) = \sum_{p \leq t} \log p = t + o(t)$. For $f(t) = 1/\log t$, we get

$$\begin{aligned}\pi(y) &= 1 + \sum_{e<n\leq y} a_n f(n) = 1 + A(y)f(y) - A(e) - \int_{t=e}^{y} A(t)\frac{1}{t}\frac{-1}{(\log t)^2}\,\mathrm{d}t \\ &= \frac{y}{\log y} + o\left(\frac{y}{\log y}\right) + \int_{t=e}^{y} \frac{A(t)}{t}\frac{1}{(\log t)^2}\,\mathrm{d}t\end{aligned}$$

We need to show that the last integral has magnitude $o(y/\log y)$. Since $A(t) = t + o(t) = O(t)$, we have $t^{-1}A(t) = O(1)$, therefore we only need to show that

$$\int_{t=e}^{y} \frac{1}{(\log t)^2}\,\mathrm{d}t = o\left(\frac{y}{\log y}\right)$$

But we can just estimate

$$\begin{aligned}\int_{e}^{y} \frac{1}{(\log t)^2}\,\mathrm{d}t &= \int_{e}^{\sqrt{y}} \frac{1}{(\log t)^2}\,\mathrm{d}t + \int_{\sqrt{y}}^{y} \frac{1}{(\log t)^2}\,\mathrm{d}t \\ &\leq \sqrt{y} + \frac{y}{(\log\sqrt{y})^2} = \sqrt{y} + 4\frac{y}{(\log y)^2} = o\left(\frac{y}{\log y}\right)\end{aligned}$$

We'll establish the assumed properties of ζ later, using modular forms of course. Here's a generalisation:

Proposition 4.7. *Fix $n \geq 1$. Suppose we are given, for any prime p, a matrix $\Phi_p \in \mathrm{GL}_n(\mathbb{C})$ all of whose eigenvalues have absolute value 1. Consider*

$$L(\{\Phi_p\}_p, s) = \prod_{p \text{ prime}} \det(1 - p^{-s}\Phi_p)^{-1}$$

Then $L(\{\Phi_p\}_p, s)$ converges absolutely in $\{\sigma > 1\}$.
Suppose further that $L(\{\Phi_p\}_p, s)$ admits a meromorphic continuation to an open neighbourhood to $\{\sigma \geq 1\}$ which is holomorphic and nonvanishing on $\{\mathrm{Re}\, s = 1\}$ with the possible exception of a pole of order δ at $s = 1$. Then

$$\sum_{p\leq x \text{ prime}} \mathrm{tr}\,\Phi_p = \delta\frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

Consequently $\pi(x)^{-1}\sum_{p\leq x \text{ prime}} \mathrm{tr}\,\Phi_p \to \delta$ as $x \to \infty$, if so inclined.

Proof. Example sheet. □

Example 4.3. Suppose $N \in \mathbb{N}, n = 1$. Take a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Then

$$L(\{\chi(p \bmod N)1_{p\nmid N}\}_p, s) = \prod_{p\nmid N}(1 - \chi(p \bmod N)p^{-s})^{-1}$$

is the Dirichlet L-function $L(\chi, s)$ of χ.
If one can show that $L(\chi, s)$ is nonvanishing on $\{\mathrm{Re}\, s = 1\}$ for all such χ, then the proposition implies the strong form of Dirichlet's theorem on primes in arithmetic progressions, that is, for any $a \in (\mathbb{Z}/N\mathbb{Z})^\times$,

$$\#\{1 \leq p \leq x : p \text{ prime}, p \equiv a \pmod{N}\} = \frac{1}{\phi(N)}\frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$$

Let $f \in S_k(\Gamma(1))$ be a normalised eigenform. For each prime p, we factorise $1 - a_p(f)X + p^{k-1}X^2 = (1-\alpha_p X)(1-\beta_p X)$. Let $\Phi_p = \operatorname{diag}(\alpha_p, \beta_p)$, then $\det(1-\Phi_p X) = 1 - a_p(f)X + p^{k-1}X^2$, in particular $L(\{\Phi_p\}_p, s) = L(f,s)$.
If we assume Theorem 3.18, we get $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$. So $p^{-(k-1)/2}\Phi_p$ has eigenvalues of absolute value 1, and its trace is just $a_p(f)/p^{(k-1)/2}$. On the other hand, $L(\{p^{-(k-1)/2}\Phi_p\}_p, s) = L(f, s+(k-1)/2)$.

Corollary 4.8. *Assume Theorem 3.18 and moreover that $L(f, s+(k-1)/2)$ is nonvanishing on $\mathrm{Re} = 1$, then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{1<p\leq x \ prime} \frac{a_p(f)}{p^{(k-1)/2}} = 0$$

Remark. The assumptions of the corollary are in fact true, but are beyond the scope of this course.

So the average value of $a_p(f)/p^{(k-1)/2}$ is zero.

Example 4.4. Ramanujan has a formula that reads

$$r_{24}(p) = \frac{16}{691}(1+p^{11}) + \frac{33152}{691}\tau(p)$$

Theorem 3.18 means that we have $|r_{24}(p) - (16/691)(1+p^{11})| = O(p^{11/2})$. The corollary, on the other hand, means that the average of $(r_{24}(p) - (16/691)(1+p^{11}))/p^{11/2}$ is zero.

We can go further and consider a family of L-functions associated with a single normalised eigenform $f \in S_k(\Gamma(1))$. These are the symmetric power L-functions associated to the representation $\mathrm{Sym}^n : \mathrm{GL}_2 \to \mathrm{GL}_{n+1}$, namely

$$L(\mathrm{Sym}^n, f, s) = L(\{\mathrm{Sym}^n \Phi_p\}_p, s) = \prod_{p \text{ prime}} \prod_{i=0}^{n} \left(1 - \alpha_p^i \beta_p^{n-i} p^{-s}\right)^{-1}$$

Certainly $L(\mathrm{Sym}^1, f, s) = L(f,s)$.

Theorem 4.9 (Sato-Tate Conjecture)**.** $a_p(f)/(2p^{(k-1)/2}) \in [-1,1]$ *are equidistributed with respect to the Sato-Tate density* $(2/\pi)\sqrt{1-t^2}\,\mathrm{d}t$*. That is, for any continuous $g : [-1,1] \to \mathbb{C}$, we have*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x \ prime} g\left(\frac{a_p(f)}{2p^{(k-1)/2}}\right) = \frac{2}{\pi}\int_{t=-1}^{1} g(t)\sqrt{1-t^2}\,\mathrm{d}t$$

Proposition 4.10. *(i) (Langlands) If, for all $n \geq 1$, $L(\mathrm{Sym}^n, f, s)$ admits an analytic continuation to $\mathbb{C}$, then Theorem 3.18 holds for f.*
(i) (Serre) If, for all $n \geq 1$, $L(\mathrm{Sym}^n, f, s)$ admits an analytic continuation to $\mathbb{C}$ which is moreover nonvanishing on $\{\mathrm{Re}\, s = 1 + n(k-1)/2\}$, then Theorem 4.9 holds for f.

Theorem 3.18 and Theorem 4.9 have both been proved, the former by P. Deligne in 1973 and the latter by a lot of people in 2010. The analytic continuation of $L(\mathrm{Sym}^n, f, s)$ and nonvanishing on the critical line are established later, by J. Newton and J. Thorne in 2019.

5 Modular Forms in General

5.1 Congruence Subgroups and their Cusps

Definition 5.1. If $N \in \mathbb{N}$, we define $\Gamma(N)$ to be the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. A subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if it contains $\Gamma(N)$ for some N.

One immediately sees that any congruence subgroup has finite index.

Example 5.1. $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z}), \Gamma(N)$ are congruence subgroups. There are two other important classes of examples, namely

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\}$$

Definition 5.2. Let $k \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ a congruence subgroup. A weakly modular function of weight k and level Γ is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ invariant under the weight k action of any $\gamma \in \Gamma$.

Example 5.2. $\mathcal{F}_0(2) = \{\tau \in \mathfrak{h} : \mathrm{Re}(\tau) \in [0,1], |\tau - 1/2| \geq 1/2\}$ is the closure of a fundamental set for $\Gamma_0(2)$. A feature of this is that none of the vertices of this hyperbolic triangle is on $\mathfrak{h}$: We can "go to infinity" by approaching any one of the three vertices, which hints that we need a better way to deal with infinity behaviours.

Definition 5.3. A cusp of a congruence subgroup Γ is a Γ-orbit (via Möbius transformations) on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \sqcup \{\infty\}$.

Lemma 5.1. $\Gamma(1)$ *acts transitively on* $\mathbb{P}^1(\mathbb{Q})$.

Proof. It suffices to show that for any $a/c \in \mathbb{Q}$ with $a, c \in \mathbb{Z}, \gcd(a,c) = 1$ we have some $\gamma \in \Gamma(1)$ sending ∞ to a/c. By Bézout's theorem, there are integers b, d such that $ad - bc = 1$. Then we just take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. □

In other words, $\Gamma(1)$ has a unique cusp, and any congruence subgroup (in fact any finite index subgroup) of it has at most finitely many cusps. Indeed, we have a $\Gamma(1)$-equivariant bijection $\Gamma(1)/\Gamma_\infty \to \mathbb{P}^1(\mathbb{Q})$ where

$$\Gamma_\infty = \mathrm{Stab}_{\Gamma(1)}(\infty) = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} : h \in \mathbb{Z} \right\}$$

So the Γ-orbits on $\mathbb{P}^1(\mathbb{Q})$ correspond to Γ-orbits of $\Gamma(1)/\Gamma_\infty$, i.e. double cosets $\Gamma\backslash\Gamma(1)/\Gamma_\infty$. But these are also Γ_∞ orbits on the finite set $\Gamma\backslash\Gamma(1)$, hence there are only finitely many (at most $[\Gamma(1) : \Gamma]$) of them.
Now, $\Gamma \cap \Gamma_\infty$ has finite index in Γ_∞ since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ if $\Gamma(N) \leq \Gamma$.

Definition 5.4. The width of the cusp ∞ of Γ is

$$h = \min \left\{ h \in \mathbb{N} : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$$

So if the width of the cusp ∞ is h, then we have $f(\tau+h) = f(\tau)$ for any weakly modular function f of weight k and level Γ. We therefore obtain an meromorphic function $\tilde{f} : D^*(0,1) \to \mathbb{C}$ such that for all $\tau \in \mathfrak{h}$, $f(\tau) = \tilde{f}(q_h)$ where $q_h = e^{2\pi i\tau/h}$.

Definition 5.5. If $\tilde{f}$ extends to a meromorphic function on $D(0,1)$. we say f is meromorphic at ∞.

Suppose f is meromorphic at ∞. Then there is a Laurent expansion $\tilde{f}(q_h) = \sum_{n\in\mathbb{Z}} a_n q_h^n$ valid in $D^*(0,\delta)$ for some $\delta > 0$, where $a_n = 0$ for sufficiently negative n. Therefore f has an expansion $f(\tau) = \sum_{n\in\mathbb{Z}} a_n q_h^n$ valid in some $\{\operatorname{Im}\tau > R\}$ for some R. We call this the q-expansion of f at ∞.

Definition 5.6. f is holomorphic (resp. vanishing) at ∞ if it is meromorphic at ∞ and $a_n = 0$ for all $n < 0$ (resp. $n \leq 0$).

What about the other cusps? For a cusp $\Gamma z, z \in \mathbb{P}^1(\mathbb{Q})$, we choose $\alpha \in \Gamma(1)$ such that $\alpha\infty = z$. Then $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup of $\Gamma(1)$ since $\Gamma(N) \leq \Gamma(1)$ is normal for all N. If f is invariant under the weight k action of Γ, then $f|_k[\alpha]$ is invariant under the weight k action of $\alpha^{-1}\Gamma\alpha$.

Definition 5.7. The width of the cusp Γz of Γ is the width of the cusp ∞ of $\alpha^{-1}\Gamma\alpha$.

Definition 5.8. Suppose f is a weakly modular function of weight k and level Γ. Then f is meromorphic (resp. holomorphic, vanishing) at Γz if $f|_k[\alpha]$ is meromorphic (resp. holomorphic, vanishing) at ∞.

Lemma 5.2. *The width of Γz as well as the notion of meromorphy, holomorphy and vanishing at Γz are well-defined.*

Proof. We've chosen a representative z of Γz and $\alpha \in \Gamma(1)$ bringing ∞ to z. Suppose we chose instead another element $\beta \in \Gamma(1)$ bringing ∞ to α, then there is some $\delta \in \Gamma_\infty$ such that $\beta = \alpha\delta$. Then $\beta^{-1}\Gamma\beta \cap \Gamma_\infty = \delta^{-1}\alpha^{-1}\Gamma\alpha\delta \cap \Gamma_\infty = \delta^{-1}(\alpha^{-1}\Gamma\alpha \cap \delta^{-1}\Gamma_\infty\delta)\delta = \alpha^{-1}\Gamma\alpha \cap \Gamma_\infty$. Therefore the width of Γz according to β is the same as the width according to α. As for meromorphy etc., we are comparing $f|_k[\alpha]$ and $f|_k[\alpha\delta]$. Suppose $f|_k[\alpha] = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\tau/h}$. If we write $\delta = \pm\left(\begin{smallmatrix}1 & M\\ 0 & 1\end{smallmatrix}\right)$, then we get

$$f|_k[\alpha\delta] = f|_k[\alpha]|_k[\delta] = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\tau/h} e^{2\pi i n m/h} (-1)^k$$

Hence meromorphy etc., of $f|_k[\alpha]$ and $f|_k[\alpha\delta]$ at ∞ are equivalent.
Suppose now that $\Gamma z = \Gamma z'$. Choose $\alpha, \alpha' \in \Gamma(1)$ bringing ∞ to z, z' respectively. Choose $\gamma \in \Gamma$ such that $\alpha' = \gamma\alpha$, and therefore $(\alpha')^{-1}\Gamma\alpha' \cap \Gamma\infty = \alpha^{-1}\Gamma\alpha \cap \Gamma_\infty$, so this choice doesn't affect the width. As for meromorphy etc., one simply observe that $f|_k[\alpha'] = f|_k[\gamma]|_k[\alpha] = f|_k[\alpha]$. $\square$

Note that the q-expansion of f at Γz, however, depends on α.

Definition 5.9. Let $k \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ a congruence subgroup. Let f be a weakly modular function of weight k and level Γ.
We say f is a modular function of weight k and level Γ if it is meromorphic at every cusp of Γ. It is a modular form of weight k and level Γ if it is holomorphic

in $\mathfrak{h}$ and at every cusp. It is a cuspidal modular form (or cusp form) of weight k and level Γ if it is a modular form and it vanishes at every cusp.
We write $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) for the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight k and level Γ.

Remark. 1. If f is weakly modular (of weight k and level Γ) and holomorphic in $\mathfrak{h}$, then f is a modular form if and only if $f|_k[\alpha]$ is holomorphic at ∞ for every $\alpha \in \Gamma(1)$.
2. In example sheet, it will be shown that $M_k(\Gamma)$ is finite dimensional. It's also possible to give an exact formula for its dimension, using Riemann-Roch (at least for $k > 1$) or otherwise.

Lemma 5.3. *Let $k \in \mathbb{Z}$ and Γ be a congruence subgroup.*
(i) If $f \in M_k(\Gamma), g \in M_l(\Gamma)$, then $fg \in M_{k+l}(\Gamma)$.
(ii) If $\Gamma' \leq \Gamma$ is another congruence subgroup, then $M_k(\Gamma) \leq M_k(\Gamma')$.
(iii) If $\Gamma' \leq \Gamma(1)$ is another congruence subgroup and $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ satisfies $\Gamma' \leq \alpha^{-1}\Gamma\alpha$, then for any $f \in M_k(\Gamma)$ (resp. $S_k(\Gamma)$), $f|_k[\alpha] \in M_k(\Gamma')$ (resp. $S_k(\Gamma')$).

Proof. (i) and (ii) are immediate.
Let's prove (iii). Observe that if $g : \mathfrak{h} \to \mathbb{C}$ is weakly modular function of weight k and level Γ holomorphic in $\mathfrak{h}$. Then g is holomorphic (resp. vanishing) at infinity if and only if $g(\tau)$ is bounded as $\tau \to \infty$ (resp. tends to 0 as $\tau \to \infty$). This is because a holomorphic $h : D^*(0,1) \to \mathbb{C}$ has a removable singularity at 0 iff h is bounded in $D^*(0,\delta)$ for some $\delta > 0$.
Now if $f \in M_k(\Gamma)$ (resp. $S_k(\Gamma)$), then $f|_k[\alpha]$ is holomorphic in $\mathfrak{h}$ and weakly modular of weight k and level Γ' because $f|_k[\alpha]|_k[\gamma'] = f|_k[\alpha\gamma'\alpha^{-1}]|_k[\alpha] = f|_k[\alpha]$ as $\alpha\Gamma'\alpha^{-1} \leq \Gamma$.
To show that $f|_k[\alpha]$ is holomorphic (resp. vanishing) at cusps, it is enough to show that $f|_k[\alpha\beta]$ is holomorphic (resp. vanishing) at ∞ for any $\beta \in \Gamma(1)$. Our previous remark means that we just need to check that it's bounded (resp. vanishing) at ∞. Write $\alpha\beta\infty = \gamma\infty$ for some $\gamma \in \Gamma(1)$. Then $\alpha\beta = \gamma\delta$ for some $\delta \in \mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Q})^+}(\infty)$. Write $\delta = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ for $a, b, d \in \mathbb{Q}, ab \neq 0$. Then

$$f|_k[\alpha\beta](\tau) = f_k[\gamma]\left(\frac{a\tau + b}{d}\right)(ad)^{k-1}d^{-k}$$

Since f is a modular form, $f|_k[\gamma]$ is bounded (resp. vanishing) at ∞, so $f|_k[\alpha\beta]$ must also be bounded (resp. vanishing) at ∞. ☐

Corollary 5.4. *Suppose $f \in M_k(\Gamma(1))$ and $N \in \mathbb{N}$, then $f(N\tau) \in M_k(\Gamma_0(N))$.*

Proof. Let $\alpha = \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $f|_k[\alpha](\tau) = f(N\tau)N^{k-1}$ For this to live in $M_k(\Gamma_0(N)$, we need only to check $\alpha\Gamma_0(N)\alpha^{-1} \leq \Gamma(1)$ by the preceding lemma. But this is clear. ☐

Example 5.3. $G_k(N\tau) \in M_k(\Gamma_0(N))$.

5.2 Jacobi's ϑ-Function

Let's consider the function $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = 1 + 2\sum_{n\geq 1} q_2^{n^2}$. This is holomorphic in $\mathfrak{h}$ and is invariant under $\tau \mapsto \tau + 2$. How does it transform under elements of $\mathrm{SL}_2(\mathbb{Z})$?

Proposition 5.5 (Poisson Summation Formula)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function. Suppose that there are $C, \delta > 0$ such that $|f(t)| \leq C/(1+|t|)^{\delta+1}$ for all $t \in \mathbb{R}$. Then*

$$\hat{f}(s) = \int_{t=-\infty}^{\infty} f(t)e^{-2\pi i s t}\,\mathrm{d}t$$

converges. If in addition that $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Proof. Define $F(t) = \sum_{n\in\mathbb{Z}} f(t,n)$, which converges absolutely and uniformly on any bounded interval by Weierstrass M-test, hence defines continuous function $\mathbb{R} \to \mathbb{C}$. We als have $F(t) = F(t+1)$.
Define $G(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi i n t}$. Again by Weierstrass M-test, this converges absolutely and uniformly on $\mathbb{R}$, so G is continuous and too has $G(t) = G(t+1)$. The claim now is $F = G$, which implies the proposition by setting $t = 0$. It suffices to check that $\hat{F}(n) = \hat{G}(n)$ for all n where

$$\hat{H}(n) = \int_{t=0}^{1} H(t)e^{-2\pi i n t}$$

But we can simply calculate, noting absolute and uniform convergence,

$$\begin{aligned}
\hat{F}(n) &= \int_{t=0}^{1} \sum_{m\in\mathbb{Z}} f(m+t)e^{-2\pi i n t}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} \int_{t=0}^{1} f(n+t)e^{-2\pi i n t}\,\mathrm{d}t \\
&= \sum_{m\in\mathbb{Z}} \int_{t=m}^{m+1} f(t)e^{-2\pi i n t}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} f(t)e^{-2\pi i n t}\,\mathrm{d}t = \hat{f}(n) \\
\hat{G}(n) &= \int_{t=0}^{1} \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{2\pi i(m-n)t}\,\mathrm{d}t = \sum_{m\in\mathbb{Z}} \hat{f}(m) \int_{t=0}^{1} e^{2\pi(m-n)t}\,\mathrm{d}t = \hat{f}(n)
\end{aligned}$$

which are the same. □

Let $f_y(t) = e^{-\pi y t^2}$, then $\vartheta(iy) = \sum_{n\in\mathbb{Z}} f_y(n)$ for $y > 0$. We have

$$\begin{aligned}
\hat{f}_y(s) &= \int_{t=-\infty}^{\infty} e^{-\pi t^2 y}e^{-2\pi i s t}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} e^{-\pi(t\sqrt{y}+is/\sqrt{y})^2}e^{-\pi s^2/y}\,\mathrm{d}t \\
&= \frac{e^{-\pi s^2/y}}{\sqrt{y}} \int_{t=-\infty}^{\infty} e^{-\pi(t\sqrt{y}+is/\sqrt{y})^2}\,\mathrm{d}t = \frac{e^{-\pi s^2/y}}{\sqrt{y}} \int_{\gamma} f(z)\,\mathrm{d}z
\end{aligned}$$

where γ is the the horizontal line $\operatorname{Im} z = 2/\sqrt{y}$ and $f(z) = e^{-\pi z^2}$. By going around a rectangular contour (easy exercise), we have

$$\hat{f}_y(s) = \frac{e^{-\pi s^2/y}}{\sqrt{y}} \int_{t=-\infty}^{\infty} e^{-\pi t^2/2} = \frac{e^{-\pi s^2/y}}{\sqrt{y}} = \frac{1}{\sqrt{y}} f_{y^{-1}}(s)$$

The preceding proposition the gives us

$$\vartheta(iy) = \sum_{n\in\mathbb{Z}} f_y(n) = \sum_{n\in\mathbb{Z}} \hat{f}_y(n) = \frac{1}{\sqrt{y}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \vartheta\left(\frac{1}{y}\right)$$

Consequently, the holomorphic functions $\vartheta(\tau)$ and $(\sqrt{\tau/i})^{-1}\vartheta(-1/\tau)$ on $\mathfrak{h}$ (we choose a branch of the square root so that it takes positive values on the positive imaginary line) agree on $i\mathbb{R}_{>0}$, so they must be equal on $\mathfrak{h}$ by identity principle. But this looks like ϑ being invariant under a certain modular action of "weight 1/2"!

How do we deal with stuff like this? For $k \in 8\mathbb{N}$, we have $\vartheta^k = \tau^{-k/2}\vartheta(-1/\tau)^k = \vartheta^k|_{k/2}[S]$.

Proposition 5.6. *For $k \in 8\mathbb{N}$, $\vartheta^k \in M_{k/2}(\Gamma)$ where $\Gamma = \Gamma(2) \cup S\Gamma(2)$.*

Proof. ϑ^k is holomorphic in $\mathfrak{h}$ and is invariant under the weight $k/2$ actions of S, T^2. So ϑ^k is invariant under the weight $k/2$ action of $\langle S, T^2\rangle$, which equals Γ (example sheet). Therefore ϑ^k is weakly modular of weight $k/2$ and levele Γ. It is holomorphic at ∞ since we can write down a q-expansion $\vartheta^k(\tau) = (1 + \sum_{n\geq 1} q_2^{n^2})^k$. It remains to show that ϑ^k is holomorphic at the other cusps.

Let's compute the cusps. They corresponds to Γ-orbits on $\mathbb{P}^1(\mathbb{Q})$, which corresponds to $\Gamma\backslash\Gamma(1)/\Gamma_\infty$. Now $\Gamma(2) \lhd \Gamma(1)$ has an isomorphism $\Gamma(2)\backslash\Gamma(1) \to \mathrm{SL}_2(\mathbb{F}_2)$. So the cusps further corresponds to $\langle S\rangle\backslash \mathrm{SL}_2(\mathbb{F}_2)/\langle T\rangle$. We can compute this the easy way, or we can feel fancy and consider the $\mathrm{SL}_2(\mathbb{F}_2)$-action on $\mathbb{P}^1(\mathbb{F}_2) = \{[1:0], [1:1], [0,1]\}$. The $\mathrm{SL}_2(\mathbb{F}_2)$-stabiliser of $[1:0]$ is just $\langle T\rangle$. So we get another bijection between $\langle S\rangle\backslash \mathrm{SL}_2(\mathbb{F}_2)/\langle T\rangle$ and $\langle S\rangle\backslash\mathbb{P}^1(\mathbb{F}_2)$, which has two elements (yay!).

If we trace back through this bunch of bijections, we find that the cusp which is not ∞ is $\Gamma\gamma\infty$ for any $\gamma \in \Gamma(1)$ such that $(\gamma \bmod 2)(1,0)^\top = (1,1)^\top$. Let's take $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, so $\gamma\infty = 1 \in \mathbb{P}^1(\mathbb{Q})$.

It remains to show that $\vartheta^k|_{k/2}[\gamma]$ is holomorphic at ∞. We have $\vartheta(\tau+1) = \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i n^2\tau}$. So $\vartheta(\tau) + \vartheta(\tau+1) = 2\vartheta(4\tau)$. We have $\gamma\tau = 1 - 1/\tau$. The identity above gives

$$\begin{aligned}\vartheta\left(1 - \frac{1}{\tau}\right) &= 2\vartheta\left(-\frac{4}{\tau}\right) - \vartheta\left(-\frac{1}{\tau}\right) \\ &= 2\sqrt{\frac{\tau}{4i}}\vartheta\left(\frac{\tau}{4}\right) - \sqrt{\frac{\tau}{i}}\vartheta(\tau) = \sqrt{\frac{\tau}{i}}\left(\vartheta\left(\frac{\tau}{4}\right) - \vartheta(\tau)\right)\end{aligned}$$

So $\vartheta(1 - 1/\tau)(\sqrt{\tau/i})^{-1} = \vartheta(\tau/4) - \vartheta(\tau)$, hence $\vartheta^k|_{k/2}[\gamma] = (\vartheta(\tau/4) - \vartheta(\tau))^k$, which means that ϑ^k is holomorphic (in fact vanishing) at this cusp. □

Remark. The fundamental domain for Γ is a hyperbolic triangle with vertices $-1, 0, \infty$ and zero angles. $0, \infty$ are the same cusp since they are Γ-conjugates, and -1 is the extra cusp.

Now to the exciting part. For any $k \in \mathbb{N}$, we have $\vartheta^k = \sum_{m\geq 0} r_k(n) q_2^m$ where $r_k(n) = \#\{n \in \mathbb{Z}^k : \sum_i n_i^2 = m\}$.

Theorem 5.7 (Ramanujan)**.** *Let $n \in \mathbb{N}$, then*

$$r_{24}(n) = \frac{65536}{691}\sigma_{11}\left(\frac{n}{2}\right) - \frac{16}{691}(-1)^n\sigma_{11}(n) - \frac{65536}{691}\tau\left(\frac{n}{2}\right) - \frac{33152}{691}(-1)^n\tau(n)$$

Remark. Here and always, we use the convention that a function is zero wherever it's undefined.

Consequently, if n is odd, then $r_{24}(n) = (16/691)\sigma_{11}(n) + (33152/691)\tau(n)$.

Proof. We just showed that $\vartheta^{24} = \sum_{n\geq 0} r_{24}(n) q_2^n \in M_{12}(\Gamma)$. If we know a basis for $M_{12}(\Gamma)$, then we can express this in terms of functions we already know well.
In example sheet, you'll show that (for any k and any congruence subgroup Γ) $\dim_{\mathbb{C}} M_k(\Gamma) \leq 1 + k[\Gamma(1) : \Gamma]/12$. So for our Γ we have $\dim_{\mathbb{C}} M_{12}(\Gamma) \leq 1 + [\Gamma(1) : \Gamma] = 1 + [\mathrm{SL}_2(\mathbb{F}_2) : \langle S \rangle] = 1 + 3 = 4$.
Let's now find four linearly independent elements of $M_{12}(\Gamma)$. $M_{12}(\Gamma(1))$ is contained in $M_{12}(\Gamma)$, so we get $\Delta, F_{12} \in M_{12}(\Gamma)$ for free. Let's just recall

$$F_{12}(\tau) = \frac{691}{65520} + \sum_{n\geq 1} \sigma_{11}(n) q^n$$

How do we get more? Let's consider $\alpha = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})^+$. We claim that $\Gamma \leq \alpha^{-1}\Gamma(1)\alpha$, or $\alpha\Gamma\alpha^{-1} \leq \Gamma(1)$. This just follows from a computation

$$\alpha \begin{pmatrix} A & B \\ C & D \end{pmatrix} \alpha^{-1} = \begin{pmatrix} A + C & (B + D - (A + C))/2 \\ 2C & D - C \end{pmatrix}$$

If $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in $\Gamma(2)$ then $A \equiv D \equiv 1 \pmod 2$ and $B \equiv C \equiv 0 \pmod 2$. If $\gamma \in S\Gamma(2)$, then $A \equiv D \equiv 0 \pmod 2$ and $B \equiv C \equiv 1 \pmod 2$. In either case, the final expression is in $\Gamma(1)$.
So $\Delta|_{12}[\alpha](\tau) = \Delta((\tau + 1)/2), F_{12}|_{12}[\alpha](\tau) = F_{12}((\tau + 1)/2)$ are both in $M_k(\Gamma)$. Let's compute them

$$\Delta\left(\frac{\tau+1}{2}\right) = \sum_{n\geq 1} \tau(n)(-1)^n q_2^n$$
$$F_{12}\left(\frac{\tau+1}{2}\right) = \frac{691}{65520} + \sum_{n\geq 1} (-1)^n \sigma_{11}(n) q_2^n$$

It's easy to check from these expressions that these are linearly independent over $\mathbb{C}$ (by viewing them as elements of $\mathbb{C}[[q_2]]/(q_2^4)$). So they form a basis for $M_{12}(\Gamma)$, and we can then very easily compute (by looking at the first few coefficients)

$$\vartheta^{24} = \frac{65536}{691} F_{12}(\tau) - \frac{16}{691} F_{12}\left(\frac{\tau+1}{2}\right) - \frac{65536}{691}\Delta(\tau) - \frac{33152}{691}\Delta\left(\frac{\tau+1}{2}\right)$$

Expanding all of them in q_2 gives the formula. □

Another application of ϑ is the meromorphic continuation of ζ.

Theorem 5.8. *Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then ξ admits a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 1, 0$ (of residues $1, -1$ respectively) and no other poles. Furthermore, it satisfies $\xi(s) = \xi(1 - s)$.*

Proof. Consider

$$X(s) = \int_{y=0}^{\infty} (\vartheta(iy) - 1) y^{s/2} \frac{\mathrm{d}y}{y}$$

Now $\vartheta(y) = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 y}$, so $|\vartheta(y) - 1| = O(e^{-\pi y})$ as $y \to \infty$, so the part

$$\int_{y=1}^{\infty} (\vartheta(iy) - 1) y^{s/2} \frac{\mathrm{d}y}{y}$$

converges absolutely for all $s \in \mathbb{C}$ and defines an entire function, using the same argument used in the proof of Proposition 4.2.
We know $\vartheta(iy) = (\sqrt{y})^{-1}\vartheta(i/y)$, so $\vartheta(iy) - 1 = (\vartheta(i/y) - 1)(\sqrt{y})^{-1} + ((\sqrt{y})^{-1} - 1) \sim 1/\sqrt{y}$ as $y \to 0$. This tells us that

$$\int_{y=0}^{1} (\vartheta(iy) - 1) y^{s/2} \frac{\mathrm{d}y}{y}$$

converges when $\operatorname{Re} s > 1$ and defines a holomorphic function there. On the other hand, for $\operatorname{Re} s > 1$,

$$\begin{aligned} X(s) &= \int_{y=0}^{\infty} 2 \sum_{n \geq 1} e^{-\pi n^2 y} y^{s/2} \frac{\mathrm{d}y}{y} = 2 \sum_{n \geq 1} \int_{y=0}^{\infty} e^{-\pi n^2 y} y^{2/s} \frac{\mathrm{d}y}{y} \\ &= 2 \sum_{n \geq 1} \pi^{-s/2} n^{-s} \Gamma(s) = 2\xi(s) \end{aligned}$$

Now let's show that X converges to a meromorphic function on $\mathbb{C}$ with the desired poles and functional equation. We have

$$\begin{aligned} \int_{y=0}^{1} (\vartheta(iy) - 1) y^{s/2} \frac{\mathrm{d}y}{y} &= \int_{y=1}^{\infty} (\vartheta(i/y) - 1) y^{-s/2} \frac{\mathrm{d}y}{y} \\ &= \int_{y=1}^{\infty} (\vartheta(iy) y^{1/2} - 1) y^{-s/2} \frac{\mathrm{d}y}{y} \\ &= \int_{y=1}^{\infty} (\vartheta(iy) - 1) y^{(1-s)/2} + (y^{(1-s)/2} - y^{-s/2}) \frac{\mathrm{d}y}{y} \end{aligned}$$

So

$$X(s) = -\frac{2}{1-s} - \frac{2}{s} + \int_{y=1}^{\infty} (\vartheta(iy) - 1)(y^{(1-s)/2} + y^{s/2}) \frac{\mathrm{d}y}{y}$$

The integral defines an entire function by the estimate $|\vartheta(iy) - 1| = O(e^{-\pi y})$. So this gives what we wanted. □

We can generalise this.

5.3 ϑ-Functions and ζ-Functions of Lattices

Let $\Lambda \leq \mathbb{R}^n$ be a lattice. Its associated ϑ function is $\vartheta_\Lambda(\tau) = \sum_{\lambda \in \Lambda} e^{\pi i \langle \lambda, \lambda \rangle \tau}$. Then ϑ_Λ is holomorphic in $\mathfrak{h}$. Moreover, we have the following proposition:

Proposition 5.9. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous such that there are $C, \delta > 0$ with $|f(t)| \leq C/(1 + \|t\|)^{n+\delta}$ for any $t \in \mathbb{R}^n$. Then*

$$\hat{f}(s) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \langle s, t \rangle} \, \mathrm{d}t$$

converges absolutely. Moreover, suppose $\Lambda \leq \mathbb{R}^n$ is a lattice such that we have $\sum_{\mu \in \Lambda^\vee} |\hat{f}(\mu)| < \infty$, where $\Lambda^\vee = \{s \in \mathbb{R}^n : \forall \lambda \in \Lambda, \langle s, \lambda \rangle \in \mathbb{Z}\}$ is its dual lattice, then

$$\sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{\operatorname{covol}(\Lambda)} \sum_{\mu \in \Lambda^\vee} \hat{f}(\mu)$$

Proof. Similar to Proposition 5.5 □

Let's apply this to $f(t) = e^{-\pi\langle t,t\rangle}$.

$$\vartheta_\Lambda(iy) = \sum_{\lambda\in\Lambda} e^{-\pi\langle\lambda,\lambda\rangle y} = \sum_{\lambda\in\sqrt{y}\Lambda} e^{-\pi\langle\lambda,\lambda\rangle}$$

So we take the lattice to be $\sqrt{y}\Lambda$. Note that $f = \hat{f}$, so

$$\vartheta_\Lambda(iy) = y^{-n/2}\operatorname{covol}(\Lambda)^{-1} \sum_{\mu\in y^{-1/2}\Lambda^\vee} f(\mu) = y^{-n/2}\operatorname{covol}(\Lambda)^{-1}\vartheta_{\Lambda^\vee}\left(\frac{i}{y}\right)$$

So by the identity principle we have

$$\vartheta_\Lambda(\tau) = \left(\sqrt{\frac{\tau}{i}}\right)^{-n/2} \operatorname{covol}(\Lambda)^{-1}\vartheta_{\Lambda^\vee}\left(-\frac{1}{\tau}\right)$$

Proposition 5.10. *Suppose $n \in 8\mathbb{N}$ and $\Lambda \leq \mathbb{R}^n$ is a lattice satisfying:*
(i) $\Lambda = \Lambda^\vee$.
(ii) Λ is even, i.e. $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$.
Then $\vartheta_\Lambda \in M_{n/2}(\Gamma(1))$.

Proof. $\vartheta_\Lambda(\tau) = \sum_{\lambda\in\Lambda} e^{\pi i\langle\lambda,\lambda\rangle\tau} = \sum_{n\geq 0} r_\Lambda(n)q^n$, where $r_\Lambda(n) = \#\{\lambda \in \Lambda : \langle\lambda,\lambda\rangle = 2n\}$. This shows that $\vartheta_\Lambda(\tau) = \vartheta_\Lambda(\tau+1)$ and $\vartheta|_{n/2}[S](\tau) = \vartheta_{\Lambda^\vee}(\tau) = \vartheta_\Lambda(\tau)$ by the formula we obtained. Since S, T generate $\Gamma(1)$, ϑ_Λ is weakly modular of weight $n/2$ and level $\Gamma(1)$. The expansion we got shows that it is holomorphic at ∞. □

Example 5.4. Consider the lattice $\Lambda_{\mathrm{E}_8} \leq \mathbb{R}^8$ which corresponds to the biggest exceptional Lie group E_8. It happens to be self-dual and even. So $\vartheta_{\Lambda_{\mathrm{E}_8}} \in M_4(\Gamma(1))$. In fact $\vartheta_{\Lambda_{\mathrm{E}_8}}(\tau) = E_4(\tau) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$. In particular, the equality $240 = a_1(E_4)$ can be interpreted as saying 240 is the number of roots in the E_8 root system.

Definition 5.10. For a lattice $\Lambda \leq \mathbb{R}^n$, its Epstein ζ-function is $\zeta_\Lambda(s) = \sum_{\lambda\in\Lambda\setminus\{0\}}\langle\lambda,\lambda\rangle^{-s}$.

Example 5.5. $\zeta_{\mathbb{Z}}(s) = \zeta(s)$.

Theorem 5.11. *Let $\xi_\Lambda(s) = \pi^{-s}\Gamma(s)\zeta_\Lambda(s)$, then:*
(i) ξ_Λ admits a meromorphic continuation to $\mathbb{C}$ with only simple poles at $s = n/2, 0$ of residues $\operatorname{covol}(\Lambda)^{-1}, -1$, respectively.
(ii) We have a functional equation $\xi_\Lambda(n/2 - s) = \operatorname{covol}(\Lambda)^{-1}\xi_{\Lambda^\vee}(s)$.

Remark. 1. $\xi(s) = \xi_{\mathbb{Z}}(s/2)$.
2. ξ_Λ usually does not have an Euler product, so we disqualify it (sad) from being an L-function.

Proof. We'll not take as much care as previously in convergence issues since it's always gonna be the same kind of arguments.
Consider

$$X_\Lambda(s) = \int_{y=0}^{\infty}(\vartheta_\Lambda(iy) - 1)y^s\,\frac{\mathrm{d}y}{y}$$

which for suitably large $\operatorname{Re} s$ has

$$\begin{aligned} X_\Lambda(s) &= \int_{y=0}^{\infty} \sum_{\lambda\in\Lambda\setminus\{0\}} e^{-\pi\langle\lambda,\lambda\rangle y} y^s \frac{\mathrm{d}y}{y} = \sum_{\lambda\in\Lambda\setminus\{0\}} \pi^{-s}\langle\lambda,\lambda\rangle^{-s}\Gamma(s) \\ &= \pi^{-1}\pi^{-s}\Gamma(s)\zeta_\Lambda(s) = \xi_\Lambda(s) \end{aligned}$$

So let's show that X_Λ defines a meromorphic function on $\mathbb{C}$ with the desired properties. As per usual,

$$\begin{aligned} X_\Lambda(s) &= \int_{y=0}^{1} (\vartheta_\Lambda(iy)-1) y^s \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} (\vartheta_\Lambda(iy)-1) y^s \frac{\mathrm{d}y}{y} \\ &= \int_{y=1}^{\infty} (\vartheta_\Lambda(i/y)-1) y^{-s} \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} (\vartheta_\Lambda(iy)-1) y^{-s} \frac{\mathrm{d}y}{y} \\ &= \int_{y=1}^{\infty} (\mathrm{covol}(\Lambda)^{-1} y^{n/2}\vartheta_{\Lambda^\vee}(iy)-1) y^{-s} \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} (\vartheta_\Lambda(iy)-1) y^s \frac{\mathrm{d}y}{y} \\ &= \frac{1}{\mathrm{covol}(\Lambda)} \int_{y=1}^{\infty} (\vartheta_{\Lambda^\vee}(iy)-1) y^{n/2-s} \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} (\vartheta_\Lambda(iy)-1) y^s \frac{\mathrm{d}y}{y} \\ &\quad + \int_{y=1}^{\infty} \frac{1}{\mathrm{covol}(\Lambda)} y^{n/2-s} - y^{-s} \frac{\mathrm{d}y}{y} \\ &= \int_{y=1}^{\infty} \frac{1}{\mathrm{covol}(\Lambda)} (\vartheta_{\Lambda^\vee}(iy)-1) y^{n/2-s} + (\vartheta_\Lambda(iy)-1) y^s \frac{\mathrm{d}y}{y} \\ &\quad - \frac{\mathrm{covol}(\Lambda)^{-1}}{n/2-s} - \frac{1}{s} \end{aligned}$$

which gives the result. □

6 Non-Holomorphic Eisenstein Series

6.1 Definition and Modularity

Modular forms are just the start of the story. A more general point of view is the study of the decomposition of $L^2(\Gamma(1)\backslash \mathrm{SL}_2(\mathbb{R}))$ as a representation of $\mathrm{SL}_2(\mathbb{R})$. It turns out that for $k \geq 2$, there is an irreducible representation D_k of $\mathrm{SL}_2(\mathbb{R})$ such that

$$S_k(\Gamma(1)) \cong \mathrm{Hom}_{\mathrm{SL}_2(\mathbb{R})}(D_k, L^2(\Gamma(1)\backslash \mathrm{SL}_2(\mathbb{R})))$$

The rest of this L^2 space are described in terms of automorphic forms. In the remainder of the course, we'll study some examples of automorphic forms, namely non-holomorphic Eisenstein series.

Definition 6.1. Let $s \in \mathbb{C}, \operatorname{Re} s > 1$. The non-holomorphic Eisenstein series with parameter s is

$$G(\tau, s) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \mathrm{Im}(\tau)^s |m\tau+n|^{-2s}$$

for $\tau \in \mathfrak{h}$.

It's easy to check that this converges absolutely and locally uniformly in $\mathfrak{h} \times \{\operatorname{Re} s > 1\}$. So it is holomorphic as a function of s. But it's not holomorphic as a function of τ.
Let's check how $G(\tau, s)$ transform under $\Gamma(1)$.

$$\begin{aligned} G(\tau,s) &= \sum_{d\in\mathbb{N}} \sum_{(m,n)\in\mathbb{Z}^2,\gcd(m,n)=d} \operatorname{Im}(\tau)^s |m\tau+n|^{-2s} \\ &= \sum_{d\in\mathbb{N}} d^{-2s} \sum_{(m,n)\in\mathbb{Z}^2,\gcd(m,n)=1} \operatorname{Im}(\tau)^s |m\tau+n|^{-2s} \\ &= 2\zeta(2s) \sum_{(m,n)\in\mathbb{Z}^2/\{\pm 1\},\gcd(m,n)=1} \operatorname{Im}(\tau)^s |m\tau+n|^{-2s} \\ &= 2\zeta(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \operatorname{Im}(\gamma\tau)^s = 2\zeta(2s)E(\tau,s) \end{aligned}$$

where $E(\tau, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \operatorname{Im}(\gamma\tau)^s$, which again converges absolutely and locally uniformly for $\tau \in \mathfrak{h}, \operatorname{Re} s > 1$. Now $E(\delta\tau, s) = E(\tau, s)$ for any $\delta \in \Gamma(1)$, so $G(\tau, s) = G(\delta\tau, s)$ for any $\delta \in \Gamma(1)$.
Let's now try to meromorphically continue $G(\tau, s)$ as a function of s. This is possible, as $G(\tau, s)$ is secretly an Epstein ζ-function: Recall $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \leq \mathbb{C}$ which we can identify as $\mathbb{R}^2$ using the $\mathbb{R}$-basis $1, i$.

Proposition 6.1. *$G(\tau, s) = \zeta_{y^{-1/2}\Lambda_\tau}(s)$ where $\tau = x + iy$.*

Proof.

$$\begin{aligned} \zeta_{y^{-1/2}\Lambda_\tau}(s) &= \sum_{\lambda\in y^{-1/2}\Lambda_\tau-\{0\}} \langle\lambda,\lambda\rangle^{-s} = \sum_{\lambda\in\Lambda_\tau-\{0\}} \langle y^{-1/2}\lambda, y^{-1/2}\lambda\rangle^{-s} \\ &= \sum_{(m,n)\in\mathbb{Z}^2-\{0\}} \operatorname{Im}(\tau)^s|m\tau+n|^{-s} = G(\tau,s) \end{aligned}$$ □

Lemma 6.2. $\operatorname{covol}(y^{-1/2}\Lambda_\tau) = 1, (y^{-1/2}\Lambda_\tau)^\vee = iy^{-1/2}\Lambda_\tau$.

Proof. $y^{-1/2}\Lambda_\tau$ has basis $xy^{-1/2} + iy^{1/2}, y^{-1/2}$, so

$$\operatorname{covol}(y^{-1/2}\Lambda_\tau) = \left|\det\begin{pmatrix} xy^{-1/2} & y^{-1/2} \\ y^{1/2} & 0 \end{pmatrix}\right| = 1$$

On the other hand, $iy^{-1/2}\Lambda_\tau$ has basis $iy^{-1/2}, y^{1/2} - iy^{-1/2}x$, which is dual to $xy^{-1/2} + iy^{1/2}, y^{-1/2}$. □

Theorem 6.3. *Let $G^*(\tau, s) = \pi^{-s}\Gamma(s)G(\tau, s)$. Then:*
(i) For fixed τ, $G^*(\tau, s)$ admits a meromorphic continuation to $\mathbb{C}$ with only simple poles at $s = 1, 0$ of residues $1, -1$, respectively.
(ii) $G^*(\tau, s) = G^*(\tau, 1 - s)$.
(iii) $G^*(\tau, s) - 1/(s(s - 1))$ extends to a C^∞ function on $\mathfrak{h} \times \mathbb{C}$.

Proof. (i), (ii): Both follow from Theorem 5.11 and our previous discussion.
(iii): We have

$$G^*(\tau,s) = \frac{1}{s(s-1)} + \int_{t=1}^{\infty} \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} e^{-\pi i|m\tau+n|^2 t/y} t^s \frac{\mathrm{d}t}{t}$$

and the second term is C^∞. □

We know $G^*(x+iy,s) = G^*(x+1+iy,s)$ by its invariant under $\Gamma(1)$, so we still have a Fourier expansion

$$G^*(x+iy,s) = \sum_{k\in\mathbb{Z}} A_k^*(y,s)e^{2\pi ikx}, A_k^*(y,s) = \int_{x=0}^{1} G^*(x+iy,s)e^{-2\pi ikx}\,\mathrm{d}x$$

Since $G^*(\tau,s)$ is C^∞ in $\mathfrak{h}\times(\mathbb{C}-\{0,1\})$, we know that $A_k^*(y,s)$ is C^∞ in $(0,\infty)\times(\mathbb{C}-\{0,1\})$, and also holomorphic as a function of s.

6.2 Computing Fourier Coefficients

Theorem 6.4. $A_0^*(y,s) = 2\xi(2s)y^s + 2\xi(2(1-s))y^{1-s}$.

Proof. Even Jack said it's a painful computation.
Both sides of this equality are holomorphic in $s\in\mathbb{C}-\{0,1\}$. By the identity principle, it suffices to show that it holds when $\operatorname{Re} s > 1$. Under this assumption, we have

$$\begin{aligned}
A_0^*(y,s) &= \int_{x=0}^{1} G^*(\tau,s)\,\mathrm{d}x = \int_{x=0}^{1}\int_{t=0}^{\infty}(\vartheta_{y^{-1/2}\Lambda_\tau}(it)-1)t^s\frac{\mathrm{d}t}{t}\\
&= \int_{x=0}^{1}\int_{t=0}^{\infty}\sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} e^{-\pi|m\tau+n|^2t/y}t^s\frac{\mathrm{d}t}{t}\,\mathrm{d}x = I_{m=0}+I_{m\neq 0}\\
I_{m=0} &= 2\sum_{n\geq 1}\int_{x=0}^{1}\int_{t=0}^{\infty} e^{-\pi n^2t/y}t^s\frac{\mathrm{d}t}{t}\,\mathrm{d}x\\
I_{m\neq 0} &= 2\sum_{m\geq 1}\int_{t=0}^{\infty}\int_{x=0}^{1}\sum_{n\in\mathbb{Z}} e^{-\pi|m\tau+n|^2t/y}t^s\,\mathrm{d}x\,\frac{\mathrm{d}t}{t}
\end{aligned}$$

We have

$$I_{m=0} = 2\sum_{n\geq 1}\int_{t=0}^{\infty} e^{-\pi n^2t/y}t^s\frac{\mathrm{d}t}{t} = 2y^s\pi^{-s}\Gamma(s)\zeta(2s) = 2\xi(2s)y^s$$

To compute $I_{m\neq 0}$, first observe that $e^{-\pi|m\tau+n|^2t/y} = e^{-\pi(mx+n)^2t/y}e^{-\pi m^2ty}$. We have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\int_{x=0}^{1} e^{-\pi(m\tau+n)^2t/y}\,\mathrm{d}x &= \sum_{n\in\mathbb{Z}}\frac{1}{m}\int_{x=n}^{n+m} e^{-\pi x^2t/y}\,\mathrm{d}x\\
&= \int_{x=-\infty}^{\infty} e^{-\pi x^2t/y} = \sqrt{\frac{y}{t}}
\end{aligned}$$

Therefore

$$\begin{aligned}
I_{m\neq 0} &= 2\sum_{m\geq 1}\int_{t=0}^{\infty} e^{-\pi m^2ty}\sqrt{y}t^{s-1/2}\frac{\mathrm{d}t}{t}\\
&= 2\sum_{m\geq 1}\pi^{1/2-s}m^{2(1/2-s)}y^{1/2-s}y^{1/2}\Gamma\left(s-\frac{1}{2}\right)\\
&= 2\pi^{(1-2s)/2}\zeta(2s-1)\Gamma\left(\frac{2s-1}{2}\right)y^{1-2}\\
&= 2\xi(2s-1)y^{1-s} = 2\xi(1-(2s-1))y^{1-s} = 2\xi(2(1-s))y^{1-s}
\end{aligned}$$ □

To compute $A_k^*(y,s)$, we introduce the Bessel function $K_s(c)$, defined for $c \in (0,\infty)$ and $s \in \mathbb{C}$ by

$$K_s(c) = \int_{t=0}^{\infty} e^{-c(t+t^{-1})} t^s \frac{\mathrm{d}t}{t} = \int_{t=1}^{\infty} e^{-c(t+t^{-1})}(t^s + t^{-s})\frac{\mathrm{d}t}{t}$$

Theorem 6.5. *If $k \neq 0$, then $A_k^*(y,s) = 2\sqrt{y}|k|^{s-1/2}\sigma_{1-2s}(|k|)K_{s-1/2}(\pi|k|y)$ where $\sigma_{1-2s}(|k|) = \sum_{m||k|} m^{1-2s}$.*

Proof. Even Jack said it's a *really* painful calculation.
Again both sides are holomorphic, so we need only to prove equality in $\{\operatorname{Re} s > 1\}$. Consider the expression

$$A_k^*(y,s) = \sum_{(m,n)\in\mathbb{Z}^2-\{(0,0)\}} \int_{t=0}^{\infty} \left(\int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y} e^{-2\pi i k x}\,\mathrm{d}x \right) e^{-\pi m^2 t y} t^s \frac{\mathrm{d}t}{t}$$

The terms with $m = 0$ vanish, so

$$A_k^*(y,s) = 2\sum_{m\geq 1} \int_{t=0}^{\infty} e^{-\pi m^2 t y} \sum_{n\in\mathbb{Z}} \left(\int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y} e^{-2\pi i k x}\,\mathrm{d}x \right) t^s \frac{\mathrm{d}t}{t}$$

We have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \int_{x=0}^{1} e^{-\pi(mx+n)^2 t/y} e^{-2\pi i k x}\,\mathrm{d}x \\
&= \frac{1}{m}\sum_{m\in\mathbb{Z}} \int_{x=n}^{n+m} e^{-\pi x^2 t y} e^{-2\pi i k x/m} e^{2\pi i k n/m}\,\mathrm{d}x \\
&= \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \frac{1}{m} e^{2\pi i k a/m} \sum_{n\in\mathbb{Z}, n\equiv a \pmod m} \int_{x=n}^{n+m} e^{-\pi x^2 t/y - 2\pi i k x/m}\,\mathrm{d}x \\
&= 1_{m|k} \int_{x=-\infty}^{\infty} e^{-\pi x^2 t/y - 2\pi i k x/m}\,\mathrm{d}x \\
&= 1_{m|k} \int_{x=-\infty}^{\infty} e^{-\pi(x t^{1/2}/y^{1/2} + i k y^{1/2}/(m t^{1/2}))^2} e^{-\pi k^2 y/(m^2 t)}\,\mathrm{d}x \\
&= 1_{m|k} e^{-\pi k^2 y/(m^2 t)} y^{1/2} t^{-1/2}
\end{aligned}$$

So

$$\begin{aligned}
A_k^*(y,s) &= 2\sum_{m\geq 1, m|k} \int_{t=0}^{\infty} e^{-\pi m^2 t y} e^{-\pi k^2 y/(m^2 t)} \sqrt{y}\, t^{s-1/2} \frac{\mathrm{d}t}{t} \\
&= 2\sqrt{y} \int_{m\geq 1, m|k} \int_{t=0}^{\infty} e^{-\pi|k|y((m^2/|k|)t + (|k|/m^2)t^{-1})} t^{s-1/2} \frac{\mathrm{d}t}{t} \\
&= 2\sqrt{y} \sum_{m\geq 1, m||k|} \left(\frac{|k|}{m^2}\right)^{s-1/2} K_{s-1/2}(\pi|k|y) \\
&= 2\sqrt{y}|k|^{s-1/2} \left(\sum_{m\geq 1, m||k|} m^{1-2s} \right) K_{s-1/2}(\pi|k|y)
\end{aligned}$$

□

To use this computation, we need to understand $K_s(c)$.

Lemma 6.6. *(i) If $c > 0$, then $K_s(c)$ is absolutely convergent and defines an entire function of s.*
(ii) If $c_0 > 0$ and $\sigma_0 < \sigma_1$, then there is some $C = C(c_0, \sigma_0, \sigma_1)$ such that for all $c \geq c_0$ and $s \in \mathbb{C}$ with $\sigma_0 \leq \operatorname{Re} s \leq \sigma_1$, we have $|K_s(c)| \leq Ce^{-c}$.
(iii) For all $s \in \mathbb{C}$, there is some $c \in (0, \infty)$ such that $K_s(c) \neq 0$.

Proof. (i) Routine.
(ii) As usual we set $\sigma = \operatorname{Re} s$. We have

$$|K_s(c)| \leq \int_{t=1}^{\infty} e^{-(t+t^{-1})}(t^\sigma + t^{-\sigma}) \frac{\mathrm{d}t}{t}$$

So the claim follows from the estimate

$$\int_{t=1}^{\infty} e^{-c(t+t^{-1})} t^\sigma \frac{\mathrm{d}t}{t} = O(e^{-c})$$

for $c > c_0$ and $\sigma \in [\sigma_0, \sigma_1]$. This is true because

$$\begin{aligned}\int_{t=1}^{\infty} e^{-C(t+t^{-1})} t^\sigma \frac{\mathrm{d}t}{t} &= \int_{t=1}^{2} e^{-C(t+t^{-1})} t^\sigma \frac{\mathrm{d}t}{t} + \int_{t=2}^{\infty} e^{-C(t+t^{-1})} t^\sigma \frac{\mathrm{d}t}{t} \\ &\leq e^{-c} \int_{t=1}^{2} t^{\sigma-1}\, \mathrm{d}t + e^{-c} \int_{t=2}^{\infty} e^{-c_0(t/2+t^{-1})} t^\sigma \frac{\mathrm{d}t}{t}\end{aligned}$$

(iii) Take the Mellin transform of $K_s(c)$, i.e.

$$\int_{c=0}^{\infty} K_s(c) c^{s_1} \frac{\mathrm{d}c}{c} = \int_{c=0}^{\infty} \int_{t=0}^{\infty} e^{-(ct+ct^{-1})} c^{s_1} t^s \frac{\mathrm{d}t\, \mathrm{d}c}{tc}$$

The change of variables $a = ct, b = c/t$ gives

$$\begin{aligned}\int_{c=0}^{\infty} K_s(c) c^{s_1} \frac{\mathrm{d}c}{c} &= \int_{a=0}^{\infty} \int_{b=0}^{\infty} e^{-(a+b)} (ab)^{s_1/2} (a/b)^{s/2} \frac{\mathrm{d}a\, \mathrm{d}b}{2ab} \\ &= \frac{1}{2} \Gamma\left(\frac{s_1 + s}{2}\right) \Gamma\left(\frac{s_1 - s}{2}\right)\end{aligned}$$

valid if $\operatorname{Re}(s_1 \pm s) > 0$. But can always choose s_1 such that this is true. Since Γ is nonvanishing, $K_s(c)$ cannot be zero for all $c \in (0, \infty)$. □

Corollary 6.7. *For any $s \in \mathbb{C} - \{0, 1\}$,*
(i) $G^*(\tau, s)$ is not the zero function on $\mathfrak{h}$.
(ii) $|G^*(\tau, s) - A_0^*(y, s)| = O(e^{-\pi y/s})$ as $y \to \infty$.
(iii) $|G^*(\tau, s)| = O(\max\{y^\sigma, y^{1-\sigma}\})$ as $y \to \infty$.

Remark. This could be compared with our estimate for the holomorphism Eisenstein series $|G_k(\tau) - G_k(\infty)| = O(e^{-2\pi y})$ as $y \to \infty$.

Proof. (i) If $G^*(\tau, s)$ were the zero function on $\mathfrak{h}$, then $A_k^*(y, s) = 0$ for all $\tau \in \mathfrak{h}$. But $A_1^*(y, s) = 2\sqrt{y} K_{s-1/2}(\pi y)$, but by the preceding lemma we know that there exists a choice for y making this nonzero, contradiction.
(ii) We have

$$|G^*(\tau, s) - A_0^*(y, s)| \leq \sum_{k \in \mathbb{Z} - \{0\}} 2\sqrt{y} |k|^{\sigma - 1/2} \sigma_{1-2\sigma}(|k|) |K_{s-1/2}(\pi |k| y)|$$

But we can find $M, N > 0$ such that $|k|^{\sigma-1/2}\sigma_{1-2\sigma}(|k|) \leq M|k|^N$ for all $k \in \mathbb{Z} - \{0\}$. If we let $C = C(1, \sigma, \sigma)$ be as in the lemma, then for $y \geq 1$,

$$|G^*(\tau, s) - A_0^*(y, s)| \leq 2\sum_{k\geq 1} 2\sqrt{y}k^N MCe^{-\pi ky} \leq O\left(\sum_{k\geq 1} e^{-\pi ky/2}\right) = O(e^{-\pi y/2})$$

as $y \to \infty$.
(iii) $|G^*(\tau, s)| \leq 2|\xi(2s)y^s + \xi(2(1-s))y^{1-s}| + O(e^{-\pi y/2})$ by (ii). □

6.3 The Prime Number Theorem

Theorem 6.8. *For any $t \in \mathbb{R}, t \neq 0$, $\zeta(1+it) \neq 0$.*

Proof. $\zeta(s) = \sum_{n\geq 1} n^{-s}$, so $\overline{\zeta(\overline{s})} = \zeta(s)$ for all $s \in \mathbb{C}$. So it suffices to show the case $t > 1$. Suppose $\zeta(1+it) = 0$.
Let $s_0 = (1+it)/2$, then $1 - s_0 = (1-it)/2$ and $A_0^*(y, s_0) = 2\xi(1+it)y^{s_0} + 2\xi(1-it)y^{1-s_0} = 0$. Consider the function

$$F(s) = \int_{\Gamma(1)\backslash\mathfrak{h}} G^*(\tau, s)\overline{G^*(\tau, s_0)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

which makes sense since G^* is invariant under $\Gamma(1)$. The integral converges for all $s \in \mathbb{C}$ since

$$\int_{\mathcal{F}} |G^*(\tau, s)||\overline{G^*(\tau, s_0)}|\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq C\int_{y=\sqrt{3}/2}^{\infty} \max\{y^\sigma, y^{1-\sigma}\}e^{-\pi y/2}\frac{\mathrm{d}y}{y^2}$$

for some $C > 0$. And the routine argument shows that F is entire. When $\operatorname{Re} s > 1$, $G(\tau, s) = 2\xi(2s)E(\tau, s)$ where recall $E(\tau, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} \operatorname{Im}(\gamma\tau)^s$. So

$$\begin{aligned} F(s) &= \int_{\Gamma(1)\backslash\mathfrak{h}} \int_{\gamma\in\Gamma_\infty\backslash\Gamma(1)} 2\xi(2s)\overline{G^*(\gamma\tau, s_0)}\operatorname{Im}(\gamma\tau)^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{x=-1/2}^{1/2}\int_{y=0}^{\infty} 2\xi(2s)\overline{G^*(\tau, s_0)}y^s\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{y=0}^{\infty} 2\xi(2s)\left(\int_{x=-1/2}^{1/2}\overline{G^*(\tau, s_0)}\,\mathrm{d}x\right)\frac{\mathrm{d}y}{y^2} \\ &= \int_{y=0}^{\infty} 2\xi(2s)\cdot A_0^*(y, s_0)\frac{\mathrm{d}y}{y^2} = 0 \end{aligned}$$

This means that $F(s) = 0$ for $\operatorname{Re} s > 1$. Since F is entire, F vanishes identically. Take $s = s_0$, then

$$0 = F(s_0) = \int_{\Gamma(1)\backslash\mathfrak{h}} |G^*(\tau, s_0)|^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

So $G^*(\tau, s_0) \equiv 0$, contradiction. □

7 Non-Examinable Lecture: Galois Representations

The Langlands program seeks relationship between modular forms/automorphic forms/modular representations and Galois representations/motives.
Since we only have one lecture, let's just tell a joke: What do the Prime Number Theorem and Fermat's Last Theorem have in common? (Punchline at the end.)
What is a Galois representation? Take $K/\mathbb{Q}$ a normal algebraic extension, possibly of infinite degree. We call $\text{Aut}(K/\mathbb{Q})$ the Galois group $\text{Gal}(K/\mathbb{Q})$ of this extension. This is naturally a topological group by takig a basis of neighbourhoods of the identity to be subgroups $\text{Gal}(K/M)$ where M/K is an intermediate field extension of $K/\mathbb{Q}$ with finite degree over $\mathbb{Q}$.

Definition 7.1. An n-dimensional Galois representation (of $K/\mathbb{Q}$ with coefficients in F) is a continuous homomorphism $\text{Gal}(K/\mathbb{Q}) \to \text{GL}_n(F)$ where F is a local field (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}_\ell, \mathbb{F}_p((t))$).

Example 7.1. Suppose $f(X) \in \mathbb{Z}[X]$ is separable and K is its splitting field, then any representation $\text{Gal}(K/\mathbb{Q}) \to \text{GL}_n(\mathbb{C})$ is a Galois representation.

Example 7.2. Suppose $E/\mathbb{Q}$ is an elliptic curve and ℓ is a prime, then we can associate a Galois representation $\rho_{E,\ell} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Q}_\ell)$.
Where does this come from? E has an algebraic group operation on it, and the group of ℓ^n-torsions in $E(\bar{\mathbb{Q}})$ is finite and isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^2$. This carries a natural action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. So we take the projective limit and get our $\rho_{E,\ell}$.
This is unramified at any $p \nmid \Delta_E \ell$ in the sense that it factors through $\text{Gal}(\mathbb{Q}_S/\mathbb{Q})$, where $S = \{p \text{ prime} : p \mid \Delta_E \ell\}$ and $\mathbb{Q}_S \subset \bar{\mathbb{Q}}$ is the maximal subextension unramified away from S.
For $p \notin S$, there is a distinguished conjugacy class of Frobenius elements $\text{Frob}_p \in \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$. And we have $\rho_{E,\ell}(\text{Frob}_p) = p + 1 - \#E(\mathbb{F}_p)$.

Example 7.3. Suppose $f \in S_k(\Gamma(1))$ be a normalised eigenform and let ℓ be a prime. As before we have a number field $K_f = \mathbb{Q}(\{a_n(f)\})$. Let λ be a prime ideal above ℓ, then Deligne showed that we have a Galois representation $\rho_{f,\lambda} : \text{Gal}(\mathbb{Q}_{\{\ell\}}/\mathbb{Q}) \to \text{GL}_2(K_{f,\lambda})$, unique up to isomorphism, such that $\rho_{f,\lambda}(\text{Frob}_p) = a_p(f)$.

An application of these is a generalisation of Kummer's criterion:

Theorem 7.1 (Kummer)**.** *If p is an odd prime, then p is regular (in the sense that $p \nmid \# \text{Cl}(\mathbb{Z}[e^{2\pi i/p}])$) iff none of the rational numbers $B_k, k = 2, 4, 6, \dots, p-3$ has numerator divisible by p.*

This is nice because it's reasonably easy to show that

Theorem 7.2. *Fermat's Last Theorem holds for all regular prime exponent p.*

p is regular iff $p \nmid \# \text{Cl}(\mathbb{Z}[e^{2\pi i/p}])$ iff the p-torsion $C_p = \text{Cl}(\mathbb{Z}[e^{2\pi i/p}])$ vanishes. On the other hand, $\text{Gal}(\mathbb{Q}(e^{2\pi i/p}/\mathbb{Q}))$ acts on $\text{CL}(\mathbb{Z}[e^{2\pi i/p}])$, hence on C_p. Since that Galois group has size $p-1$, we have a decomposition $C_p = \bigoplus_\chi C_{p,\chi}$ where the sum is taken over hommorphisms $\chi : \text{Gal}(\mathbb{Q}(e^{2\pi i/p}/\mathbb{Q})) \cong (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{F}_p^\times$ and

$$C_{p,\chi} = \{\sigma \in C_p : \forall \sigma \in \text{Gal}(\mathbb{Q}(e^{2\pi i/p}/\mathbb{Q})), \sigma(a) = \chi(\sigma)a\}$$

Theorem 7.3 (Herbrand-Ribet)**.** *If p is an odd prime and $2 \leq k \leq p-3$, then p divides the numerator of B_k iff $C_{p,\chi_k} \neq 0$ where $\chi_k : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{F}_p^\times, b \mapsto b^{1-k}$.*

The hard part in the proof is to show that if p divides the numerator of B_k then $C_{p,\chi_k} \neq 0$. The starting point of this proof is to observe that the normalised eigenform $F_k = -B_k/(2k) + \sum_{n \geq 1} \sigma_{k-1}(n)q^n$ is congruent modulo p to a cuspidal normalised eigenform f. So we obtain a Galois representation $\rho_{f,\mathfrak{p}} : \mathrm{Gal}(\mathbb{Q}_{\{p\}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\mathfrak{p}})$ for some $\mathfrak{p}$ above p. This has the property that $\mathrm{tr}\, \rho_{f,\mathfrak{p}}(\mathrm{Frob}_\ell) \equiv \sigma_{n-1}(\ell) \pmod{\mathfrak{p}}$ for any prime $\ell \neq p$.
What do the Prime Number Theorem and Fermat's Last Theorem have in common? They are both proved by looking at cuspidal eigenforms.