Complex Manifolds *

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This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Complex Manifolds* in Lent 2023. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

Contents

*Based on the lectures under the same name taught by Dr. A. Kovalev in Lent 2023.

0 Complex Analysis in Several Variables

Recall:

Definition 0.1. A smooth (n-dimensional) real manifold M is a Hausdorff, second-countable topological space with a collection of charts, which are homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ (where $U_\alpha \subset M, V_\alpha \subset \mathbb{R}^n$ are open) such that $M = \bigcup_\alpha U_\alpha$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth wherever defined.

In this course, we concern a modification where we replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and "smooth" by "complex analytic". Inevitably, we need some multivariate complex analysis.

Definition 0.2. For an open set $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is holomorphic if for every $a \in U$, there is a power series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ valid on a disk around a.

Equivalently, it is C^1 (viewed with the identification of $\mathbb{C}$ with $\mathbb{R}^2$) and satisfies the Cauchy-Riemann equations $\partial f/\partial \bar{z} = 0$ on U, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Just to remind you that if f is just smooth, we have $f(z) = f(a) + \partial f/\partial z|_{z=a}(z-a) + \partial f/\partial \bar{z}|_{z=a}\overline{(z-a)} + o(|z-a|)$ as $z \to a$.
Recall also the Cauchy integral formula, which says that if $f : U \to \mathbb{C}$ is smooth, then

$$f(z) = \frac{1}{2\pi i}\int_{|w-z|=r} \frac{f(w)}{w-z}\,\mathrm{d}w$$

whenever $\{|w-z| \le r\} \subset U$.
Now let's look at the case with more variables.

Definition 0.3. For an open set $U \subset \mathbb{C}^n$ and C^1 function $f : U \to \mathbb{C}$, f is holomorphic if $f_j(z) = f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$ is holomorphic in z for all $j = 1, \ldots, n$ and fixed $z_1, \ldots, \hat{z}_j, \ldots, z_n$, wherever it makes sense.
We say f is holomorphic on a set $S \subset \mathbb{C}^n$ if it is holomorphic on an open set containing S.

Equivalently, $\partial f/\partial \bar{z}_j = 0$ for all j (sometimes also written as $\bar{\partial} f = 0$) where $\partial/\partial \bar{z}_j = (1/2)(\partial/\partial x_j + i\partial/\partial y_j)$.

Definition 0.4. A polydisc in $\mathbb{C}^n$ is an open set of the form $\Delta = \Delta_1 \times \cdots \times \Delta_n$ where Δ_j are open disks in $\mathbb{C}$.

Polydiscs form a basis for the standard Euclidean topology on $\mathbb{C}^n$.

Theorem 0.1 (Cauchy integral formula in several variables)**.** *Suppose f is holomorphic on the closure of a polydisc Δ, then*

$$f(z) = \frac{1}{(2\pi)^n}\int_{\partial\Delta_j, j=1,\ldots,n} \frac{f(w)}{(w_1 - z_1)\cdots(w_n - z_n)}\,\mathrm{d}w$$

Proof. Repeatedly integrate. ∎

Remark. We are not integrating over $\partial\Delta$, but a (proper) submanifold of it.

We still have power series expansions.

Theorem 0.2. *$f : U \to \mathbb{C}$ is holomorphic if and only if there is a power series expansion*

$$f(z) = \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{1}{i_1!\cdots i_n!} \left.\frac{\partial^{i_1+\cdots+i_n} f}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}\right|_{z=a} (z-a_1)^{i_1}\cdots(z-a_n)^{i_n}$$

valid on any polydisc in U centered at $a \in U$.

More generally, $f : U \to V, V \subset \mathbb{C}^m$ is holomorphic if each component of it is holomorphic.

Definition 0.5. f is biholomorphic if it is bijective and f, f^{-1} are both holomorphic.

Theorem 0.3. *f is biholomorphic iff it is bijective and holomorphic.*

Definition 0.6. The complex Jacobian of a holomorphic $f = (f_1, \ldots, f_m)$ at w is

$$J(f)_w = \left(\left.\frac{\partial f_k}{\partial z_j}\right|_{z=w} \right)_{k,j}$$

For every w, this defines a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^m$. If this is surjective, we say w is a regular point (for f). If $u \in \mathbb{C}^m$ has $f^{-1}(u)$ consisting exclusively of regular points, we say u is a regular value for f.

Remark. Let's go back to the case where $n = m = 1$. Suppose $f = u + iv$. In addition to its complex Jacobian which is just a value, we also have its real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

which is similar (as a matrix with complex entries) to

$$\begin{pmatrix} \partial f/\partial z & \partial f/\partial \bar{z} \\ \partial \bar{f}/\partial z & \partial \bar{f}/\partial \bar{z} \end{pmatrix}$$

Suppose f is holomorphic, then this is just $\mathrm{diag}(\partial f/\partial z, \overline{\partial f/\partial z})$.
What about in high dimensions? If $f : U \to \mathbb{C}^m$ is holomorphic, then $J_{\mathbb{R}}(f)$ is similar to the block-diagonal matrix with blocks $J(f), \overline{J(f)}$. When $m = n$, we can take determinants and see that $\det J_{\mathbb{R}}(f) = |\det J(f)|^2 \geq 0$. In particular, if $J(f)$ is nonsingular, then $\det J_{\mathbb{R}}(f) > 0$.

Let's just quote a few more theorems for later use.

Theorem 0.4 (Holomorphic inverse function theorem)**.** *Let $U, V \subset \mathbb{C}^n$ be open and $f : U \to V$ holomorphic. Suppose $z \in U$ is regular for f (equivalently, $J(f)$ is nonsingular), then (and only then) there exists neighbourhoods $U_0 \ni z, V_0 \ni f(z)$ such that $f|_{U_0} : U_0 \to V_0$ is biholomorphic.*

Theorem 0.5 (Holomorphic implicit function theorem)**.** *Let $U \subset \mathbb{C}^m$ open and $f : U \to \mathbb{C}^n$ holomorphic with $m \geq n$. Suppose $w \in U$ has $(\partial f_k \partial z_j|_{z=w})|_{j,k\leq n}$ nonsingular and $f(w) = 0$. Then there are some open $U_1 \subset \mathbb{C}^{m-n}$ and $U_2 \subset \mathbb{C}^n$ such that $w \in U_1 \times U_2 \subset U$ and that there is a unique $g : U_1 \to U_2$ such that $f^{-1}(0) \cap (U_1 \times U_2)$ is the graph of g.*

1 Complex Structures

Definition 1.1. A complex n-manifold (or n-fold) M is a Hausdorff, second countable topological space equipped with an atlas of complex coordinate charts $\{\phi_i : U_i \to V_i\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of M, V_i are open in $\mathbb{C}^n$, each ϕ_i is a homeomorphism, and each $\phi_j \circ \phi_i^{-1}$ is holomorphic wherever defined.

The components of ϕ_i are known as (complex/holomorphic) local coordinates. Given a complex n-fold, it has the natural structure of a real $2n$-fold $M_{\mathbb{R}}$ by the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. This is known as the "underlying" real manifold of M.

Definition 1.2. Let M, N be complex manifolds and $\{\phi_i\}_{i\in I}, \{\psi_\alpha\}_{\alpha\in A}$ their respective cooordinate charts. A continuous map $f : M \to N$ is holomorphic if each $\psi_\alpha \circ F \circ \phi_i^{-1}$ is holomorphic wherever defined.
f is an isomorphism if it has a two-sided holomorphic inverse. M, N are biholomorphic (or isomorphic) if there exists an isomorphism between them.

It follows from complex analysis that f is an isomorphism if and only if it is a holomorphic bijection.

Proposition 1.1. *If M is a compact connected complex manifold, then the only holomorphic functions on M (i.e. holomorphic maps $M \to \mathbb{C}$) are constants.*

Proof. Suppose $f : M \to \mathbb{C}$ is holomorphic, then $|f|$ is a continuous real-valued function on a compact domain, which must attain a maximum, say at $p \in M$. For any chart $\phi : U \to V \subset \mathbb{C}$ around p, then $f|_U$ must be constant by maximum modulus principle in several complex variables (example sheet). Since M is compact, it is covered by finitely many charts. The path-connectedness of M then leads to the conclusion (or you can also use identity principle if so wished). □

Remark. The proposition is a generalisation of Liouville's theorem (by applying it to $M = \mathbb{C}P^1$, which we haven't defined but will soon).

Example 1.1. 1. Open subsets of $\mathbb{C}^n$ are complex manifolds in the obvious way.
2. Complex 1-folds are known as Riemann surfaces. They are classified by the uniformisation theorem: Every Riemann surface is isomorphic to one of $\mathbb{C}P^1, \mathbb{C}, \mathbb{C}/\mathbb{Z} \cong \mathbb{C} \setminus \{0\}$, a complex torus $\mathbb{C}/\Lambda$ for a lattice $\Lambda \leq \mathbb{C}$, or a quotient of the form Δ/Γ where Δ is the open unit disk in $\mathbb{C}$ and Γ is a subgroup of the group of Möbius transformations acting on Δ properly discontinuously.
3. More generally, properly discontinuous actions give rise to quotient complex manifolds (example sheet).
4. The complex projective space $\mathbb{C}P^n$ is constructed as follows: As a set, it is the set of 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$. We have a natural surjection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$, which induces a quotient topology on $\mathbb{C}P^n$, which as one easily sees is Hausdorff and second-countable. Every point of $\mathbb{C}P^n$ can be referred to in terms of homogeneous coordinates $[z_0 : \cdots : z_n] = \pi(z_0, \ldots, z_n)$. Note that $[z_0 : \cdots : z_n] = [w_0 : \cdots : w_n]$ iff there is some $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda z_i = w_i$ for every i.

Let $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\} \subset \mathbb{C}P^n$ which is open. Then we get charts $\phi_i : U_i \to \mathbb{C}^n, [z_0 : \cdots : z_n] \mapsto z_i^{-1}(z_0, \ldots, \hat{z}_i, \ldots, z_n)$, which has

$$\phi_j \circ \phi_i^{-1}(w_1, \ldots, w_n) = \frac{1}{w_j}(w_1, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, \hat{w}_j, \ldots, w_n)$$

Therefore they give $\mathbb{C}P^n$ the structure of a complex n-fold.
5. We can make S^2 the underlying real manifold of a complex manifold by diffeomorphically identifying it with $\mathbb{C}P^1$ via stereographic projection.

Remark. In the study of real manifolds, we have Whitney's embedding theorem telling us we can embed every real manifold into $\mathbb{R}^N$ for some N. However, compact complex manifolds almost never embed into $\mathbb{C}^N$ for any N due to Proposition 1.1.
A cool thing about projective spaces is that a lot of complex manifolds can be embedded in some $\mathbb{C}P^N$, such as every compact Riemann surface. We say a complex manifold is projective if it admits a holomorphic embedding into some projective space.

Example 1.2. 1. A complex torus is a quotient $\mathbb{C}^g/\Lambda$ for a lattice $\Lambda \leq \mathbb{C}^g$. The complex structure on it is given by continuous local inverses of the quotient map $\mathbb{C}^g \to \mathbb{C}^g/\Lambda$. Suppose $(\phi_i, D_i), (\phi_j, D_j)$ are two such charts with D_i, D_j polydiscs, then $\phi_j \circ \phi_i^{-1}(z) = z + \lambda_{ij}(z)$ for some $\lambda_{ij} : D_i \cap D_j \to \Lambda$. But λ_{ij} must be continuous and Λ is discrete, so λ_{ij} is constant. Therefore $\phi_j \circ \phi_i^{-1}$ is just translation by λ_{ij} and therefore holomorphic.
$\mathbb{C}^{2n}/\Lambda$ is diffeomorphic to $(S^1)^{2n}$ as real manifolds, but there are uncountably many non-biholomorphic complex n-torus for all $n > 0$. It is a hard fact that complex n-tori are generally not projective for $n > 1$.
2. The Hopf surface $H^2 = \mathbb{C}^2 \setminus \{0\}/\{z \sim 2z\}$ is a complex surface. It is diffeomorphic to $S^1 \times S^3$. We'll show later that H^2 is not projective. In general, one can similarly define a Hopf manifold $H^n = \mathbb{C}^n \setminus \{0\}/\{z \sim 2z\}$ of every dimension $n > 0$. At some point you must've shown that H^1 is biholomorphic to a complex torus.
3. Complex Grassmannians. Start from an n-dimensional complex vecter space V. For $k < n$ we define, on the level of sets, $\mathrm{Gr}_k(V)$ to be the collection of k-dimensional linear subspaces of V. We'll give it a complex structure, which shall have $\mathrm{Gr}_1(V) \cong \mathbb{C}P^{n-1}$.
Each W is given noncanonically (after choosing a basis) by a $k \times n$ matrix with complex entries. We can diagonalise a non-singular $k \times k$ part, leaving $k(n-k)$ "free parameters" which gets us to embed $\mathrm{Gr}_k(V)$ into a projective space (the "Plücker embedding"). See example sheet for details.
It turns out that $\mathrm{Gr}_k(V)$ is compact. A sketch of the proof is as follows: Let $(\mathbb{C}^{k,n})^* \subset \mathbb{C}^{kn}$ be the (open) set of linearly independent k-tuples in $\mathbb{C}^n$. We get a map $\pi : (\mathbb{C}^{k,n})^* \to \mathrm{Gr}(k,n) = \mathrm{Gr}_k(\mathbb{C}^n)$. The topology on $\mathrm{Gr}(k,n)$ turns out (or one can also define it like this, I suppose) to be the quotient topology via π. Let $(\mathbb{C}^{k,n})^*_\circ \subset (\mathbb{C}^{k,n})^*$ be the subset of orthonormal k-tuples, which is compact and has image $\mathrm{Gr}(k,n)$ under π.
4. A complex Lie group is a group with the structure of a complex manifold such that the group operations are holomorphic. Examples of this include $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C})$. None of these is compact. Indeed, all compact complex Lie groups are commutative (example sheet).

Note that $\mathrm{U}(n)$ is in general not a complex manifolds (indeed it doesn't even have even real dimension when n is odd).

2 The (Anti)Holomorphic (Co)Tangent Bundle

Suppose M is a complex n-fold. For $p \in M$, write $z_j = x_j + iy_j$ to denote the local complex coordinates. The real tangent space of M at p is $T_pM = \mathrm{Span}_{\mathbb{R}}\{\partial/\partial x_j, \partial/\partial y_j\}_{j=1,\dots,n}$.
Set $J_p \in \mathrm{GL}(T_pM) \subset \mathrm{End}\, T_pM$ given by $\partial_{x_j} \mapsto \partial_{y_j}, \partial_{y_j} \mapsto -\partial_{x_j}$. So $J_p^2 = -I$.
We can complexify $T_pM \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Span}_{\mathbb{C}}\{\partial/\partial x_j, \partial/\partial y_j\}_{j=1,\dots,n}$. The $\mathbb{C}$-linear extension $J_p \in \mathrm{GL}(T_pM \otimes_{\mathbb{R}} \mathbb{C})$ still has $J_p^2 = -1$. The possible eigenvalues of J_p are $\pm i$. Let $T_p^{1,0}M$ and $T_p^{0,1}M$ be the eigenspaces of $i, -i$ respectively.

Definition 2.1. $T_p^{1,0}M$ is known as the holomorphic tangent space and $T_p^{0,1}M$ the antiholomorphic tangent space.

The complex conjugation on $T_pM \otimes_{\mathbb{R}} \mathbb{C}$ is given by $\lambda e \mapsto \bar{\lambda} e$ for any $e \in T_pM, , \lambda \in \mathbb{C}$. This is an $\mathbb{R}$-linear automorphism of $T_pM \otimes_{\mathbb{R}} \mathbb{C}$. On the other hand, $J(\lambda e) = \lambda J(e)$, so J commutes with complex conjugation, and hence complex conjugation interchanges the holomorphic and antiholomorphic tangent spaces.

Proposition 2.1. *(i) $T_pM \otimes_{\mathbb{R}} \mathbb{C} = T_p^{1,0}M \oplus T_p^{0,1}M$, thus $\dim_{\mathbb{C}} T_p^{1,0}M = \dim_{\mathbb{C}} T_p^{0,1}M = n$.*
(ii) J_p and hence the decomposition in (i) are independent of the choice of coordinates.

Proof. (i) After the change of basis $\partial/\partial z_j = (1/2)(\partial/\partial x_j - i\partial/\partial y_j), \partial/\partial \bar{z}_j = (1/2)(\partial/\partial x_j + i\partial/\partial y_j)$, we see that $J(\partial/\partial z_j) = i\partial/\partial z_j, J(\partial/\partial \bar{z}_j) = -i\partial/\partial \bar{z}_j$ for all j.
(ii) Recall that the real tangent spaces is equivalently the space of derivations acting on $C^\infty(M, \mathbb{R})$. The complexified tangent spaces similarly acts on $C^\infty(M, \mathbb{C})$. And now $T_p^{1,0}M$ consists of those vanishing on holomorphic functions in $C^\infty(M, \mathbb{C})$, and $T_p^{0,1}M$ consist of those vanishing on antiholomorphic functions (i.e. whose conjugate is holomorphic). Since the $\pm i$ eigenspace decomposition of J are canonically defined, J itself must also be canonically defined. In fact, J is a C^∞ section of $\mathrm{End}(TM)$. □

We can describe the (anti)holomorphic tangent spaces more explicitly as $T_p^{1,0}M = \{v - iJv : v \in T_pM\}, T_p^{0,1}M = \{v + iJv : v \in T_pM\}$, both of which naturally receive ($\mathbb{R}$-linear) isomorphisms from T_pM.
The Jacobians express the transformation law for $\partial/\partial x_j, \partial/\partial y_j$ as a change of coordinates. More precisely,

Lemma 2.2. *On overlap of complex coordinate neighbourhoods with coordinates $z_j, w_j, j = 1, \dots, n$, we have*

$$\frac{\partial}{\partial w_k} = \sum_{j=1}^n \frac{\partial z_j}{\partial w_k} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{w}_k} = \sum_{j=1}^n \frac{\partial \bar{z}_j}{\partial \bar{w}_k} \frac{\partial}{\partial \bar{z}_j}$$

Proposition 2.3. *Every real manifold underlying a complex manifold has a canonical orientation.*

Proof. $\det J_{\mathbb{R}}(f) = |J(f)|^2 > 0$ wherever f is biholomophic. In particular, this applies when f gives a change-of-coordinates. □

Consider $\coprod_{p\in M} T_p^{1,0}M = T^{1,0}M$. This is a well-defined complex vector bundle by a procedure similar to that of the tangent bundle of a real manifold. It is a subbundle of $TM \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 2.2. $T^{1,0}M$ is the holomorphic vector bundle of M.

Sections of $T^{1,0}M$ form a subspace of the space of complex vector fields.

Definition 2.3. $\xi \in \Gamma(T^{1,0}M)$ is a holomorphic vector field if for any holomorphic $f \in C^\infty(M, \mathbb{C})$, ξf is again holomorphic.

Similarly we can define the antiholomorphic vector bundle $T^{0,1}M$. It's easy to see that for any open $U \subset M$ and $f : U \to \mathbb{C}$, f is holomorphic iff $\xi f = 0$ for any $\xi \in \Gamma(T^{0,1}M)$.

Remark. Recall the standard faithful representation $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{R}^{2n}$, which is given by the injective group homomorphism $\phi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$ via replacing each complex entry $a + ib$ with a 2×2 block

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$\phi(\mathrm{GL}(n, \mathbb{C}))$ consists of all matrices commuting with J, which is a block-diagonal matrices with blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Suppose f expresses a holomorphic change-of-coordinates. Then the Cauchy-Riemann equation for f is simply the relation $J_{\mathbb{R}}(f) \in \phi(\mathrm{GL}_n(\mathbb{C}))$. Thus a holomorphic atlas induces a reduction of the structure group of TM from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{C})$, i.e. this makes TM a complex vector bundle isomorphic to $T^{1,0}M$. More explicitly, the isomorphism is given by sending $v \in TM$ to $v - iJv \in T^{1,0}M$. Locally, $a\partial/\partial x_k + b\partial/\partial y_k$ is mapped to $2(a + b)\partial/\partial z_k$.
$J \in \Gamma(\mathrm{End}\, TM)$ is called the almost complex structure on a complex manifold M. In general, an almost complex structure is such a section which squares to $-I$.

Recall that if $f : M \to N$ is a smooth map between real manifolds, then there is a notion of a differential, which is a linear map $(\mathrm{d}f)_p : T_pM \to T_{f(p)}N$ for each $p \in M$. Consider the complex linear extension $(\mathrm{d}f)_p : T_pM \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(p)}N \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 2.4. *For a smooth map $f : M \to N$ between complex manifolds, the followings are equivalent:*
(i) f is holomorphic.
(ii) $(\mathrm{d}f)_p : T_pM \otimes_{\mathbb{R}} \mathbb{C} \to T_{f(p)}N \otimes_{\mathbb{R}} \mathbb{C}$ interchanges the almost complex structure of M and that of N, i.e. $\mathrm{d}f \circ J_M = J_N \circ \mathrm{d}f$.
(iii) $\mathrm{d}f(T^{1,0}M) \subset T^{1,0}N$.
(iv) $\mathrm{d}f(T^{0,1}M) \subset T^{0,1}N$.

Proof. The statements are local, so we may assume WLOG that M, N are open sets in complex vector spaces.
(i) $\implies$ (iii): If f is holomorphic, then it satisfies Cauchy-Riemann which

happens iff $(\mathrm{d}f)_p$ can always be expressed in some basis by $J_{\mathbb{R}}(f)$ consisting of 2×2 blocks of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$T_p^{1,0}$ is spanned over $\mathbb{C}$ by e_k which is the vector with 1 at $2k-1$-th entry and $-i$ at $2k$-th entry. But then $J_{\mathbb{R}}(f)e_k$ must still have type $(1,0)$.
(ii) $\implies$ (i): Write each 2×2 block of $\mathrm{d}f$ in the form

$$B_{kl} = \begin{pmatrix} c_{2k-1,2l-2} & c_{2k-1,2l} \\ c_{2k,2l-1} & c_{2k,2l} \end{pmatrix}$$

The condition of (ii) is saying that each B_{kl} commutes with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ which is just saying that B_{kl} has the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ for some a, b. So f satisfies the Cauchy-Riemann equations.
The other implications are clear. □

We can also complexify cotangent spaces by a similar procedure. On $T_p^*M\otimes_{\mathbb{R}}\mathbb{C}$, (the dual of) J acts via $\mathrm{d}x_j \mapsto -\mathrm{d}y_j, \mathrm{d}y_j \mapsto \mathrm{d}x_j$ for all j. We again change coordinates to reach the basis $\mathrm{d}z_k = \mathrm{d}x_k + i\,\mathrm{d}y_k, \mathrm{d}\bar{z}_k = \mathrm{d}x_k - i\,\mathrm{d}y_k$, which are the $i, -i$ eigenvectors, respectively. And we have the transformation law

$$\mathrm{d}w_k = \sum_i \frac{\partial w_k}{\partial z_j}\,\mathrm{d}z_j, \mathrm{d}\bar{w}_k = \sum_j \frac{\partial \bar{w}_k}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j$$

Similar to the tangent case, we get the holomorphic cotangent bundle $(T^*M)^{1,0}$ and antiholomorphic cotangent bundle $(T^*M)^{0,1}$ whose direct sum is $T^*M\otimes_{\mathbb{R}}\mathbb{C}$. Their fibres are spanned by the covectors $\mathrm{d}z_j$ and $\mathrm{d}\bar{z}_j$, the duals to $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$, respectively. In particular, $\langle \mathrm{d}z_k, \partial/\partial z_l\rangle = \delta_{kl}$ and $\langle \mathrm{d}z_k, \partial/\partial \bar{z}_l\rangle = 0$ for all k, l.

3 Dolbeault Cohomology

Proposition 3.1. *We have a decomposition*

$$\bigwedge^r (T^*M\otimes_{\mathbb{R}}\mathbb{C}) = \bigoplus_{p+q=r} \bigwedge^{p,q}(T^*M\otimes_{\mathbb{R}}\mathbb{C})$$

where $\bigwedge^{p,q}(T^*M\otimes_{\mathbb{R}}\mathbb{C}) = \bigwedge^p (T^*M)^{1,0}\wedge \bigwedge^q (T^*M)^{0,1}$.

This is known as the type decomposition. Note that $\overline{\bigwedge^{p,q}} = \bigwedge^{q,p}$.

Definition 3.1. The space of sections of $\bigwedge^r(T^*M\otimes\mathbb{C})$ are denoted $\Omega^r(M)^{\mathbb{C}}$, of $\bigwedge^r(T^*M)$ are denoted $\Omega^r(M)$, and of $\bigwedge^{p,q}(T^*M\otimes\mathbb{C})$ are denoted $\Omega^{p,q}(M)$.

In local coordinates, forms in $\Omega^{p,q}(M)$ are

$$\sum_{I,I'} a_{I,I'}\,\mathrm{d}z_{i_1}\wedge\cdots\wedge \mathrm{d}z_{i_p}\wedge \mathrm{d}\bar{z}_{i'_1}\wedge\cdots\wedge \mathrm{d}\bar{z}_{i'_q}$$

where $a_{I,J}$ are smooth functions in the coordinate neighbourhood, and the sum is over multi-indices $I = i_1\cdots i_p, I' = i'_1\cdots i'_q$. Our good ol' almost complex structure J extends its action to $\Omega^{p,q}(M)$ via $J\phi = i^{p-q}\phi$.
There are few particular subspaces: $\Omega^{p,p}(M)\cap\Omega^{2p}(M) = \Omega^{p,p}_{\mathbb{R}}(M)$ are the real (i.e. invariant under complex conjugation) (p,p)-forms.

Definition 3.2. Let $n = \dim_{\mathbb{C}} M$. The canonical line bundle over M is $K_M = \bigwedge^{n,0}(T^*M \otimes \mathbb{C}) = \bigwedge^n (T^*M)^{1,0}$.

Its transition functions are of the form $\det(\partial w_k/\partial z_j)_{j,k=1,\dots,n}$. They are holomorphic by construction.
The ordinary exterior derivative $\mathrm{d} : \Omega^0(M)^{\mathbb{C}} \to \Omega^1(M)^{\mathbb{C}} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ decomposes as $\mathrm{d} = \partial + \bar{\partial}$, where $\partial = \pi^{1,0} \circ \mathrm{d}, \bar{\partial} = \pi^{0,1} \circ \mathrm{d}$. Here, $\pi^{p,q} : \Omega^*(M)^{\mathbb{C}} \to \Omega^{p,q}(M)$ is the projection to the (p,q)-component along other components in the type decompositon. Locally, for a function $f \in C^\infty(M, \mathbb{C})$ we have

$$\partial f = \sum_k \frac{\partial f}{\partial z_k} \,\mathrm{d}z_k, \bar{\partial} f = \sum_k \frac{\partial f}{\partial \bar{z}_k} \,\mathrm{d}\bar{z}_k$$

More generally, for $\alpha \in \Omega^{p,q}(M)$, we write $\partial\alpha = (\pi^{p+1,q} \circ \mathrm{d})\alpha, \bar{\partial}\alpha = (\pi^{p,q+1} \circ \mathrm{d})\alpha$. For $\alpha \in \Omega^{p,0}(M)$, $\bar{\partial}\alpha = 0$ iff locally α has the form $\sum_I f_I \,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$ for holomorphic f_I's. In this case, we call α a holomorphic p-form. Holomorphic 1-forms are called holomorphic differentials.

Lemma 3.2. *On a complex manifold M:*
(i) For any $\eta \in \Omega^{p,q}(M)$, we have $\mathrm{d}\eta = \partial\eta + \bar{\partial}\eta$.
(ii) $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.
(iii) $\bar{\partial}(\xi \wedge \eta) = \bar{\partial}\xi \wedge \eta + (-1)^{p+q}\xi \wedge \bar{\partial}\eta$ for any $\xi \in \Omega^{p,q}(M)$.

Proof. (i) It suffices to show this in local coordinates. But we just have $\mathrm{d}(f \,\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_{I'}) = (\partial f + \bar{\partial} f) \,\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_{I'}$.
(ii) $\mathrm{d}^2 = 0$.
(iii) WLOG $\eta \in \Omega^{p',q'}(M)$. But in this case the statement just follows from taking the $(p+p'+1, q+q')$ and $(p+p', q+q'+1)$ components of $\mathrm{d}(\xi \wedge \eta)$. □

Corollary 3.3. $\mathrm{d}(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$.

Sometimes it's convenient to replace the complex operators $\partial, \bar{\partial}$ with real operators d and $\mathrm{d}^{\mathrm{c}} = i(\bar{\partial} + \partial)$. We can get back home by the formulae

$$\partial = \frac{1}{2}(\mathrm{d} + i\mathrm{d}^{\mathrm{c}}), \bar{\partial} = \frac{1}{2}(\mathrm{d} - i\mathrm{d}^{\mathrm{c}})$$

A straightforward check shows that $(\mathrm{d}^c)^2 = 0$ and $\mathrm{d}\mathrm{d}^{\mathrm{c}} = -\mathrm{d}^{\mathrm{c}}\mathrm{d} = 2i\partial\bar{\partial}$.
Recall that if we have a smooth map $f : M \to N$, we have the pullback $f^* : \Omega^1(N) \to \Omega^1(M)$ by $\langle f^*\alpha, X\rangle = \langle \alpha, (\mathrm{d}f)X\rangle$ for any vector field X on M.
Extending linearly, we get $f^* : \Omega^1(N)^{\mathbb{C}} \to \Omega^1(M)^{\mathbb{C}}$. Suppose f is holomorphic and $\alpha \in \Omega^{1,0}(N)$ (this also works analogously for $\Omega^{0,1}(N)$). So $J\alpha = i\alpha$. Then $\langle Jf^*\alpha, X\rangle = \langle f^*\alpha, JX\rangle = \langle \alpha, (\mathrm{d}f)JX\rangle = \langle \alpha, J(\mathrm{d}f)X\rangle = \langle i\alpha, (\mathrm{d}f)X\rangle = \langle if^*\alpha, X\rangle$. So $Jf^*\alpha = if^*\alpha$. Therefore $f^*\alpha \in \Omega^{1,0}(M)$.
Similarly,

Proposition 3.4. *The pullback by holomorphic maps preserve the type decomposition. That is, for $\alpha \in \Omega^{p,q}$ and f holomorphic, $f^*\alpha \in \Omega^{p,q}$.*

Remark. The converse is also true (exercise).

Now, $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$ for smooth f. If f happens to be holomorphic, then $(\bar{\partial} \circ f^*)\eta = (\pi^{p,q+1} \circ \mathrm{d} \circ f^*)\eta = (\pi^{p,q+1} \circ f^* \circ \mathrm{d})\eta = (f^* \circ \pi^{p,q+1} \circ \mathrm{d})\eta = (f^* \circ \bar{\partial})\eta$.
Thus,

Proposition 3.5. $\bar{\partial} \circ f^* = f^* \circ \bar{\partial}$, $\partial \circ f^* = f^* \circ \partial$.

Definition 3.3. The Dolbeault cohomology of a complex manifold M is

$$H^{p,q}(M) = \frac{\ker \bar{\partial}|_{\Omega^{p,q}(M)}}{\bar{\partial}(\Omega^{p,q-1}(M))}$$

Corollary 3.6. *If $f : M \to N$ is holomorphic, then f^* descends to a complex linear map on cohomology $H^{p,q}(N) \to H^{p,q}(M)$.*

The formula $(g \circ f)^* = f^* \circ g^*$ is of course valid on both $\Omega^{p,q}$ and $H^{p,q}$. In particular, biholomorphic complex manifolds have isomorphism Dolbeault cohomology.

Remark. 1. $\bigoplus_{p+q=r} H^{p,q}(M)$ need not in general be $H^r_{\mathrm{dR}}(M)$.
2. By contrast to H^*_{dR}, $H^{*,*}$ are not topological invariants. They are invariants of a holomorphic structure on M, though.

Lemma 3.7 ($\bar{\partial}$-Poincaré Lemma in One Variable)**.** *For $D = D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ and $g \in C^\infty(D(a, r+\delta))$ for some $\delta > 0$, we have*

$$f(z) = \frac{1}{2\pi i} \int_D \frac{g(w)}{w-z} \, \mathrm{d}w \wedge \mathrm{d}\bar{w}$$

is a member of $C^\infty(D)$ and $\bar{\partial} f = g$ on D.

To prove this, we need to slightly extend Cauchy integral formula.

Lemma 3.8. *Consider $F \in C^\infty(D(a, r+\delta))$ and let z be such that $|z-a| < r$. Then*

$$F(z) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{F(w)}{w-z} \, \mathrm{d}w + \frac{1}{2\pi i} \int_{|z-a|<r} \frac{\partial F}{\partial \bar{w}}(w) \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}$$

Proof. Consider the 1-form $\eta = (2\pi i(w-z))^{-1} F(w) \, \mathrm{d}w$ on $D_\epsilon = D(a,r) \setminus D(z,\epsilon)$. We have

$$\mathrm{d}\eta = -\frac{1}{2\pi i} \frac{\partial F}{\partial \bar{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}$$

Stokes' theorem gives

$$\int_{D_\epsilon} \mathrm{d}\eta = \int_{\partial D_(a,r)} \eta - \int_{\partial D(z,\epsilon)} \eta$$

Now

$$\int_{\partial D(z,\epsilon)} \eta = \frac{1}{2\pi} \int_0^{2\pi} F(z + \epsilon e^{i\theta}) \, \mathrm{d}\theta \to F(z)$$

as $\epsilon \downarrow 0$. Poles of order 1 are "integrable": $\mathrm{d}w \wedge \mathrm{d}\bar{w} = -2i \, \mathrm{d}x \wedge \mathrm{d}y = -2ir \, \mathrm{d}r \wedge \mathrm{d}\theta$, hence the estimate

$$\left| \frac{\partial F}{\partial \bar{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z} \right| = \left| \frac{\partial F}{\partial \bar{w}} \frac{2 \, dx \wedge \mathrm{d}y}{r} \right| = 2 \left| \frac{\partial F}{\partial \bar{w}} \, dr \wedge \mathrm{d}\theta \right|$$

And so we can safely send ϵ to zero in the 2-dimensional integral, and the proof is done. □

Proof of Lemma 3.7. We use a partition-of-unity argument. For $z_0 \in D$, consider $D_0 = D(z_0, 2\epsilon) \subset D$. We can always write $g = g_1 + g_2$ where $g_1|_{|z-z_0|\geq 2\epsilon} = 0, g_2|_{|z-z_0|\leq\epsilon} = 0$. Then

$$f_2(z) = \frac{1}{2\pi i}\int_D \frac{g_2(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

is well-defined for z near z_0 and is a proper integral. Differentiating under the integral sign, we conclude $\partial f_2/\partial \bar{z} = 0$.
g_1 has compact support, so

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i}\int_D \frac{g_1(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\bar{w} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w)}{w-z}\,\mathrm{d}w \wedge \mathrm{d}\bar{w} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(u+z)}{u}\,\mathrm{d}u \wedge \mathrm{d}\bar{u} = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g_1(z+re^{i\theta})}{e^{i\theta}}\,\mathrm{d}r \wedge \mathrm{d}\theta \end{aligned}$$

which is well-defined and smooth in z. Hence $f = f_1 + f_2$ has

$$\frac{\partial f}{\partial \bar{z}}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g_1}{\partial \bar{z}}(z+re^{i\theta})e^{-i\theta}\,\mathrm{d}r \wedge \mathrm{d}\theta = \frac{1}{2\pi i}\int_D \frac{\partial g_1}{\partial \bar{w}}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}$$

By the preceding lemma, we have

$$\begin{aligned} g_1(z) &= \frac{1}{2\pi i}\int_{|w-a|=r} \frac{g_1(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{|w-a|<r} \frac{\partial g_1}{\partial \bar{w}}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z} \\ &= \frac{1}{2\pi i}\int_{|w-a|<r} \frac{\partial g_1}{\partial \bar{w}}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z} \end{aligned}$$

for z near z_0. So $\partial f_1/\partial \bar{z}|_{z=z_0} = g_1(z_0) = g(z_0)$ and $\partial f/\partial \bar{z}|_{z=z_0} = \partial f_1/\partial \bar{z}|_{z=z_0}$ as $\partial f_2/\partial \bar{z} = 0$. ☐

Theorem 3.9 ($\bar{\partial}$-Poincaré Lemma). *Let $D = D(a_1, r_1) \times \cdots \times D(a_n, r_n) \subset \mathbb{C}^n$ be a polydisc (allowing $r_k = \infty$ for any or all k). Then $H^{p,q}(D) = 0$ whenever $q \geq 1$.*

Proof. Fix $q \geq 1$. Start with a form $\phi \in \Omega^{p,q}(D)$ and suppose $\bar{\partial}\phi = 0$. WLOG $p = 0$, for if η has type $(0, q)$ and $\mathrm{d}z_I$ has type $p, 0$, then $\bar{\partial}(\eta \wedge \mathrm{d}z_I) = \bar{\partial}\eta \wedge \mathrm{d}z_I$ which is identically zero iff $\bar{\partial}\eta = 0$.
For a polydisc D_0 with same centre and radii $\epsilon_k < r_k$, we shall show that there is some $\psi \in \Omega^{0,q-1}(D_0)$ such that $\bar{\partial}\psi = \phi|_{D_0}$. To do this, we're gonna successively integerate $\mathrm{d}\bar{z}_m, \mathrm{d}\bar{z}_{m-1}$, and so on. Suppose only $\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_k$ occur in ϕ. We have $\phi = \mathrm{d}\bar{z}_k \wedge \phi_1 + \phi_2$ where ϕ_1, ϕ_2 do not have $\mathrm{d}\bar{z}_k$. Since $\bar{\partial}\phi = 0$, we have $\partial\phi_1/\partial\bar{z}_l = 0$ for all $l > k$, where $\phi_1 = \sum_I \phi_I\,\mathrm{d}\bar{z}_I$ for a multi-index $I \subset \{1, \ldots, k-1\}$. Set

$$\eta_I = \frac{1}{2\pi i}\int_{|w_k - a_k|\leq\epsilon_k} \phi_I(\ldots, w_k, \ldots)\frac{\mathrm{d}w_k \wedge \mathrm{d}\bar{w}_k}{\bar{w}_k - z_k}$$

Then $\partial\eta_I/\partial\bar{z}_k = \phi_I$. Also, $\partial\eta_I/\partial\bar{z}_l = 0$ for all $l > k$ as $\partial\phi_1/\partial\bar{z}_l = 0$. This means that η_I is holomorphic in $z_{k+1}, \ldots, z_n$. And so $\phi + (-1)^q\bar{\partial}(\sum_k \eta_I\,\mathrm{d}\bar{z}_{I\setminus\{k\}})$ is still $\bar{\partial}$-closed, but it does not involve any $\mathrm{d}\bar{z}_i$ for $i \geq k$, whence induction.
So we know the existence of such $\psi \in \Omega^{0,q-1}(D_0)$ with $\bar{\partial}\psi = \phi|_{D_0}$. To extend

it to D, we consider $D_m = D(a_k, \epsilon_k^{(m)})$ such that $\bar{D}_m \subset D_{m+1}$ and $\epsilon_k^{(m)} \to r_k$ as $m \to \infty$ for all k. Our procedure gives $\psi_m \in \Omega^{0,q-1}(D_m)$ with $\bar{\partial}\psi_m = \phi|_{D_m}$. Assume first that $q \geq 2$. We'll show that ψ_m can be chosen in a way that it eventually stabilises on any compact subset of D.
After constructing ψ_m with $\bar{\partial}\psi_m = \phi|_{D_m}$, we take any $\alpha \in \Omega^{0,q-1}(D_{m+1})$ such that $\bar{\partial}\alpha = \phi_{D_{m+1}}$. Then $\bar{\partial}(\alpha - \psi_m) = 0$ on D_m. So there is some β such that $(\psi_m - \alpha)|_{D_{m-1}} = \bar{\partial}\beta$. Now let $\psi_{m+1} = \alpha + \bar{\partial}\beta$, then $\bar{\partial}\psi_{m+1} = \phi|_{D_{m+1}}$ and $\psi_{m+1} = \psi_m$ on D_{m-1}.
So ψ_m eventually stabilises on any compact subset of D, therefore converges to some $\psi \in \Omega^{0,q-1}(D)$ with $\bar{\partial}\psi = \phi$.
For $q = 1$, we proceed with a slightly different argument. Given $\psi_m \in C^\infty(D)$ with $\bar{\partial}\psi_m = \phi|_{D_m}$ and choose $\alpha \in C^\infty(D)$ with $\bar{\partial}\alpha = \phi|_{D_{m+1}}$. Now $\psi_m - \alpha$ is a holomorphic function on D_m, so it has a power series expansion around $a = (a_1, \dots, a_n)$ converging uniformly on D_{m-1}. So we can choose a polynomial β such that $\sup_{\bar{D}_{n-1}} |(\psi_m - \alpha) - \beta| < 2^{-m}$. Set now $\psi_{m+1} = \alpha + \beta$. Then $\bar{\partial}\psi_{m+1} = \bar{\partial}\alpha = \phi|_{D_{m+1}}$. Now $\psi_{m+1} - \phi_m$ is a holomorphic function on D_m with $\sup_{\bar{D}_{m-1}} |\psi_{m+1} - \psi_m| < 2^{-m}$.
So $(\psi_k)_{k=1,2,\dots}$ is uniformly Cauchy on D_{m-1} for all m, therefore $\psi_k \to \psi$ for some ψ, uniformly on any compact subsets of D. Then $\bar{\partial}\psi = \phi$ on D. □

$H^{p,0}(\mathbb{C}^n)$ is infiniite-dimensional, but whenever M is a compact complex manifold M, each $H^{p,q}(M)$ is finite-dimensional. This is not obvious at all in general. Although it's fairly easy to show that $H^{0,0}(M) \cong \mathbb{C}$ is M is compact and connected.

4 Almost Complex Manifolds

Definition 4.1. An almost complex manifold is a smooth manifold M equipped with an almost complex structure $J \in \Gamma(\operatorname{End} TM)$ such that $J^2 = -I$.

Lemma 4.1. *Let $J \in \operatorname{End} \mathbb{R}^m$ be such that $J^2 = -I$, then:*
(i) $J \in \mathrm{GL}(m, \mathbb{R})$.
(ii) $m = 2n$ is even.
(iii) $\{A \in \mathrm{GL}(m, \mathbb{R}) : AJA^{-1} = J\}$ is a subgroup of $\mathrm{GL}(m, \mathbb{R})$ isomorphic to $\mathrm{GL}(n, \mathbb{C})$.
Thus the map $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C}) \to \{J \in \operatorname{End} \mathbb{R}^{2n} : J^2 = -I\}, [S] \mapsto SJ_0S^{-1}$ is a bijection parameterising almost complex structures on $\mathbb{R}^{2n}$, where J_0 is the block-diagonal matrix with blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Proof. Don't even tell me you need a proof of (i) and (ii). To show (iii), simply observe that J cannot have real eigenvalues, so v, Jv are always linearly independent for $v \neq 0$. And for any $e_2 \in \operatorname{Span}_{\mathbb{R}}\{v, Jv\}$, Je_2 is linearly independent to v, Jv, e_2. Proceed as such gives a basis $v, Jv, e_2, Je_2, \dots$, which reveals that $J \sim J_0$. □

Remark. The lemma implies that the choice of J is equivalent to the choice of a $\mathrm{GL}(n, \mathbb{C})$-structure on TM.

Back to the case of an almost complex manifold. Using the argument above, we know that T_pM always has a basis of the form $e_1, Je_1, \ldots, e_n, Je_n$ which of course extend to a local frame field around p. Let $e_1^*, Je_1^*, \ldots, e_n^*, Je_n^*$ be the dual coframe field. $\epsilon = Je_1^* \wedge e_1^* \wedge \cdots \wedge Je_n^* \wedge e_n^*$ recovers the orientation induced on M by J.

Example 4.1. Suppose M is a complex manifold with local complex coordinates z_j, then

$$\epsilon = \frac{i^n}{2^n} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

Note that $-J$ is another almost complex structure on M, which gives the same orientation if n is even, and opposite orientation if n is odd.

Definition 4.2. The torsion (or the Nijenhuis tensor) of an almost complex structure J is $N_J \in \Gamma(\mathrm{Hom}(\bigwedge^2 TM, TM))$ such that on local vector fields we have

$$N_J(X,Y) = 2([JX, JY] - [X,Y] - J[X, JY] - J[JX, Y])$$

If $N_J = 0$, we say J is torsion-free (or integrable).

Remark. 1. N_J is $C^\infty(M)$-linear. This can be checked by calculation, for example by using the formula $[fX, Y] = f[X,Y] + (Yf)X$.
2. Locally, $N_J(\partial_i, \partial_j) = \sum_k N_{ij}^k \partial_k$ where these (N_{ij}^k) depend on (J_j^i) and $(\partial_k J_j^i)$

Theorem 4.2 (Newlander-Nirenberg Theorem)**.** *J arises from an atlas of local complex coordinates if and only if J is torsion-free.*

Proof. We'll show the easy part, which is the "only if" direction. Suppose we have complex local coordinates, then $\partial/\partial x_\alpha, \partial/\partial y_\alpha, J\partial/\partial x_\alpha, J\partial/\partial y_\alpha$ must all have constant coefficients (indeed, if $J = J_0$ we simply have $J\partial/\partial x_\alpha = \partial/\partial y_\alpha, J\partial/\partial y_\alpha = -\partial/\partial x_\alpha$), so the Lie brackets just vanish. □

Remark. The "if" direction is hard, but slightly easier with an additional real analyticity assumption.

Now, when we defined $T^{1,0}$ and $T^{0,1}$ on a complex manifold M, every one of our references to the complex structure of M are made through J. So they make sense for any almost complex manifold (after all $J \sim J_0$). The same is true for $\bigwedge^{p,q}(T^*M)^{\mathbb{C}}$, $\pi^{p,q}$, ∂ and $\bar{\partial}$.

Proposition 4.3. *Suppose (M,J) is an almost complex manifold. Then we have* $\mathrm{d}(\Omega^{p,q}(M)) \subset \Omega^{p-1,q+2}(M) + \Omega^{p,q+1}(M) + \Omega^{p+1,q}(M) + \Omega^{p+2,q-1}(M)$.

Proof. Obviously $\mathrm{d}\Omega^{0,0} \subset \Omega^{0,1} + \Omega^{1,0}$, $\mathrm{d}\Omega^{1,0} \subset \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0}$ and $\mathrm{d}\Omega^{0,1} \subset \Omega^{0,2} + \Omega^{1,1} + \Omega^{2,0}$.
Every (p,q)-form has the form $\sum_i \epsilon_1^{(i)} \wedge \cdots \wedge \epsilon_{p+q}^{(i)}, \epsilon_j^{(i)} \in \Omega^{0,1} + \Omega^{1,0}$. So the result follows from the product rule for d. □

Theorem 4.4. *For an almost complex manifold (M,J), the followings are equivalent:*
(i) For $Z, W \in \Gamma(T^{1,0}M)$, we have $[Z,W] \in \Gamma(T^{1,0}(M))$.
(ii) For $Z, W \in \Gamma(T^{0,1}M)$, we have $[Z,W] \in \Gamma(T^{0,1}(M))$.
(iii) $\mathrm{d}\Omega^{1,0}(M) \subset \Omega^{1,1}(M) + \Omega^{2,0}(M)$ *and* $\mathrm{d}\Omega^{0,1}(M) \subset \Omega^{0,2}(M) + \Omega^{1,1}(M)$.
(iv) $\mathrm{d}\Omega^{p,q}(M) \subset \Omega^{p+1,q}(M) + \Omega^{p,q+1}(M)$ *for any p, q.*
(v) $N_J = 0$, i.e. J is integrable.

Proof. (i) $\iff$ (ii): For $Z \in T^{1,0}M$ iff $\bar{Z} \in T^{0,1}M$ and the Lie bracket is a real linear operator.
(i), (ii) $\implies$ (iii): We know that $\mathrm{d}\omega(Z, W) = Z\omega(W) - W\omega(Z) - \omega([Z, W])$. So if $\omega \in \Omega^{1,0}$ and $Z, W \in T^{0,1}$, then $\mathrm{d}\omega$ has an $(0, 2)$ component. SImilarly for the $(1, 0)$ part.
(iii) $\implies$ (i): If $Z, W \in T^{1,0}, \omega \in \Omega^{0,1}$, then $\mathrm{d}\omega(Z, W) = 0$ and so $\omega([Z, W]) = 0$. Therefore $[Z, W] \in T^{1,0}$.
(iii) $\implies$ (iv): It's the same argument as in the proof of the preceding proposition, except using the stronger assumption in (iii).
(iv) $\implies$ (iii): Clear.
(i) $\iff$ (v): We can write a general $(1, 0)$-vector field in the form $X - iJX$ for some real vector field X. Consider $Z = [X - iJX, Y - iJY] = -[JX, JY] + [X, Y] + iJ(J[X, JY]) + iJ(J[JX, Y])$. It's straightforward to check that $2(Z + iJZ) = -N_J(X, Y) - iJN_J(X, Y)$. The right hand side vanishes iff $Z \in T^{1,0}$. □

Remark. 1. Existence of J is essentially a problem in algebraic topology.
2. Integrability of J is a problem in nonlinear (first-order) differential equations, which is harder.
3. On a real surface, every almost complex sturcture comes from a complex structure. You'll also show in example sheet that a complex structure on a surface is equivalent to the choice of a conformal equivalence class of a Riemannian metric.

5 Submanifolds and Subvarieties

Recall that $Y \subset X$ is an embedded submanifold of a smooth manifold X if Y is a manifold and the inclusion map $\iota : Y \to X$ is smooth with injective derivatives everywhere, and is a homeomorphism onto its image. Then (and only then) locally around any $y \in Y$, Y is the preimage of a regular value of some smooth map.

Definition 5.1. Suppose X is a complex manifold and $Y \subset X$ is an embedded smooth submanifold with even real dimension $\dim_{\mathbb{R}} Y = 2k$. We say Y is a (k-dimensional) complex submanifold of X iff there is a set of complex coordinate charts $\phi_i : U_i \to \mathbb{C}^n$ such that $Y \subset \bigcup_i U_i$ and $\phi_i(U_i \cap Y) = \phi(U_i) \cap (\mathbb{C}^k \times 0_{n-k})$.

Remark. 1. If Y is a complex submanifold of X, then Y naturally has the structure of a complex manifold, since $\{\phi_i\}_i$ restricts to give a holomorphic atlas. We write $\mathrm{codim}_{\mathbb{C}}(Y, X) = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$.
2. For closed submanifolds, the condition $Y \subset \bigcup_i U_i$ can be replaced by $X = \bigcup_i U_i$. In general, the latter condition is stronger (e.g. $Y = \mathbb{C} \setminus \{0\} \subset X = \mathbb{C}$).
3. We can rephrase the definition as $\forall y \in Y, \exists F : V_y \to \mathbb{C}^{n-k}$ holomorphic, with V_y open around y, such that $\mathrm{rank}_{\mathbb{C}} J(F)_y = k$ and $F^{-1}(0) = Y \cap V_y$.
4. The inclusion $\iota : Y \to X$ is a holomorphic map iff for all $y \in Y$ we have $T^{1,0}_y Y \subset T^{1,0}_y X$.
5. Complex projective manifolds are compact complex submanifolds of $\mathbb{CP}^N$.

Definition 5.2. $Y \subset X$ is called a subvariety (or, more precisely, an analytic subvariety) of a complex manifold X if Y is closed in X and for every $p \in Y$,

there is some open neighbourhood $p \in U_p \subset Y$ such that $U_p \cap Y = f^{-1}(0)$ for some holomorphic $f : U_p \to \mathbb{C}^m$. p is a smooth (or nonsingular) point if there is a choice of f with $J(f)_p = m$. It is a singular point otherwise.
The singular locus Y^{s} of Y is the set of all singular points on Y, and the smooth locus Y^* is the set of smooth points of Y.

By the implicit function theorem, every connected component of Y^* is a complex submanifold of X. Y is irreducible if we cannot write $Y = Y_1 \cup Y_2$ such that Y_1, Y_2 are subvarieties, neither of which equals to Y.

Definition 5.3. Suppose $Y \subset X$ is a subvariety. Suppose Y is irreducible, then we write $\operatorname{codim}(Y, X)$ to denote $\operatorname{codim}(Y^*, X)$. In general, $\operatorname{codim}(Y, X) = m$ means that every component of Y^* has codimension m.
If $Y \subset X$ has codimension 1, we say Y is a(n analytic) hypersurface.

You heard subvariety. That calls for some commutative algebra.

Definition 5.4. For a complex manifold X, a germ at $x \in X$ is an equivalence class of function elements, i.e. pairs (f, U) with $f : U \to \mathbb{C}$ holomorphic, such that $x \in U$, and $(f, U) \sim (g, V)$ if there is some open $W \subset U \cap V$ with $f|_W = g|_W$. We denote the set of all germs at x by $\mathcal{O}_{X,x}$.

We sometimes write $[f, x]$, $[f]$, or simply f, to represent a germ at x, with $f : U \to \mathbb{C}$ holomorphic with $x \in U$. We get a linear map $\mathcal{O}_{X,x} \to \mathbb{C}$ via evaluation $[f, x] \mapsto f(x)$.
$\mathcal{O}_{X,x}$ is a local ring with unique maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} : f(x) = 0\}$. This is because f is a unit in $\mathcal{O}_{X,x}$ whenever $f(x)$ is nonzero.
We write $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0}$. For a complex n-fold X, we have $\mathcal{O}_{X,x} \cong \mathcal{O}_n$ by using a local chart.
We quote the following results without proof.

Theorem 5.1. *$\mathcal{O}_n$ is a unique factorisation domain.*

Remark. In case you forgot about algebra completely, a ring R is a unique factorisation domain if it has no zerodivisor and every element in R can be written as a finite product of irreducible elements (i.e. elements which can not be written as a product of two non-units), uniquely up to reordering and multiplying with units.

Theorem 5.2 (Weak Nullstellensatz)**.** *Let $f, g \in \mathcal{O}_n$ with f irreducible. Represent both of them by holomorphic functions on some open set $U \ni 0$. If g vanishes on $f^{-1}(0) \cap U$, then f divides g in $\mathcal{O}_n$.*

Definition 5.5. Let $U \subset \mathbb{C}^n$ be open. $V \subset U$ is a thin subset iff V is locally contained in the vanishing locus of a nonconstant holomorphic function.

Remark. Thin subsets are not necessarily subvarieties, as we don't ask V to be closed in U.

Theorem 5.3. *(a) If $f \in \mathcal{O}_n$ is irreducible, then there is a thin subset $V \subset \mathbb{C}^n$ and some open U around 0 such that f is irreducible in $\mathcal{O}_{\mathbb{C}^n,z}$ for all $z \in U \setminus V$.*
(b) If $f, g \in \mathcal{O}_n$ are coprime (i.e. they are both nonzero and every common divisor of them is a unit). Then there are U, V as in (a) such that f, g are coprime in $\mathcal{O}_{\mathbb{C},z}$ for all $z \in U \setminus V$.

Remark. Note that this is false if we don't remove the thin subset V. Indeed, $y^2 - xz^3$ is irreducible in $\mathcal{O}_3$ but not in $\mathcal{O}_{\mathbb{C},(x,0,0)}$ for any small $x \neq 0$.

Definition 5.6. Let $Y \subset X$ be a hypersurface and suppose $p \in Y$. Then there is a square-free $f = [(f, U_p)] \in \mathcal{O}_{X,p}$ such that $Y \cap U_p = f^{-1}(0) \cap U_p$. Any such f is known as a locally defining function of Y around p.
At $p \neq Y$, we take any unit f in $\mathcal{O}_{X,p}$ and call it a locally defining function of Y arond p.

Suppose f, g are locally defining functions for Y near p and $f = f_1 \cdots f_l, g = g_1 \cdots g_m$ with $f_i, g_j \in \mathcal{O}_{X,p}$ irreducible. Then by Theorem 5.2 (and the uniqueness part of Theorem 5.1) we know that $m = l$ and $f_i = g_i$ possibly after rearrangement.

Theorem 5.4. *Let $Y \subset X$ be a hypersurface. Then:*
(i) Y^* is open and dense in Y.
(ii) Y^* is connected if Y is irreducible.
(iii) Y^{s} is contained in a subvariety of X of codimension at least 2.

6 Holomorphic Vector Bundles

Let X be a complex manifold.

Definition 6.1. A holomorphic vector bundle of rank k over X is a complex manifold E together with a holomorphic submersion $\pi : E \to X$ such that for all $x \in X$, the fibre $E_x = \pi^{-1}(x)$ is a k-dimensional complex vector space and every $y \in X$ has an open neighbourhood U and a biholomorphic map Φ_U such that

$$\begin{array}{ccc} \pi^{-1}U & \xrightarrow{\phi_U} & U \times \mathbb{C}^k \\ \downarrow \pi & \swarrow \mathrm{pr}_1 & \\ U & & \end{array}$$

commutes and $\phi_U|_{E_x} : E_x \to \mathbb{C}^k$ is a complex linear isomorphism for all $x \in U$.

Remark. 1. E is known as the total space, and π the bundle projection, of the vector bundle. Such U is called a trivialising neighbourhood of y, and ϕ_U a local trivialisation.
2. One can think of a holomorphic vector bundle as a smooth vector bundle (of rank $2k$, fixing a smooth identification of $\mathbb{R}^{2k}$ by $\mathbb{C}^k$) equipped with a system of holomorphic local trivialisations covering X.

Let U_α, U_β be trivialising neighbourhoods (with trivialisations ϕ_α, ϕ_β) with nonempty intersection. Then $\phi_\beta \circ \phi_\alpha^{-1}(x, v) = (x, \psi_{\beta\alpha}(x)v)$ for some "holomorphic transition functions" $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$.
It also makes sense to speak of holomorphic local sections of E.

Definition 6.2. For an open $U \subset X$, a section of E on U is a holomorphic $s : U \to E$ such that $\pi \circ s$ is the identity on U.

Locally, we can express such a section by a holomorphic function to $\mathbb{C}^k$.
For holomorphic vector bundles $E, \tilde{E}$, we may construct $E \oplus \tilde{E}, E \otimes \tilde{E}, \bigwedge^r E, E^\vee$, and so on, analogously to the real case. They are all holomorphic vector bundles. With them, one can make more constructions such as $\mathrm{End}\, E = E \otimes E^*, \det E = \bigwedge^k E$.

Definition 6.3. Two holomorphic vector bundles $E, \check{E}$ are isomorphic if there exists a biholomorphic $F : E \to \check{E}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & \check{E} \\ \downarrow & \swarrow & \\ X & & \end{array}$$

commutes, and F induces complex linear isomorphisms on fibres.

Remark. Just like in the real case, any holomorphic vector bundle is determined up to isomorphism by its transition functions. Conversely, if one has a set of holomorphic transition functions which are compatible (i.e. cocycle conditions and what not), then they glue to a holomorphic vector bundle.

Definition 6.4. Suppose $\pi : E \to X$ is a holomorphic vector bundle and $f : Y \to X$ is a holomorphic map, then the pullback of E along f is the vector bundle $f^*\pi : f^*E \to Y$ such that there is a holomorphic map F making the diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & X \end{array}$$

commute, and for each holomorphic trivialisation of E over $U \subset X$, there is a holomorphic trivialisation of f^*E over $f^{-1}(U)$ on which F becomes $(b, v) \in f^{-1}(U) \times \mathbb{C}^k \mapsto (f(b), v) \in U \times \mathbb{C}^k$.

Remark. Suppose $\{U_\alpha\}_\alpha$ is a covering set of trivialising neighbourhood for X with transition functions $\psi_{\beta\alpha}$ Then f^*E is trivialised over $\{f^{-1}(U_\alpha)\}_\alpha$ with transition functions $\psi_{\beta\alpha} \circ f$.

Example 6.1. 1. $T^{1,0}X, (T^*X)^{1,0}, \bigwedge^p(T^*X)^{1,0}, K_X$ are all holomorphic vector bundles, since their transition functions are compositions of complex Jacobians for the local coordinates and some known holomorphic functions. Their holomorphic sections are called holomorphic vector fields, holomorphic 1-forms, and so on.
2. Let $Y \subset X$ be a complex submanifold. Then the inclusion $i : Y \to X$ is holomorphic and i^*E is a holomorphic vector bundle over Y, which we'll denote by $E|_Y$.

We shall mostly be interested in holomorphic line bundles for the rest of this course.

Proposition 6.1. *Isomorphism classes of holomorphic line bundles over X form an abelian group under $\otimes$.*

Proof. Let $L, \tilde{L}$ be holomorphic line bundles with transition functions $\psi_{\beta\alpha}, \tilde{\psi}_{\beta\alpha}$ (WLOG over the same trivialising neighbourhoods $\{U_\alpha\}_\alpha$), respectively. Then $L \otimes \tilde{L}$ has transition functions $\psi_{\alpha\beta}\tilde{\psi}_{\alpha\beta} = \tilde{\psi}_{\alpha\beta}\psi_{\alpha\beta}$, hence $L \otimes \tilde{L} \cong \tilde{L} \otimes L$. The identity is the trivial line bundle $\mathrm{pr}_1 : X \times \mathbb{C} \to X$, and inverses are given by duals: Indeed, L^* has transition functions $(\psi_{\beta\alpha}^\top)^{-1} = \psi_{\beta\alpha}^{-1}$. $\square$

Definition 6.5. The group as in the proposition is known as the Picard group $\text{Pic}(X)$ of X.

Corollary 6.2. *If h is a holomorphic map $Y \to X$, then the pullback h^* gives a homomorphism* $\text{Pic}(X) \to \text{Pic}(Y)$

Example 6.2. The tautological line bundle $\mathcal{O}(-1)$ is a holomorphic line bundle over $\mathbb{C}P^n$ (when $n = 2$, this is sometimes known as the Hopf bundle). It is constructed as follows: Start with the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n, (z_0, \ldots, z_n) \to [z_0 : \cdots : z_n]$. Then the preimage of a point is isomorphic to $\mathbb{C} \setminus \{0\}$, so we want to construct $E = \mathcal{O}(-1)$ such that $\mathbb{C}^{n+1} \setminus \{0\}$ is E removing (the image of) its zero section.
Let's work out the "trivialisations" on π and see if we can extend the corresponding "transition functions" so that we get some transition functions that allow gluing.
Let $U_i = \{z_i \neq 0\} \subset \mathbb{C}P^n$, then the "trivialisations" are $\phi_i^{-1}([z_0 : \cdots : z_n], w) = (w/z_i)(z_0, \ldots, z_n)$ and $\phi_j(\zeta_0, \ldots, \zeta_n) = ([\zeta_0\zeta_j : \cdots : 1_j : \cdots : \zeta_n/\zeta_j], \zeta_j) \in U_j \times \mathbb{C}$. So $\phi_{ji}(z) = z_j/z_i \in \text{GL}(1, \mathbb{C})$ and satisfies the cocycle condition, hence glue to a holomorphic line bundle.
$\mathcal{O}(1) = \mathcal{O}(-1)^*$ is known as the hyperplane bundle. We further define, for any positive integer m, $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ and $\mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m}$. By convention, $\mathcal{O}(0) = \mathcal{O} = \underline{\mathbb{C}} = \mathbb{C}P^n \times \mathbb{C}$ is the trivial line bundle.
Thus we get a group homomorphism $\mathbb{Z} \to \text{Pic}(\mathbb{C}P^n)$. We shall see that this is an isomorphism during the rest of this course.

To study line bundles more easily, we introduce the notion of divisors.

7 Divisors

Definition 7.1. A meromorphic function f on X is a function defined away from a nowhere dense subset of X that's locally a quotient of holomorphic functions. More precisely, there exists an open cover $X = \bigcup_i U_i$ and holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$ such that $f|_{U_i} = g_i/h_i$ and $g_i h_j = g_j h_i$ on $U_i \cap U_j$.

Remark. More precisely, a meromorphic function is an equivalence class of these, where two such functions are equivalent iff they agree away from a nowhere dense subset of X.

Remark. If $\dim X = 1$, any meromorphic function on X can be extended to a holomorphic map $X \to \mathbb{C}P^1$. There is NOT an analogy of this in higher dimensions. Indeed, take $X = \mathbb{C}^n, n > 1$ and consider the meromorphic function $f(z) = z_j/z_i$ for $j \neq i$ (i.e. $g(z) = z_j, h_i = z_i$). Then g/h is undefined on $\{z_j = z_i = 0\}$ and f has no possible continuous extension there.

Definition 7.2. A divisor on X is a locally finite formal sum $D = \sum_j a_j Y_j$ of irreducible hypersurfaces $Y_j \subset X$, with $a_j \in \mathbb{Z}$. D is effective (written $D \geq 0$) if $a_j \geq 0$ for all j.

Remark. Here, "locally finite" means that for any $p \in X$, there is some open $U \ni x$ such that $U \cap Y_j = \varnothing$ for all but finitely many j. In particular, if X is compact, then this is a finite sum.

One can obviously add divisors.

Definition 7.3. $\operatorname{Div}(X)$ is the abelian group of all divisors on X.

We'll assume from now on that X is compact.
For a divisor D, the compactness of X means that there is a finite open cover $\{U_\alpha\}_\alpha$ of X, and for all α we have a holomorphic function $f_{\alpha,j} : U_\alpha \to \mathbb{C}$ locally cutting out Y_j. We can then assign to D a collection of data $\{(U_\alpha, f_\alpha)\}_\alpha$ where $f_\alpha = \prod_{j=1}^N f_{\alpha,j}^{a_j}$ are viewed as the locally defining functions of D. If $D \geq 0$, each f_α is holomorphic on U_α. In general, each f_α is meromorphic.
Let $Y \subset X$ be an irreducible hypersurface and $p \in Y^*$. Let $f \in \mathcal{O}_{X,p}$ be (the germ of) a locally defining function of Y.

Definition 7.4. For $g \in \mathcal{O}_{X,p}$ nonzero, the order $\operatorname{ord}_{Y_p}(g)$ of g at p is the maximal $a \in \mathbb{Z}$ such that $f^a \mid g$ in $\mathcal{O}_{X,p}$.

This is well-defined by Theorem 5.1.

Lemma 7.1. $\operatorname{ord}_{Y,p}(g)$ *is almost locally constant as a function of* p*, in the sense that there is some open* $U \ni p$ *and a thin set* V *of codimension at least* 2 *in* X *such that* $\operatorname{ord}_{Y,p}(g) = \operatorname{ord}_{Y,q}(g)$ *for all* $q \in U \setminus V$*.*

Proof. Theorem 5.3. □

Definition 7.5. For an irreducible hypersurface $Y \subset X$ and holomorphic function g, we set $\operatorname{ord}_Y(g) = \operatorname{ord}_{Y,p}(g)$ for any $p \in Y^*$ away from the thin set as in the lemma.

This is well-defined by Theorem 5.4 and the fact that Y^{s} is thin in Y and $\operatorname{codim}(V, X) \geq 2$. It's easy to see that $\operatorname{ord}_Y(gh) = \operatorname{ord}_Y(g) + \operatorname{ord}_Y(h)$.

Definition 7.6. Suppose $F \neq 0$ is a meromorphic function. For an irreducible hypersurface Y, pick an open set U with nonempty intersection with Y on which F is locally given by g/h. Then we set $\operatorname{ord}_Y(F) = \operatorname{ord}_Y(g) - \operatorname{ord}_Y(h)$.

If $\operatorname{ord}_Y(F) = d > 0$, we say F has a zero of order d along Y; If $\operatorname{ord}_Y(F) = -d < 0$, we say F has a pole of order d along Y.

Definition 7.7. For a meromorphic function $f \neq 0$ on X, its divisor is defined as

$$(f) = \sum_{Y \text{ irred. hypersurface on } X} \operatorname{ord}_Y(f) Y$$

Any divisor of this form is known as a principal divisor.

Remark. 1. One can check that (f) is indeed a divisor.
2. Recall that the compactness assumption on X means that every divisor is a finite sum of hypersurfaces.
3. By Hartog's lemma (example sheet), $(f) \geq 0$ iff f is holomorphic.
4. For $f, g \neq 0$, we have $(fg) = (f) + (g), (f/g) = (f) - (g)$.
5. If $\dim X = 1$, then $\operatorname{Div}(X)$ is the free abelian group on X. For $\dim X > 1$, it's possible that $\operatorname{Div}(X) = 0$. But nontrivial divisors always exist wehn X is projective. Roughly, if X sits inside $\mathbb{C}P^N$ for some N, then $Y = X \cap H$ is a divisor on X for a general hyperplane H in $\mathbb{C}P^N$. They are called hyperplane sections.

Definition 7.8. Two divisors D, E are linearly equivalent if $D-E$ is a principal divisor.

Suppose $F : Z \to X$ is a holomorphic maps between compact connected complex manifolds. Let $D = \sum_i a_i Y_i$ be a divisor on X. Suppose $F(Z)$ is not contained in any Y_i with $a_i \neq 0$. Then we can formulate the pullback $F^*D \in \mathrm{Div}(Z)$ as follows: Take a finite open cover $\{U_\alpha\}_\alpha$ such that for all i, α, we can take $f_{\alpha,i} : U_\alpha \to \mathbb{C}$ cutting out Y_i locally. Recall that we can assign to D the data $\{(U_\alpha, f_\alpha)\}_\alpha$ where $f_\alpha = \prod_j f_{\alpha,j}^{a_j}$. Then F^*D shall be the divisor on Z corresponding to the data $\{F^{-1}(U_\alpha), f_\alpha \circ F\}_\alpha$.

Remark. If $D = Y$ is a irreducible hypersurface, F^*D needs not be irreducible, and can even have multiplicities.

For a divisor D with associated data $\{(U_\alpha, f_\alpha)\}_\alpha$, we can also define $\psi_{\beta\alpha} = f_\beta/f_\alpha : U_\alpha \cap U_\beta \to \mathbb{C}$, which is holomorphic and misses zero. They satisfy the cocycle condition $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. So they give a set of gluing data for a holomorphic line bundle $[D] \in \mathrm{Pic}(X)$.

Definition 7.9. $[D]$ is called the associated line bundle to D.

Definition 7.10. A section of a holomorphic line bundle $L \to X$ is holomorphic if it is expressed by holomorphic functions on trivialising neighbourhoods.

If s_α is the local form of a holomorphic section of $L \to X$ on U_α, then we have $s_\beta = \psi_{\beta\alpha}s_\alpha$ on $U_\alpha \cap U_\beta$ for all α, β.

Proposition 7.2. *(i)* $[D]$ *does not depend on the choice of* f_α.
(ii) Suppose $F : Z \to X$ *is a holomorphic map, then* $F^*[D] = [F^*D]$.
(iii) $[D + D'] = [D] \otimes [D']$, *so we get a homomorphism* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.
(iv) D *is principal iff* $[D]$ *is trivial, so the homomorphism in (iii) factors through an injective homomorphism* $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$.

Proof. (i) If $\tilde{f}_\alpha = h_\alpha f_\alpha$ is another choice with h_α necessarily holomorphic and nonzero on U_α, then $\tilde{\psi}_{\beta\alpha} = \psi_{\beta\alpha}(h_\beta/h_\alpha)$. So the two line bundles are related by tensoring with the line bundle L defined by the transition data $(h_\beta/h_\alpha)_{\beta\alpha}$. But L has a nonzero holomorphic section h locally given by h_α, so L is (holomorphically) trivial, i.e. the two line bundles are isomorphic.
(ii) (iii) Clear from the form of transition functions.
(iv) Suppose $D = (f)$, then we can take $f_\alpha = f|_{U_\alpha}$. Then the transition functions are just the identities.
Conversely, if $[D]$ is trivial, then it has a nonzero holomorphic section s. Write $s_\alpha : U_\alpha \to \mathbb{C}^\times$ for its local form. Then $s_\beta = \psi_{\beta\alpha}s_\alpha$, so essentially $s_\beta/s_\alpha = \psi_{\beta\alpha} = f_\beta/f_\alpha$. So we can patch together the local data f_α/s_α to get a global meromorphic function f on X whose divisor is D. □

We want to understand the homomorphism as in part (iv) of the preceding proposition. We need some extra work to establish this.

Definition 7.11. A meromorphic section of $L \in \mathrm{Pic}(X)$ is a section (defined away from a nowhere dense set) that's locally given by meromorphic functions.

Proposition 7.3. *Suppose* s, s' *are two meromorphic sections with* s' *not identically zero, then* $s = fs'$ *for some meromorphic* f. *Conversely, if* s *is a meromorphic section and* f *is a meromorphic function, then* fs *is a meromorphic section.*

Remark. In particular, if s' is a meromorphic section that's not identically zero, then $f \mapsto fs'$ is an isomorphism between the vector spaces of meromorphic functions on X and meromorphic sections of L. This works on any open $U \subset X$.

Let s be a nonzero meromorphic section of L and U_α, U_β trivialising neighbourhoods of L. We write $s_\alpha = s|_{U_\alpha}, s_\beta = s|_{U_\beta}$. Then $s_\alpha / s_\beta = \psi_{\alpha\beta}$ is a nonvanishing holomorphic function on $U_\alpha \cap U_\beta$. So for any irreducible hypersurface Y in X, we have $\operatorname{ord}_Y(s_\alpha) = \operatorname{ord}_Y(s_\beta)$.
Therefore $\operatorname{ord}_Y(s)$, and therefore $(s) = \sum_Y \operatorname{ord}_Y(s) Y \in \operatorname{Div}(X)$, are well-defined. And of course $(s) \geq 0$ iff s is a holomorphic section.
Suppose $L = [D]$ for some $D \in \operatorname{Div}(X)$. Let $\{(U_\alpha, f_\alpha)\}_\alpha$ be the locally defining functions of D. Then $f_\beta = \psi_{\beta\alpha} f_\alpha$ by definition of $[D]$. The local data $\{f_\alpha\}_\alpha$ gives rise to a section s of $[D]$. And $(s) = D$ by definition, so we might as well also write $[(s)] = [D]$. In general, if s is any nonzero meromorphic section of a line bundle L, then $L = [(s)]$. If $\tilde{s}$ is another nonzero section, then $\tilde{s} = fs$ for some meromorphic f and so $(\tilde{s}) = (s) + (f)$.
Conversely, given $L \in \operatorname{Pic}(X)$, this gives a bijection between the set of divisors D with $[D] = L$ and the set of nonzero meromorphic sections of L up to a nonzero scalar. Therefore the image of the homomorphism in Proposition 7.2 is the set of holomorphic line bundles which admits a nonzero meromorphic section.

Remark. For a divisor D, we can consider the vector space $\mathcal{L}(D)$ which collects zero and meromorphic functions f such that $D + (f) \geq 0$. Then $\mathcal{L}(D)$ is essentially just the space of holomorphic sections on $[D]$. It is a fact that if X is compact then $\mathcal{L}(D)$ is finite-dimensional.

8 The First Chern Class

Consider a complex line bundle $L \to X$ over a smooth manifold X. Denote by $\mathrm{d}_A : \Gamma(L) \to \Gamma(T^*X \otimes L)$ the invariant derivative associated to a connection A on L. Locally, $\mathrm{d}_A s_\alpha = \mathrm{d}s_\alpha + A_\alpha s_\alpha$ for $A_\alpha \in \Omega^1(U_\alpha)^{\mathbb{C}}$ the local forms of A, transforming as $A_\beta = A_\alpha + \psi_{\beta\alpha} \mathrm{d}(\psi_{\beta\alpha}^{-1})$ (note that we are in the rank 1 case). More generally, d_A can be regarded as a map $\Omega^r_X(L) \to \Omega^{r+1}_X(L)$.
The curvature $F(A) \in \Omega^2_X(\operatorname{End} L) = \Omega^2(X)^{\mathbb{C}}$ (noting $\operatorname{End} L = L \oplus L^*$ is topologically trivial) of a connection A is defined by the identity $\mathrm{d}_A \mathrm{d}_A s = F(A)s$. Locally, $F(A)|_{U_\alpha} = \mathrm{d}A_\alpha$.

Remark. This does NOT mean that $F(A)$ is exact: The A_α's do not in general glue to a 1-form on X since they have a different transformation law.

Recall that the difference a of two connections lives in $\Omega^1(X)^{\mathbb{C}}$, and their curvatures differ by $\mathrm{d}a$. Thus we can identify a class $[F(A)] \in H^2(X, \mathbb{C}) \cong H^2_{\mathrm{dR}}(X) \otimes \mathbb{C}$ which depends only on L. This is almost the first Chern class, except we need some extra steps.
Fix a fixed Hermitian inner product on L, which is just a Hermitian norm, represented locally by $h_\alpha : U_\alpha \to \mathbb{R}_{>0}$ computing the norm of $1 \in \mathbb{C}$ on fibres.

Definition 8.1. A connection A is unitary if for all $s, \hat{s} \in \Gamma(L)$, we have $\mathrm{d}\langle s, \hat{s} \rangle = \langle \mathrm{d}_A s, \hat{s} \rangle + \langle s, \mathrm{d}_A \hat{s} \rangle$.

In a unitary local trivialisation (i.e. where the Hermitian inner product becomes the standard norm), any unitary connection A is purely imaginary, i.e.

$iA_\alpha \in \Omega^1(U_\alpha)$.

Definition 8.2. For a unitary connection A on L, $c_1(L) = [iF(A)/(2\pi)] \in H^2_{\mathrm{dR}}(X)$ is called the first Chern class of L.

Example 8.1. $c_1(\mathbb{C}_X) = 0$.

Proposition 8.1. *$c_1(L \otimes \tilde{L}) = c_1(L) + c_1(\tilde{L})$. In particular, $c_1(L^*) = -c_1(L)$.*

Proof. Let $s \in \Gamma(L), \tilde{s} \in \Gamma(\tilde{L})$, then locally $s_\alpha \cdot \tilde{s}_\alpha = (s \otimes \tilde{s})|_{U_\alpha}$. Let $A, \tilde{A}$ be connections on $L, \tilde{L}$ respectively. Then $\mathrm{d}_{A\otimes\tilde{A}}(s\otimes\tilde{s}) = (\mathrm{d}s + As)\otimes\tilde{s} + s\otimes(\mathrm{d}\tilde{s} + \tilde{A}\tilde{s})$. So locally $\mathrm{d}_{A\otimes\tilde{A}}(s_\alpha\tilde{s}_\alpha) = \mathrm{d}(s_\alpha\tilde{s}_\alpha) + (A_\alpha + \tilde{A}_\alpha)s_\alpha\tilde{s}_\alpha$. Hence $\mathrm{d}_{A\otimes\tilde{A}}(\mathrm{d}_{A\otimes\tilde{A}}(s\otimes\tilde{s}) = (F(A) + F(\tilde{A}))(s\otimes\tilde{s})$, hence the proposition. □

Remark. 1. For a general (not necessarily unitary) connection A, we know that $iF(A)$ is the sum of a closed real 1-form and an exact complex 1-form.
2. A unitary connection always exists, but is not unique in general. However, there is a "best" unitary connection to choose.

Proposition 8.2 (Chern connection)**.** *Suppose L is a holomorphic line bundle together with a Hermitian inner product. Then there is a unique unitary connection A on L such that, on holomorphic local trivialisations $U_\alpha \subset X$, we have $A_\alpha \in \Omega^{1,0}(U_\alpha)$.*

Proof. We show uniqueness first. For this, let's restrict to $U = U_\alpha$, where we use the shorthand $A = A_\alpha$. Also let $h = h_\alpha : U \to \mathbb{R}_{>0}$ be the Hermitian norm. The choice of a local trivialisation on U is the choice of a holomorphic section $e : U \to \mathbb{C}$ with $e \equiv 1$. Now any other C^∞ section over U is given by $s = \lambda e$ for some smooth function $\lambda : U \to \mathbb{C}$.
Let's now impose the conditions. For A to be unitary, we have

$$\begin{aligned}\mathrm{d}|s|^2 &= \langle \mathrm{d}_A s, s\rangle + \langle s, \mathrm{d}_A s\rangle = \langle (d\lambda + A\lambda)e, \lambda e\rangle + \langle \lambda e, (\mathrm{d}\lambda + A\lambda)e\rangle \\ &= h\bar{\lambda}\,\mathrm{d}\lambda + h\lambda\,\mathrm{d}\bar{\lambda} + h\lambda\bar{\lambda}(A + \bar{A})\end{aligned}$$

But $\mathrm{d}|s|^2 = \mathrm{d}(\lambda\bar{\lambda}h) = h\lambda\,\mathrm{d}\bar{\lambda} + h\bar{\lambda}\,\mathrm{d}\lambda + \lambda\bar{\lambda}\,\mathrm{d}h$. So we must have $A + \bar{A} = h^{-1}\,\mathrm{d}h$, so $A^{1,0} = h^{-1}\,\partial h = \partial \log h$. This gives uniqueness.
For existence, note that the formula $\partial \log h_\alpha$ gives a local Chern connection on U_α. Now suppose U_β is another trivialisation. Then the transition functions $\psi_{\alpha\beta}$, being holomorphic, has $\mathrm{d}\psi_{\alpha\beta} = \partial\psi_{\alpha\beta}$. So $h_\beta = \psi_{\alpha\beta}\bar{\psi}_{\alpha\beta}h_\alpha$. Thus

$$A_\beta = \partial \log h_\beta = \partial \log h_\alpha + \frac{\partial \psi^{-1}_{\beta\alpha}\bar{\psi}_{\beta\alpha}}{\psi^{-1}_{\beta\alpha}\bar{\psi}_{\beta\alpha}} = A + \psi_{\beta\alpha}\,\mathrm{d}\psi^{-1}_{\beta\alpha}$$

which is the transformation law we need. □

Corollary 8.3. *The curvature of the Chern connection is*

$$F(A) = \bar{\partial}\partial \log |e|^2 = \frac{i}{2}\mathrm{dd^c} \log |e|^2$$

where e is any local nonvanishing holomorphic section.

Note also that $F(A)$ need not be $\bar{\partial}$-exact.

Corollary 8.4.
$$\frac{i}{2\pi}[F(A)] = c_1(L) \in H^{1,1}(X)$$

Now let's see what happens when X is compact and connected. Suppose $Y \subset X$ is an analytic hypersurface, then Y too is compact. For any $\phi \in \Omega^{2n-2}(X)$ with $\mathrm{d}\phi = 0$, we can send $[\phi] \in H^{2n-2}_{\mathrm{dR}}(X)$ to

$$\int_{Y^*} \phi \in \mathbb{R}$$

where the orientation on Y^* is given by the complex structure. This is a well-defined linear functional. Moreover, by Poincaré duality, there is some $\eta_Y \in H^2_{\mathrm{dR}}(X)$ (the Poincaré dual of Y) such that

$$\int_{Y^*} \phi = \int_X \eta_X \wedge \phi$$

In general, for any $D = \sum_i a_i Y_i \in \mathrm{Div}(X)$, we can define $\eta_D = \sum_i a_i \eta_{Y_i} \in H^2_{\mathrm{dR}}(X)$. So, given a divisor D, we can on one hand obtain a cohomology class η_D, and on the other hand obtain a cohomology class which is the first Chern class of the line bundle it defines.

Proposition 8.5. $\eta_D = c_1([D])$.

Corollary 8.6. $c_1([D])$ *in fact lives in the image of the natural homomorphism* $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}) = H^2_{\mathrm{dR}}(X)$.

Remark. Sadly, this homomorphism is not injective, so our definition of c_1 contains slighly less information than the topological definition of Chern classes.

To prove Proposition 8.5, we need to show that for any closed $\phi \in \Omega^{2n-2}(X)$, we have
$$-\frac{1}{2\pi i}\int_X F(A) \wedge \phi = \sum_i a_i \int_{Y_i^*} \phi$$

where A is a Chern connection on $[D]$ for some Hermitian inner product on it, and $D = \sum_i a_i Y_i$.

We may reduce to the case where $D = Y$ is just a hypersurface. As X is compact, we can cover it by finitely many trivialising neighbourhoods $X = \bigcup_\alpha U_\alpha$ for $[D]$. Let f_α be the local defining functions for $D = Y$ on U_α, i.e. $Y = (s)$ for a meromorphic section s of $[D]$ given by f_α.

For $\epsilon > 0$, let $X(\epsilon) \subset X$ be the collection of $p \in X$ such that $|s(p)| > \epsilon$ where $|\cdot|$ is the chosen Hermitian norm. Then

$$\int_X F(A) \wedge \phi = \frac{i}{2} \lim_{\epsilon \to 0} \int_{X(\epsilon)} (\mathrm{dd^c} \log |s|^2) \wedge \phi$$

By Stokes' Theorem, this can be written as

$$-\frac{i}{2} \lim_{\epsilon \to 0} \int_{\partial X(\epsilon)} (\mathrm{d^c} \log |s|^2) \wedge \phi$$

with the obvious choice of orientation that makes Stokes' Theorem work. Now

$$|s|^2|_{U_\alpha \cap (X \setminus X(\epsilon))} = f_\alpha \bar{f}_\alpha h_\alpha$$

where h_α is the local expression for the Hermitian norm. Then

$$\mathrm{d^c} \log |s|^2 = i(\bar\partial - \partial)\log(f_\alpha \bar f_\alpha h_\alpha) = i(\bar\partial \log \bar f_\alpha - \partial \log f_\alpha + (\bar\partial - \partial) h_\alpha)$$

The volume ("surface area"?) of $\partial X(\epsilon)$ tends to 0 as $\epsilon \to 0$.
We may assume the compact set $\bar U_\alpha$ is contained in a trivialising neighbourhood. ∇h_α is bounded there. So

$$\lim_{\epsilon\to 0} \int_{\partial X(\epsilon)\cap U_\alpha} (\mathrm{d^c} \log h_\alpha) \wedge \phi = 0$$

Also, ϕ is a real differential form, so

$$\int_{\partial X(\epsilon)\cap U_\alpha} (\bar\partial \log \bar f_\alpha) \wedge \phi = \overline{\int_{\partial X(\epsilon)\cap U_\alpha} (\partial \log f_\alpha) \wedge \phi}$$

Combining everything, we conclude

$$-\frac{i}{2} \lim_{\epsilon\to 0} \int_{\partial X(\epsilon)\cap U_\alpha} (\mathrm{d^c} \log |s|^2) \wedge \phi = \lim_{\epsilon\to 0} -i \operatorname{Im} \int_{\partial X(\epsilon)\cap U_\alpha} (\partial \log f_\alpha) \wedge \phi$$

To simplify this integral further, we observe the following: We can extend $f_\alpha(z)$ to local coordinates $z_1 = f_\alpha(z), \ldots, z_n$. Also we can assume U_α is a coordinate polydisc for this. Then we reach

$$-\frac{i}{2} \lim_{\epsilon\to 0} \int_{\partial X(\epsilon)\cap U_\alpha} (\mathrm{d^c} \log |s|^2) \wedge \phi = -i \operatorname{Im} \lim_{\epsilon\to 0} \int_{|z_1| = \epsilon/\sqrt{h_\alpha}} \frac{\mathrm{d}z_1}{z_1} \wedge \phi_1(z_1, \ldots)$$

where $\phi = \tilde\phi + \phi_1$ where $\tilde\phi$ collects all terms in ϕ containing $\mathrm{d}z_1, \mathrm{d}\bar z_1$. By the residue theorem, this is

$$-i2\pi \int_{z_1 = 0} \phi_1(0, z_2, \ldots, z_n)$$

Patching together the U_α's gives the desired conclusion.

9 Line Bundles on Projective Spaces

Example 9.1. Let X be a connected compact Riemann surface. Any $D \in \mathrm{Div}(X)$ is a finite formal sum of points on X. For any $P \in X$, $[P] \in H_0(X, \mathbb{Z})$ is a generator, and $P \mapsto [P]$ extends to a homomorphism $\deg : \mathrm{Div}(X) \to \mathbb{Z}$, which is explicitly given by $\sum_P a_P P \mapsto \sum_P a_P$.
Now $\eta_P \in H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ is a generator. We can write $\eta_P = [\phi]$ where $\phi \in \Omega^2(X)$ has

$$\int_X \phi = 1$$

Suppose now that L is a holomorphic line bundle over X. Its degree, in this instance, is defined as

$$\deg L = \langle c_1(L), [X] \rangle = -\frac{1}{2\pi i} \int_X F(A)$$

where A is some connection on L and $[X] \in H_2(X, \mathbb{Z})$ is the fundamental class of X. Proposition 8.5 tells us that if $L = [D]$ then $\deg L = \deg D$.
Now $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ is a surjective homomorphism, so, for each value of c_1, there exists a holomorphic line bundle on X with nontrivial meromorphic sections.
Let's now specialise to the case where $X = \mathbb{C}P^1$ and see what happens. Nonconstant holomorphic maps $\mathbb{C}P^1 \to \mathbb{C}P^1$ are precisely the rational functions. And every rational function as such has the same number of zeros and poles, counted with multiplicity. Conversely, given collections of desired zeros and poles that are equally many, we can construct a rational function with exactly those zeros and poles. So two divisors on $\mathbb{C}P^1$ are linearly equivalent precisely when they have the same degree.
Recall the Hopf bundle $\pi : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C}P^1$ which extends to the tautological line bundle $\mathcal{O}(-1)$. Then $[z_0 : z_1] \mapsto (z_0/z_1, 1)$ induces a meromorphic section of $\mathcal{O}(-1)$. Locally, s looks like $s_1 \equiv 1$ over $U_1 = \{z_1 \neq 0\}$ and $s_0 = 1/z$ over $U_0 = \{z_0 \neq 0\}$. So s has no zeros and has a unique simple pole at $[1 : 0]$. Aaaaaand so $\deg \mathcal{O}(-1) = \deg s = -1$. In general, $\deg \mathcal{O}(k) = k$ for all $k \in \mathbb{Z}$. This allows us to classify line bundles on $\mathbb{CP}^1$ in view of the next proposition.

Proposition 9.1. *If a holomorphic line bundle $L \to \mathbb{C}P^1$ has $c_1(L) = 0$, then L is holomorphically trivial.*

Proof. First of all, L is trivial as a smooth bundle: Since $c_1(L) = 0$, $F(A)$ is exact for all connections A on L. We can then make a choice with $F(A) = 0$. This means that we always have $\mathrm{d}A_\alpha = 0$ on a local trivialisation U_α.
In the case of $\mathbb{C}P^1$, we can take the trivialising open cover to be $U_0 = \{|z| < R\}, U_1 = \{|z| > 1/R\}$ for $R > 1$. By the classical Poincaré lemma, A_α is exact on U_α, i.e. $A_\alpha = \mathrm{d}a_\alpha$ on U_α. Then $\mathcal{A}_\alpha + \psi^{-1}\,\mathrm{d}\psi = 0$ for $\psi = e^{-a_\alpha}$.
WLOG $A_0 = A_1 = 0$, so $\mathrm{d}\psi_{01}$ and hence ψ is constant on $U_0 \cap U_1$. So we can extend ψ to U_0 and obtain a global trivialisation.
To show that L is holomorphically trivial, we need only to produce a holomorphic nonvanishing section. We already know that there is a diffeomorphism $L \to \mathbb{C}P^1 \times \mathbb{C}$. Choose a Hermitian norm on the fibres of L and let A be the Chern connection. So $\mathrm{d}_A = \partial_A + \bar{\partial}_A$. And we write $\mathrm{d} + A = (\partial + A') + (\bar{\partial} + A'')$. A section s is holomorphic iff $\bar{\partial}_A s = 0$. For $f : \mathbb{C}P^1 \to \mathbb{C}$, we test sections of the form $s = e^f$. For $0 = \bar{\partial}_A s = \bar{\partial} s + A'' s$, it is necessary and sufficient that $\bar{\partial} f = -A''$. We claim that $H^{0,1}(\mathbb{C}P^1) = 0$, which means that we can always solve this equation.
Take the open cover $\mathbb{C}P^1 = U_0 \cup U_1$ where $U_0 = \mathbb{C}, U_1 = \mathbb{C}P^1 \setminus \{0\}$. By Theorem 3.9, there are some $f_j : U_j \to \mathbb{C}$ with $\bar{\partial} f_j = -A''|_{U_j}$. So $\bar{\partial}(f_1 - f_0) = 0$ on $\mathbb{C} \setminus \{0\}$, therefore we might write a Laurent series

$$f_1 - f_0 = \sum_{n=-\infty}^{\infty} c_n z^n$$

valid for $z \in \mathbb{C} \setminus \{0\}$. Then just put

$$f = \begin{cases} f_0 + \sum_{n \geq 0} c_n z^n & \text{on } U_0 \\ f_1 - \sum_{n \leq -1} c_n z^n & \text{on } U_1 \end{cases}$$

which is well-defined on $\mathbb{C}P^1$ and $\bar{\partial} f = -A''$. □

Corollary 9.2. *Every line bundle on $\mathbb{C}P^1$ has the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$. In particular,* $\operatorname{Pic}\mathbb{C}P^1 \cong \mathbb{Z}$.

Remark. 1. In fact $\operatorname{Pic}\mathbb{C}P^n \cong \mathbb{Z}$ for all n. It is generated by $\mathcal{O}(1)$ which has a holomorphic section whose divisor is a hyperplane. Also, via the isomorphism $H^2(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$ we have $c_1(\mathcal{O}(k)) = k$.
2. In example sheet, you will have computed $E = \mathbb{C}/\Lambda$ has Picard group $\operatorname{Pic}(E) \cong \mathbb{Z} \oplus E$.

Definition 9.1. Let X be a complex n-manifold and $Y \subset X$ a nonsingular hypersurface. Then the normal bundle of Y is $N_{Y/X} = (T^{1,0}X)|_Y / T^{1,0}(Y)$. Its dual $N^*_{Y/X}$ is called the conormal bundle.

On fibres over p, $N^*_{Y/X}$ takes value $\{\alpha \in (T^*_p X)^{1,0} : \alpha|_{T^{1,0}_p Y} = 0\}$.
Now let f_α be a locally defining function of Y over $U_\alpha \subset X$. Then $(\mathrm{d}f_\alpha)_p \in (T^*_p X)^{1,0}$ for all $p \in U_\alpha$, and moreover $df_\alpha|_{T^{1,0}(Y \cap U_\alpha)} = 0$. But $(\mathrm{d}f_\alpha)_p \neq 0$ for all $p \in Y \cap U_\alpha$. So $\mathrm{d}f_\alpha$ defines a local nonvanising holomorphic section, i.e. a trivialisation, of $N^*_{Y/X}$.
Recall that $\psi_{\alpha\beta} = f_\alpha / f_\beta$ are transition functions of $[Y]$. What are transition functions of $N^*_{Y/X}$? We have $\mathrm{d}f_\alpha = \mathrm{d}(\psi_{\alpha\beta} f_\beta) = (\mathrm{d}\psi_{\alpha\beta}) f_\beta + \psi_{\alpha\beta}\,\mathrm{d}f_\beta = \psi_{\alpha\beta}\,\mathrm{d}f_\beta$. Hence for sections s of $N^*_{Y/X}$, we have $s_\beta\,\mathrm{d}f_\beta = s_\alpha\,\mathrm{d}f_\alpha$ which is equivalent to say $s_\alpha = \psi^{-1}_{\alpha\beta} s_\beta$.
This means that $[Y]|_Y \otimes N^*_{Y/X}$ is holomorphically trivial.

Theorem 9.3 (Adjunction Formula, vol. 1)**.** $N^*_{Y/X} = [-Y]|_Y$.

Since $\mathrm{d}f_\alpha$ is nonvanishing on Y, f_α extends to a local complex coordinates $f_\alpha, \zeta_2, \ldots, \zeta_n$ on a neighbourhood U_α. Any holomorphic local section of K_X on U_α has the form $h\,\mathrm{d}f_\alpha \wedge \mathrm{d}\zeta_2 \wedge \cdots \wedge \mathrm{d}\zeta_n$. Suppose U_β is another such neighbourhood with $U_\alpha \cap U_\beta \neq \varnothing$, then we have $f_\beta = G_{\beta\alpha}(f_\alpha, \zeta_2^{(\alpha)}, \ldots, \zeta_n^{(\alpha)}) f_\alpha$ for some $G_{\beta\alpha}(0, \zeta^{(\alpha)}) = \psi_{\beta\alpha}(\zeta^{(\alpha)}) \neq 0$. On $U_\alpha \cap U_\beta \cap Y$, we have $\zeta^{(\beta)} = F_{\alpha\beta}(\zeta^{(\alpha)})$ for some $F_{\beta\alpha}$.
We conclude that $K_X|_Y = N^*_{Y/X} \otimes K_Y$.

Theorem 9.4 (Adjunction Formula, vol. 2)**.** $K_Y = (K_X \otimes [Y])|_Y$.

We can apply these most easily to projective manifolds, in particular hypersurfaces in $\mathbb{C}P^n$. To do this, we need to compute the canonical bundle on $\mathbb{C}P^n$. Let $[z_0 : \cdots : z_n]$ be the homogeneous coordinates on $\mathbb{C}P^n$ and $U_i = \{z_i \neq 0\}$, $H_i = \{z_i = 0\}$. On U_0, we have complex coordinates $w_i = z_i / z_0$. Now $\omega = w_1^{-1}\,\mathrm{d}w_1 \wedge \cdots \wedge w_n^{-1}\,\mathrm{d}w_n$ is a meromorphic section of $K_{\mathbb{CP}^n}$ over U_0, and $\operatorname{ord}_{H_j} \omega = -1$ for each $j = 1, \ldots, n$.
How about its order on H_0? Choose any $j > 0$. Then on U_j we have coordinates $\tilde{w}_k = z_k / z_j$ for $k \neq j$, and the change-of-coordinates is $w_i = \tilde{w}_i / \tilde{w}_0$ for $i \neq j, 0$ and $w_j = 1/\tilde{w}_0$.
So $w_i^{-1}\,\mathrm{d}w_i = \tilde{w}_i^{-1}\,\mathrm{d}\tilde{w}_i - \tilde{w}_0^{-1}\,\mathrm{d}\tilde{w}_0$ for $i \neq j, 0$ and $w_j^{-1}\,\mathrm{d}w_j = -\tilde{w}_0^{-1}\,\mathrm{d}\tilde{w}_0$. Therefore

$$\omega = (-1)^j \tilde{w}_0^{-1}\,\mathrm{d}\tilde{w}_0 \wedge \cdots \wedge \widehat{\tilde{w}_j^{-1}\,\mathrm{d}\tilde{w}_j} \wedge \cdots \wedge \tilde{w}_n^{-1}\,\mathrm{d}\tilde{w}_n$$

So $\operatorname{ord}_{H_0} \omega = -1$ too. Hence ω has a simple pole along each H_i (and is nonvanishing and holomorphic elsewhere).
Now all H_i are linearly equivalent (indeed, for any homogeneous irreducible

polynomial $p(z_0, \ldots, z_n)$ of degree k, the hypersurface given by its vanishing locus is linearly equivalent to kH_0.). Hence $K_{\mathbb{C}P^n} = \mathcal{O}(-n-1)$ noting that $\mathcal{O}(-1) = [-H_0]$.

10 Blow-Ups

Let $\Delta \subset \mathbb{C}^n$ be a polydisc centred at the origin. We put $\tilde{\Delta} = \{(z, w) \in \Delta \times \mathbb{C}P^{n-1} : z_i w_j = z_j w_i\}$.
$\tilde{\Delta}$ is made into a complex manifold as follows: For each standard chart $h_j : U_j \subset \mathbb{C}P^{n-1} \to \mathbb{C}^{n-1}$, we put $\hat{h}_j : (\Delta \times U_j) \cap \tilde{\Delta} \to \mathbb{C}^n, (z, w) \mapsto (h_j(w), z_j)$.

Definition 10.1. The map $\sigma : \tilde{\Delta} \to \Delta, (z, w) \mapsto z$ is known as the blow-up of Δ at the origin.

Sometimes we just call $\tilde{\Delta}$ the blow-up.

Remark. 1. If $n = 1$, then $\tilde{\Delta}$ and σ is a biholomorphism. For general n, σ maps $\tilde{\Delta} \setminus \sigma^{-1}(0)$ biholomorphically onto $\Delta \setminus \{0\}$. And $\sigma^{-1}(0)$ is a submanifold of $\tilde{\Delta}$ isomorphic to $\mathbb{C}P^{n-1}$.
2. If we let $\Delta = \mathbb{C}^n$, then the second projection $\tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$ is essentially the tautological line bundle on $\mathbb{C}P^{n-1}$.

We can of course extend this notion of blow-up to any complex n-manifold X. For $x \in X$, we choose a chart $\phi : U \to \Delta$ around x. Then we define $\tilde{X} = (X \setminus \{x\}) \cup_{\phi^{-1} \circ \sigma} \tilde{\Delta}$ (i.e. identifying $\tilde{\Delta} \setminus \sigma^{-1}(0) \cong U \setminus \{x\}$). This is a well-defined complex manifold equipped with a map $\sigma = \sigma_x : \tilde{X} \to X$.

Definition 10.2. $\sigma_x : \tilde{X} \to X$ is known as the blow-up of X at x. By construction, it restricts to a biholomorphism $\tilde{X} \setminus \sigma_x^{-1}(x) \cong X \setminus \{x\}$. $E = \sigma_x^{-1}(x) \cong \mathbb{C}P^{n-1}$ is known as the exceptional divisor of this blow-up, which is an element of $\mathrm{Div}(\tilde{X})$.

Lemma 10.1. *$\sigma_x : \tilde{X} \to X$ is independent of the choice of the chart ϕ.*

Proof. To give a chart is the same as to give a set of complex local coordinates. Suppose $(z_i)_i$ are the coordinates associated to ϕ and $(z'_j = f_j(z))_j$ an alternative set of coordinates. Consider $w'_j = \sum_j (\partial f_j / \partial z_i)(0) w_i$. We claim that $F : (z, w) \mapsto (z', w')$ defines a map $\tilde{\Delta} \to \tilde{\Delta}'$ which is biholomorphic and makes the diagram

$$\begin{array}{ccc} \tilde{\Delta} & \xrightarrow{F} & \tilde{\Delta}' \\ \sigma \downarrow & & \downarrow \sigma' \\ \Delta & \xrightarrow{f} & \Delta' \end{array}$$

commute. This implies the lemma.
If f is given by a linear $A = (A^i_j) \in \mathrm{GL}(n, \mathbb{C})$, then

$$z'_i w'_j = \sum_{k,l} A^k_i A^l_j z_k w_l = \sum_{k,l} A^k_i A^l_j z_l w_k = z'_j w'_i$$

So everything follows.
Knowing this, we can assume WLOG that $(\partial f_j / \partial f_i)(0) = \delta_{ij}$. Consequently $w'_j = w_j$ for all j. Note that

$$(\hat{h}'_i \circ \hat{h}_i^{-1})(w_1, \ldots, w_{n-1}, z) = (w_1, \ldots, w_{n-1}, z + \text{higher order terms})$$

Thus F has derivative I at x, so it is biholomorphic. □

Proposition 10.2. $[E]|_E = \mathcal{O}_{\mathbb{C}P^{n-1}}(-1)$.

Proof. Look at transition functions. □

Proposition 10.3. *Let $\sigma : \tilde{X} \to X$ be the blow-up of X at $x \in X$. Then $K_{\tilde{X}} = \sigma^* K_X \otimes [(n-1)E]$.*

Proof. We'll show this with the assumption that K_X has nontrivial meromorphic sections. So there is a nonzero meromorphic $(n, 0)$-form ω on X. Then $\sigma^*\omega$ is a meromorphic $(n, 0)$-form on $\tilde{X}$.
Since σ is biholomorphic away from E, the zeros and poles of $\sigma^*\omega$ away from E are the zeros and poles of ω, both counted with multiplicities. So we just need to know what happens on E.
Near $x \in X$, we write $\omega = f(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ with $\bar{\partial} f = 0$. In local coordinates, $\sigma|_{(\Delta \times U_j) \cap \tilde{\Delta}}$ sends $(v_1, \ldots, v_{n-1}, z)$ to $(zv_1, \ldots, z, \ldots, zv_{n-1})$. Therefore

$$\begin{aligned}\sigma^*\omega = \sigma^*(f(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n) &= (f \circ \sigma)\,\mathrm{d}(zv_1) \wedge \cdots \wedge \mathrm{d}z \wedge \cdots \wedge \mathrm{d}(zv_{n-1}) \\ &= z^{n-1}(f \circ \sigma)\,\mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}z \wedge \cdots \wedge \mathrm{d}v_{n-1}\end{aligned}$$

We thus get an extra zero of order $n-1$ along $E \cap ((\Delta \times U_1) \cap \tilde{\Delta}) = \{z = 0\}$. □

Remark. If X was a complex surface, then we already know that

$$-1 = \deg[E]|_E = \int_E c_1(E) = n = E \cdot E$$

where $\cdot$ is the topological intersection pairing. So E has self-intersection -1, which means that we cannot deform it on X as a complex manifold.

Perhaps it's time to talk about connected sums.
Suppose M_1, M_2 are smooth m-manifolds. Fix $p_1 \in M_1, p_2 \in M_2$. Write $\phi_i : U_i \subset M_i \to \mathbb{R}^m$ for local charts around p_i, with $\phi_i(U_i) \supset B_3 = \{x : \|x\| < 3\}$.
Consider $\xi : x \mapsto x/\|x\|^2$ for $1/2 < \|x\| < 2$, say. This is an orientation-reversing diffeomorphism on the spherical shell $\{1/2 < \|x\| < 2\}$.

Definition 10.3. The connected sum $M_1 \# M_2$ of M_1, M_2 at p_1, p_2 is $(M_1 \setminus \phi_1^{-1}(\{\|x\| < 1/2\})) \cup_{\phi_2^{-1} \circ \xi \circ \phi_1} (M_2 \setminus \phi_2^{-1}(\{\|x\| < 1/2\}))$

So basically we stretched out a tube around p_1, p_2, cut them open, and fitted them together.

Remark. $M_1 \# M_2$ is orientable whenever M_1, M_2 are.

Proposition 10.4. *The blow-up $\tilde{X} \to X$ at $x \in X$ is diffeomorphic to $X \# \overline{\mathbb{C}P^n}$ at $x \in X$ and any $p \in \overline{\mathbb{C}P^n}$.*

Here, $\overline{\mathbb{C}P^n}$ is just $\mathbb{C}P^n$ equipped with the reverse orientation.

Proof. WLOG $X = \Delta$ is a ball $\{\|x\| \le 2\}$ and $x = 0$. We want to show that $\tilde{\Delta}$ is diffeomorphic to $\overline{\mathbb{C}P^n} \setminus K$ for some small closed smooth coordinate ball K.
It's convenient to identify $\overline{\mathbb{C}P^n} = \{[\bar{z}_0 : z_1 : \cdots : z_n]\}$. Let $\phi : (1 : z) \mapsto \mathbb{C}^n$ and $K = \phi^{-1}(\{\|z\| \le 1/2\}) = \{[\bar{z}_0 : z] : \|z\| > (1/2)|z_0|\}$.
We put $\psi : \overline{\mathbb{C}P^n} \setminus K \to \tilde{\Delta}, [\bar{z}_0 : z] \mapsto ((z_0/\|z\|^2)z, \Pi(z))$ where $\Pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$ is the quotient map. This is an orientation-revering diffeomorphism.
Finally, note that if $|z_0|/2 < \|z\| < 2|z_0|$, then $\sigma \circ \psi(1 : z) = \xi(z)$. So this is indeed the connected sum. □

11 Hermitian and Kähler Metric

Definition 11.1. A Hermitian metric on a complex manifold X is positive-definite Hermitian inner product h on $T^{1,0}X$ such that $h_p : T_p^{1,0}X \times T_p^{1,0}X \to \mathbb{C}$ is smooth in p. That is, for any smooth sections A, B of $T^{1,0}X$, we have $h(A, B) \in C^\infty(X, \mathbb{C})$.

In local coordinates, we write $h = \sum_{i,j} h_{i\bar{j}}(z)\,\mathrm{d}z_i\,\mathrm{d}\bar{z}_j$ where the local defining functions $h_{i\bar{j}} = h(\partial/\partial z_i, \partial/\partial z_j)$ are smooth and $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$.
By the way, a Riemannian metric g is J-invariant if $g(JA, JB) = g(A, B)$ for all $A, B \in TX^{\mathbb{R}}$.

Proposition 11.1. *There is a natural identification between Hermitian metrics on a complex manifold X and J-invariant Riemannian metrics on $X^{\mathbb{R}}$ (i.e. X but viewed as a real manifold).*

Proof. Recall that we have a map $\gamma : T_xX \to T_x^{1,0}X$ sending e to $e - iJe$ which is an isomorphism of $\mathbb{R}$-vector spaces. Then $\gamma(Je) = i\gamma(e)$.
Suppose we are given a Hermitian metric h. Define g by $g(u, v) = (1/2)\operatorname{Re} h(u - iJu, v - iJv)$. Since h is Hermitian, this is a Riemannian metric which is J-invariant since $h(iA, iB) = h(A, B)$.
Conversely, suppose g is a J-invariant Riemannian metric. We can extend g to $T_xX \otimes_{\mathbb{R}} \mathbb{C}$. Call the extension h_g. So $h_g(\lambda u, \mu v) = \lambda\bar{\mu}g(u, v)$ for any $u, v \in T_xX$ and $\lambda, \mu \in \mathbb{C}$. Then $h = h_g|_{T_x^{1,0}X \times T_x^{1,0}X}$ is a Hermitian metric.
It's easy to check that these two procedures are inverse to each other. $\square$

In local coordinates $z_1, \ldots, z_n$ (and we write $\partial/\partial z_j = (1/2)(\partial/\partial x_j - i\partial/\partial y_j)$), this correspondence is given by

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = g\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right) = 2\operatorname{Re} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = 2\operatorname{Re} h_{i\bar{j}}$$

what if we take the imaginary part of h?

Definition 11.2. The fundamental form of a Hermitian metric h is $\omega(u, v) = -(1/2)\operatorname{Im} h(u - iJu, v - iJv)$.

Proposition 11.2. *ω is is a $(1, 1)$-form. Furthermore, if g is the J-invariant Riemannian metric corresponding to h, then $\omega(u, v) = g(Ju, v)$. In particular, any two of ω, g, J determine the third.*

Proof. ω is a real 2-form since h is Hermitian. It has type $(1, 1)$ iff $\omega(Ju, Jv) = \omega(u, v)$, which holds since, under the correspondence $u \mapsto u - iJu$, Ju is mapped to $i(u - iJu)$. Finally, $-(1/2)\operatorname{Im} h(u - iJu, v - iJv) = (1/2)\operatorname{Re} h(i(u - iJv), v - iJv) = (1/2)\operatorname{Re}(Ju + iu, v - iJv) = g(Ju, v)$. $\square$

Remark. In local coordinates,

$$g = 2\sum_{i,j}\big((\operatorname{Re} h_{i\bar{j}})(\mathrm{d}x_i\,\mathrm{d}x_j + \mathrm{d}y_i\,\mathrm{d}y_j) + (\operatorname{Im} h_{i\bar{j}})(\mathrm{d}x_i\,\mathrm{d}y_j - \mathrm{d}x_j\,\mathrm{d}y_i)\big)$$

Indeed, we already know what's up with the first term. As for the second term, we observe that

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = 2\operatorname{Re} h\left(\frac{\partial}{\partial z_i}, i\frac{\partial}{\partial z_j}\right) = 2\operatorname{Im} h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$
$$= -\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right)$$

Lemma 11.3. *In local coordinates, $\omega = i\sum_{i,j} h_{i\bar{j}}\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$.*

Proof. $i\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j = i(\mathrm{d}x_i \wedge \mathrm{d}x_j + \mathrm{d}y_i \wedge \mathrm{d}y_j) + (\mathrm{d}x_i \wedge \mathrm{d}y_j + \mathrm{d}x_j \wedge \mathrm{d}y_i)$. By our calculation in the preceding remark, $\omega(\partial/\partial x_i, \partial/\partial x_j) = -2\operatorname{Im} h_{i\bar{j}} = \omega(\partial/\partial y_i, \partial/\partial y_j)$ and $\omega(\partial/\partial x_i, \partial/\partial y_j) = g(\partial/\partial y_i, \partial/\partial y_j) = 2\operatorname{Re} h_{i\bar{j}}$. Therefore

$$\begin{aligned}\sum_{i,j} h_{i\bar{j}} i(\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j) &= \sum_i h_{i\bar{i}} i(\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i) + \sum_{i<j} 2(\operatorname{Re} h_{i\bar{j}}) i(\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j) \\ &= \sum_{i,j} 2(\operatorname{Re} h_{i\bar{j}})\,\mathrm{d}x_i \wedge \mathrm{d}y_j \\ &\qquad - 2\sum_{i<j} (\operatorname{Im} h_{i\bar{j}})(\mathrm{d}x_i \wedge \mathrm{d}x_j + \mathrm{d}y_i \wedge \mathrm{d}y_j) \\ &= \omega\end{aligned}$$

□

Corollary 11.4. *For any $a \in T^{1,0}X$, we have $-i\omega(a, \bar{a}) = \sum_{i,j} h_{i\bar{j}} a_i \bar{a}_j > 0$ if $a \neq 0$.*

Definition 11.3. A real $(1,1)$-form σ is positive if $-i\sigma(a, \bar{a}) > 0$ for any $a \in T^{1,0}X$ nonzero. If this happens, we write $\sigma > 0$.
For a complex line bundle L, we say $c_1(L) > 0$ if $c_1(L)$ is represented by a positive $(1,1)$-form.

By what we have done, any positive $(1,1)$-form ω determines a Hermitian metric h on X.

Definition 11.4. For a complex manifold X, we write $c_1(X) = -c_1(K_X)$.

Definition 11.5. If $c_1(X) > 0$, we call X a Fano manifold. If $c_1(X) = 0$, we call X a Calabi-Yau manifold.

If $f : Y \to X$ is a holomorphic immersion, i.e. $(\mathrm{d}f)^{\mathbb{C}} = T^{1,0}_y Y \to T^{1,0}_{f(y)} X$ is injective, and g is a Riemannian metric on X, then f^*g is a well-defined Riemannian metric on Y and f^*g is J-invariant if g is.
In particular, a Hermitian metric is induced on any immersed complex submanifold. Locally, we can find complex coordinates $z_1, \ldots, z_n$ around $f(y) \in X$ such that Y is locally given by $Y = \{z_{k+1} = \cdots = z_n = 0\}$ and f is locally given by $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$. From here it's clear that the fundamental form of f^*h is simply $f^*\omega$.

Definition 11.6. A Hermitian manifold (X, ω) is called a Kähler manifold if $\mathrm{d}\omega = 0$, in which case ω is called a Kähler form and h a Kähler metric.

Example 11.1. 0. On $\mathbb{C}^n$, the standard Hermitian metric $h = (1/2)\sum_j \mathrm{d}z_j\,\mathrm{d}\bar{z}_j$ is Kähler since $\omega = (1/2)\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ is closed.

1a. The standard metric on $\mathbb{C}^n$ gives a well-defined metric on any complex torus $\mathbb{C}^n/\Lambda$ where $\Lambda \leq \mathbb{C}^n$ is a lattice. And ω descends to a Kähler form on $\mathbb{C}^n/\Lambda$.
1b. Any never-zero 2-form on a Riemann surface that induces the same orientation as a complex structure is a closed positive $(1,1)$-form. So any Hermitian metric on a Riemann surface is automatically Kähler.
2. $\mathbb{C}P^n$ has a Kähler metric. Consider the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$. Let $V_j = \{z \in \mathbb{C}^{n+1} : z_j = 1\}$ be an affine hyperplane. Then $\pi(V_j) = U_j = \{z_j \neq 0\} \subset \mathbb{C}P^n$.
Locally, we can write done the form

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2 \in \Omega^{1,1}(V_j)$$

Let's show that these patch together globally.
Recall that the change-of-coordinates takes $z \in V_j$ to $fz \in V_k$ where $f = z_j/z_k$ is holomorphic and nonvanishing on $V_j \setminus \{z_k = 0\}$. Then

$$\frac{i}{2\pi}\partial\bar{\partial}\log\|fz\|^2 = \frac{i}{2\pi}\partial\bar{\partial}(\log\|z\|^2 + \log(f\bar{f}))$$

Now

$$\partial\bar{\partial}\log(f\bar{f}) = \partial\frac{f\bar{\partial}\bar{f}}{f\bar{f}} = -\frac{\bar{\partial}\partial\bar{f}}{\bar{f}} = 0$$

So ω is compatible with these change-of-coordinates, whence we have a global $(1,1)$-form on $\mathbb{C}P^n$, which is closed by its local description.
Now for any $T \in U(n+1)$ induces a biholomorphic automorphism of $\mathbb{C}P^n$ (which is just a projective transformation) with $T^*\omega = \omega$. It also preserves $\|\cdot\|$ on $\mathbb{C}^{n+1}$. The $U(n+1)$-action on $\mathbb{C}P^n$ is transitive, so these tell us that it suffices to check the positivity of ω at one point.

$$\begin{aligned}\omega|_{U_0} &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+\sum_{j=1}^n z_j\bar{z}_j\right) = \frac{i}{2\pi}\partial\frac{\sum_{j=1}^n z_j\,\mathrm{d}\bar{z}_j}{1+\sum_{j=1}^n z_j\bar{z}_j} \\ &= \frac{i}{2\pi}\left(\frac{\sum_{j=1}^n \mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j}{1+\sum_{j=1}^n z_j\bar{z}_j} - \frac{\left(\sum_{j=1}^n \bar{z}_j\,\mathrm{d}z_j\right)\wedge\left(\sum_{j=1}^n z_j\,\mathrm{d}\bar{z}_j\right)}{(1+\sum_{j=1}^n z_j\bar{z}_j)^2}\right)\end{aligned}$$

Yeah... Let's look at the point $[1:0:\cdots:0]$. Then this simplifies to the expression $(i/(2\pi))\sum_{j=1}^n \mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j$ which is positive.
This Kähler metric on $\mathbb{C}P^n$ is called the Fubini-Study metric.
3. More generally, any positive $(1,1)$-form representing $c_1(X)$ on a Fano manifold X makes X a Kähler manifold.
4. Any complex submanifold of a Kähler manifold is again Kähler, as ω pulls back to a closed positive $(1,1)$-form. Consequently, any projective complex manifold is Kähler.

Let (M,g) be an oriented Riemannian n-manifold. The volume form of (M,g) is an n-form $\mathrm{vol}_g \in \Omega^n(M)$ locally given by $\mathrm{vol}_g = \omega_1\wedge\cdots\wedge\omega_n$ where ω_j form a local orthonormal coframe field of 1-forms for g.
In local coordinates, if $g = \sum_{i,j} g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$, then

$$\mathrm{vol}_g = \sqrt{\det((g_{ij})_{i,j})}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$$

If M is compact, then we can define its volume to be

$$\operatorname{vol} M = \int_M \operatorname{vol}_g(M)$$

Now let (X, h) be a Hermitian complex manifold, then we get a J-invariant Riemannian metric g associated to it and a fundamental form ω. Recall from two seconds ago that $\omega(-, \cdot) = g(J-, \cdot)$.
Near every $x \in X$, we can (basically by the Gram-Schmidt process) find an "adapted" local coframe field $\omega_1, \epsilon_1, \ldots, \omega_n, \epsilon_n$ in T^*X orthonormal with respect to g and such that $\epsilon_k = -J\omega_k, \omega_k = J\epsilon_k$. Then $\omega_1 + i\epsilon_1, \ldots, \omega_n + i\epsilon_n$ is a (complex) local coframe field orthogonal with respect to h. They are i-eigenvectors of J, so they actually form a coframe field on $T^{1,0}X$.
Thus

$$h = \frac{1}{2}\sum_k (\omega_k + i\epsilon_k) \otimes (\omega_k - i\epsilon_k), g = \sum_k (\omega_k \otimes \omega_k + \epsilon_k \otimes \epsilon_k), \omega = \sum_k \omega_k \wedge \epsilon_k$$

One might also write $\omega = \sum_k \omega_k \otimes \epsilon_k - \epsilon_k \otimes \omega_k$ for more clarity.
Anyways, collecting what we've done gives

Proposition 11.5. *The volume form of a Hermitian manifold is* $\operatorname{vol}_g = \omega^n/n!$.

If $Y \subset X$ is a complex submanifold with dimension d, then recall that $\omega|_Y$ is a fundamental form for the induced Hermitian metric on Y. So $\omega^d/d!$ is the volume form on Y, hence

Corollary 11.6 (Wirtinger)**.** *Suppose Y is compact, then*

$$\operatorname{vol} Y = \frac{1}{d!}\int_Y \omega^d$$

.

Note that if ω is closed, then the right hand side depends only on the topological type of Y, hence the left hand side has to as well. This is shocking – the left hand side is a volume!
Also, $[\omega] \in H^2_{\mathrm{dR}}(X)$ is nonzero since

$$\int_X \omega^n = n! \operatorname{vol}(X) \neq 0$$

And in general $[\omega^k] \neq 0$ for all $k = 1, \ldots, n$ by Stokes' Theorem. Thus $\dim H^{2k}_{\mathrm{dR}} > 0$ for all such k.
If $Y \subset X$ is a compact complex submanifold, then

$$\int_Y \omega^d \neq 0$$

means $[Y] \neq 0$ in $H_{2d}(X, \mathbb{Z})$, i.e. the cycle Y is nontrivial.

12 Hodge Theory

Suppose (M, g) is an oriented Riemannian manifold with dimension m. The dual inner product on T^*M extends to $\bigwedge^r T^*M$ by setting $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_r} : 1 \leq$

$i_1 < \cdots < i_r \leq m\}$ to be an orthonormal basis for $\bigwedge^r T_x^* M$, where $\omega_1, \ldots, \omega_m$ is an orthonormal basis for $T_x^* M$. In particular, $\mathrm{vol}_g = \omega_1 \wedge \cdots \wedge \omega_m$ has unit norm.

Definition 12.1. The Hodge star operator $* : \bigwedge^r T_x^* M \to \bigwedge^{n-r} T_x^* M$ is a linear map satisfying the relation $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g \mathrm{vol}_g$.

The Hodge star is uniquely determined by $*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_r}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-r}}$, where $i_1, \ldots, i_r, j_1, \ldots, j_{m-r}$ is an even permutation of $1, \ldots, m$. In addition, $*^2 \alpha = (-1)^{r(m-r)} \alpha$ for $\alpha \in \bigwedge^r$. One can also check that $*$ varies smoothly, and so is gives a map $* : \Omega^r(M) \to \Omega^{m-r}(M)$.
Now suppose (X, h) is a Hermitian complex n-fold. Write g for the corresponding J-invariant Riemannian metric. Then $\dim_{\mathbb{R}} X = 2n$ and so $* : \Omega^r(X) \to \Omega^{2n-r}(X)$. Write $\omega_1, \epsilon_1, \ldots, \omega, \epsilon_n$ be an adapted local orthonormal coframe field (recall that adapted means $J\omega_k = -\epsilon_k, J\epsilon_k = \omega_k$ for all k). $(1,0)$-forms are spanned by $\{\epsilon_k + i\epsilon_k\}_k$, and $(0,1)$-forms by $(\omega_k - i\epsilon_k)_k$. We have

$$\|(\omega_{k_1} + i\epsilon_{k_1}) \wedge \cdots \wedge (\omega_{k_p} + i\epsilon_{k_p}) \wedge (\omega_{l_1} - i\epsilon_{l_1}) \wedge \cdots \wedge (\omega_{l_q} - i\epsilon_{l_q})\|_h^2 = 2^{p+q}$$

More precisely, this is the Hermitian extension of g to $\bigwedge^r T^* M \otimes \mathbb{C}$, still denoted by h. So this means that the Hermitian inner product extends to $\bigwedge^{p,q} T^* M$ for $p, q \leq n$.
We can also extend the Hodge star to $* : (\bigwedge^r T^* X)^{\mathbb{C}} \to (\bigwedge^{2n-r} T^* X)^{\mathbb{C}}$.

Lemma 12.1. $\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_h \mathrm{vol}_g$,

Proof. This is certainly true if α, β are scalar multiples of real r-forms, since $*$ is a real operator and $\bar{\mu}\bar{\beta} = \overline{\mu\beta}$. The result follows by linearity. ☐

Corollary 12.2. *The Hodge star restricts to* $* : \Omega^{p,q}(X) \to \Omega^{n-q,n-p}(X)$. *Furthermore,* $*^2|_{\Omega^{p,q}(X)} = (-1)^{p+q}$.

Definition 12.2. Write $\mathrm{d}^* = - * \mathrm{d} * : \Omega^r(X) \to \Omega^{r-1}(X)$ with the convention $\Omega^{-1}(X) = \{0\}$. The Laplacian is $\Delta = \Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \Omega^r(X) \to \Omega^r(X)$. Both of these extend $\mathbb{C}$-linearly to complex differential forms.

Example 12.1. When $X = \mathbb{C}^n$ and h is the standard Euclidean inner product, then

$$\Delta|_{\Omega^0} = -4 \sum_{k=1}^{n} \frac{\partial}{\partial z_k} \frac{\partial}{\partial \bar{z}_k}$$

Definition 12.3. $\partial^* = - * \bar{\partial} *, \bar{\partial}^* = - * \partial *$.

Thus $\bar{\partial} : \Omega^{p,q} \to \Omega^{p-1,q}, \bar{\partial}^* : \Omega^{p,q} \to \Omega^{p,q-1}$ and we have the identities $\mathrm{d}^*|_{\Omega^{p,q}} = \partial^* + \bar{\partial}^*, (\partial^*)^2 = 0 = (\bar{\partial}^*)^2, \partial^* \bar{\partial}^* = -\bar{\partial}^* \partial^*$.

Definition 12.4. The L^2-inner product is

$$\langle \epsilon, \eta \rangle_{X,h} = \int_X \langle \epsilon, \eta \rangle_h \mathrm{vol}_g = \int_X \xi \wedge *\bar{\eta}$$

This makes $\Omega^r(X)^{\mathbb{C}}, \Omega^{p,q}(X)$ into pre-Hilbert spaces (in general not complete).

Proposition 12.3. $\partial^*, \bar\partial^*$ *are the formal adjoints of* $\partial, \bar\partial$ *with respect to the* L^2*-inner product. That is,*

$$\int_X \langle \partial\alpha, \beta\rangle_h \operatorname{vol}_g = \int_X \langle \alpha, \partial^*\beta\rangle_h \operatorname{vol}_g$$
$$\int_X \langle \bar\partial\alpha, \beta\rangle_h \operatorname{vol}_g = \int_X \langle \alpha, \bar\partial^*\beta\rangle_h \operatorname{vol}_g$$

for compactly supported forms α, β *of compatible type.*

Proof. Let's prove the $\bar\partial$ version. The ∂ version would be analogous.
Suppose $\alpha \in \Omega^{p,q-1}(X), \beta \in \Omega^{p,q}(X)$ are compactly supported. Then

$$\begin{aligned}\int_X \langle \bar\partial\alpha, \beta\rangle_h \operatorname{vol}_g &= \int_X \bar\partial\alpha \wedge *\bar\beta = \int_X \bar\partial(\alpha \wedge *\bar\beta) - (-1)^{p+q-1}\alpha \wedge \bar\partial * \bar\beta \\ &= \int_X \mathrm{d}(\alpha \wedge *\bar\beta) + (-1)^{p+q} \int_X \alpha \wedge \overline{\partial * \beta} \\ &= \int_X \alpha \wedge *\overline{(- * \partial * \beta)} = \int_X \langle \alpha, \bar\partial^*\beta\rangle \operatorname{vol}_g\end{aligned}$$

□

Corollary 12.4. *(i)*

$$\int_X \langle \mathrm{d}\alpha, \beta\rangle \operatorname{vol}_g = \int_X \langle \alpha, \mathrm{d}^*\beta\rangle \operatorname{vol}_g$$

(ii) Let $(\mathrm{d^c})^* = - * \mathrm{d^c} *$*, then*

$$\int_X \langle \mathrm{d^c}\alpha, \beta\rangle \operatorname{vol}_g = \int_X \langle \alpha, (\mathrm{d^c})^*\beta\rangle \operatorname{vol}_g$$

Definition 12.5. Write $\Delta_\partial = \partial\partial^* + \partial^*\partial$ for the ∂-Laplacian and $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ for the $\bar\partial$-Laplacian. They are endomorphisms of $\Omega^{p,q}(X)$.

Each $\Delta, \Delta_\partial, \Delta_{\bar\partial}$ is formally self-adjoint. Although $\Delta = \Delta_{\mathrm{d}}$ does not in general act on (p,q)-forms.

Definition 12.6. An r-form α is (d-)harmonic if $\Delta\alpha = 0$, ∂-harmonic if $\Delta_\partial\alpha = 0$, and $\bar\partial$-harmonic if $\Delta_{\bar\partial}\alpha = 0$. We denote by $\mathcal{H}^r(X)$ the spaces of harmonic r-forms, $\mathcal{H}^{p,q}_\partial(X)$ for the space of ∂-harmonic (p,q)-forms, and $H^{p,q}_{\bar\partial}(X)$ the space of $\bar\partial$-harmonic (p,q)-forms.

Proposition 12.5. *Suppose* X *is a compact Hermitian manifold. Then:*
(i) $\Delta\alpha = 0$ *iff* $\mathrm{d}\alpha = 0$ *and* $\mathrm{d}^*\alpha = 0$.
(i) $\Delta_\partial\alpha = 0$ *iff* $\partial\alpha = 0$ *and* $\partial^*\alpha = 0$.
(i) $\Delta_{\bar\partial}\alpha = 0$ *iff* $\bar\partial\alpha = 0$ *and* $\bar\partial^*\alpha = 0$.

Proof. For (iii), this follows from the calculation $0 = \langle \Delta_{\bar\partial}\alpha, \alpha\rangle_{X,h} = \|\bar\partial^*\alpha\|^2_{X,h} + \|\bar\partial\alpha\|^2_{X,h}$.
(i) and (ii) are similar. □

Remark. We have $\Delta_{\bar\partial}\alpha = \overline{\Delta_\partial\bar\alpha}$.

Theorem 12.6 (Hodge Decomposition vol. 1). *Let (X, h) be a compact Hermitian manifold. Then:*
(i) $\mathcal{H}^{p,q}_{\bar\partial}(X)$ is finite-dimensional for all p, q, and there is an L^2-orthogonal decomposition $\Omega^{p,q}(X) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \Delta_{\bar\partial}(\Omega^{p,q}(X)) = \mathcal{H}^{p,q}_{\bar\partial}(X) \oplus \bar\partial\Omega^{p,q-1}(X) \oplus \bar\partial^*\Omega^{p,q+1}(X)$.
(ii) $\mathcal{H}^r(X)$ is finite dimensional for all r, and there is an L^2-orthogonal decomposition $\Omega^r(X)^{\mathbb{C}} = \mathcal{H}^r(X)^{\mathbb{C}} \oplus \Delta\Omega^r(X)^{\mathbb{C}} = \mathcal{H}^r(X)^{\mathbb{C}} \oplus \mathrm{d}\Omega^{r-1}(X)^{\mathbb{C}} \oplus \mathrm{d}^*\Omega^{r+1}(X)^{\mathbb{C}}$.

Proof. Omitted. □

Remark. 1. In fact, (ii) holds for any compact oriented Riemannian manifold.
2. The proof makes use of the theory of elliptic differential operators.

Suppose (X, h) is a compact Hermitian manifold. For $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X)$, it is in particular $\bar\partial$-closed, hence defines a class $[\alpha] \in H^{p,q}(X)$. This gives a $\mathbb{C}$-linear map $\mathcal{H}^{p,q}_{\bar\partial}(X) \to H^{p,q}(X)$.

Corollary 12.7. *$\mathcal{H}^{p,q}_{\bar\partial}(X) \to H^{p,q}(X)$ is an isomorphism.*

Proof. Surjectivity: For $\alpha \in \Omega^{p,q}(X)$, we can decompose $\alpha = \alpha_0 + \bar\partial\alpha_1 + \bar\partial\alpha_2$ by the preceding theorem for some $\alpha_0 \in \mathcal{H}^{p,q}_{\bar\partial}(X)$. So $\bar\partial\alpha = 0$ iff $\bar\partial\bar\partial^*\alpha_2 = 0$. But the second condition implies that $\|\bar\partial^*\alpha_2\|^2_{X,h} = 0$, and so $[\alpha] = [\alpha_0]$.
Injectivity: Suppose $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X)$ has $\alpha = \bar\partial\gamma$, then $\|\bar\partial\gamma\|_{X,h} = \langle\bar\partial^*\alpha, \gamma\rangle = 0$, thus $\alpha = 0$. □

Similarly, we get a $\mathbb{C}$-linear isomorphism $\mathcal{H}^r(X) \to H^r_{\mathrm{dR}}(X)^{\mathcal{C}}$.

Definition 12.7. The Hodge numbers of X are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X) = \dim_{\mathcal{C}} \mathcal{H}^{p,q}_{\bar\partial}(X)$.

In particular, $h^{p,0}(X)$ is the dimension of the space of holomorphic p-forms on X. If $p = n = \dim X$, $p_g(X) = h^{n,0}(X)$ is the dimension of the space of holomorphic sections of K_X, also known as the geometric genus of X. When X is a compact Riemann surface, $p_g(X)$ computes the topological genus of X.
For a (p,q)-form α, $\alpha \in \mathcal{H}^{p,q}_{\bar\partial}(X)$ iff $*\bar\alpha \in \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$. This gives an $\mathbb{R}$-linear isomorphism $\mathcal{H}^{p,q}_{\bar\partial}(X) \cong \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$. Furthermore, we have a pairing $\mathcal{H}^{p,q}_{\bar\partial}(X) \times \mathcal{H}^{n-p,n-q}_{\bar\partial}(X) \to \mathbb{C}$ via

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta \in \mathbb{C}.$$

This is a non-degenerate complex bilinear form. Indeed,

$$\int_X \alpha \wedge *\bar\alpha = \|\alpha\|^2_{X,h} > 0$$

when $\alpha \neq 0$. The existence of this pairing is known as the Kodaira-Serre duality, which may be alternatively written as the natural isomorphism $\mathcal{H}^{p,q}_{\bar\partial}(X) \cong \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)^\vee$. In particular, $h^{p,q}(X) = h^{n-p,n-q}(X)$.
If X is connected, then this tells us that $h^{n,n} = 1$. The isomorphism $\mathcal{H}^{n,n}_{\bar\partial}(X) \to \mathbb{C}$ is simply given by

$$\alpha \mapsto \int_X \alpha$$

On a general Hemitian manifold, there is no simple relation between $H^{p,q}, \mathcal{H}^{p,q}_{\bar\partial}$ and $H^r_{\mathrm{dR}}, \mathcal{H}^r$. Life is better then the metric is Kähler.
Let ω be a Kähler form on X.

Definition 12.8. The Lefchetz operator L sends $\alpha \in \Omega^r(X)$ to $\alpha \wedge \omega \in \Omega^{r+2}(X)$. Its pointwise adjoint $\Lambda : \Omega^{r-2}(X) \to \Omega^r(X)$ has the property that $\langle \Lambda\alpha, \beta\rangle_g = \langle \alpha, \mathrm{L}\beta\rangle_g$ (pointwise).

Lemma 12.8. $\Lambda = *^{-1}\mathrm{L}*$. *More explicitly, if* $\alpha \in \Omega^r(X)$, *then* $\Lambda\alpha = (-1)^r * \mathrm{L} * \alpha$.

Proof. For α, β of suitable degrees, we have $\langle \mathrm{L}\beta, \alpha\rangle_h \operatorname{vol}_g = (\beta \wedge \omega) \wedge *\bar\alpha = \beta \wedge * *^{-1} (\omega \wedge *\bar\alpha) = \langle \beta, *^{-1}\mathrm{L} * \alpha\rangle \operatorname{vol}_g$. □

Theorem 12.9 (Kähler Identities)**.** *Suppose* (X, ω) *is Kähler, then:*
(i) $[\Lambda, \bar\partial] = -i\partial^*, [\Lambda, \partial] = i\bar\partial^*$.
(ii) $[\bar\partial, \mathrm{L}] = [\partial, \mathrm{L}] = 0, [\partial^*, \Lambda] = [\bar\partial^*, \Lambda] = 0$.
(iii) $[\Lambda, \mathrm{d}] = -(\mathrm{d}^c)^*, [\mathrm{L}, \mathrm{d}^*] = \mathrm{d}^c$.
(iv) $[\bar\partial^*, \mathrm{L}] = i\partial, [\partial^*, \mathrm{L}] = -i\bar\partial$.

Proof. Clearly we really just need to prove $[\Lambda, \bar\partial] = -i\partial^*$ and $[\partial, \mathrm{L}] = 0$. The second relation follows immediately from $\partial\omega = 0$ since this means that $\partial(\alpha \wedge \omega) = (\partial\alpha) \wedge \omega$. The first relation has a long-ish proof, so let's omit the details. The gist is to first do it for $\mathbb{C}^n$ (with the standard Kähler form) by messy calculations, and extend it to the general case. This extension is possible since the Kähler condition implies the existence of complex local coordinates with $h_{i\bar j} = \delta_{ij}, \operatorname{grad} h_{i\bar j} = 0$. □

Theorem 12.10. *On a Kähler manifold* X*, we have* $\Delta = 2\Delta_{\bar\partial} = 2\Delta_\partial$*. In particular,* $\Delta(\Omega^{p,q}(X)) \subset \Omega^{p,q}(X)$.

So the notion of a harmonic form on a Kähler manifold does not depend on "which" harmonic form we chose. We will just write $\mathcal{H}^{p,q}(X)$ for $\mathcal{H}^{p,q}_{\bar\partial}(X) = \mathcal{H}^{p,q}_\partial(X)$.

Proof. We have $i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial = 0$ by Theorem 12.9. So

$$\begin{aligned}\Delta &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= \Delta_\partial + \partial\bar\partial^* + \bar\partial\partial^* + \Delta_{\bar\partial} + \bar\partial^*\partial + \partial^*\bar\partial = \Delta_\partial + \Delta_{\bar\partial}\end{aligned}$$

It remains to show that $\Delta_\partial = \Delta_{\bar\partial}$. By Theorem 12.9, $-i\Delta_\partial = \partial[\Lambda, \bar\partial] + [\Lambda, \bar\partial]\partial = -\bar\partial[\Lambda, \partial] - [\Lambda, \partial]\bar\partial = -i\Delta_{\bar\partial}$. □

Fix a compact Kähler manifold X.

Theorem 12.11 (Hodge Decomposition vol. 2)**.** *Suppose* X *is a compact Kähler manifold, then* $H^r_{\mathrm{dR}}(X) \otimes \mathbb{C} \cong \bigoplus_{p+q=r} H^{p,q}(X)$ *and* $\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}(X)}$.

Proof.

$$\begin{aligned}H^r_{\mathrm{dR}}(X) \otimes \mathbb{C} \cong \mathcal{H}^r(X) \otimes \mathbb{C} &\cong \bigoplus_{p+q=r} (\mathcal{H}^r(X) \otimes \mathbb{C}) \cap \Omega^{p,q}(X) \\ &= \bigoplus_{p+q=r} \mathcal{H}^{p,q}_{\bar\partial}(X) \cong \bigoplus_{p+q=r} H^{p,q}(X)\end{aligned}$$

Also $\overline{(\mathcal{H}^r(X) \otimes \mathbb{C}) \cap \Omega^{p,q}(X)} = (\mathcal{H}^r(X) \otimes \mathbb{C}) \cap \Omega^{q,p}(X)$. □

Corollary 12.12. *Every nonzero holomorphic p-form on X is* d*-close but never* d*-exact.*

Proof. $\{\alpha \in \Omega^{p,0} : \bar{\partial}\alpha = 0\} = \mathcal{H}^{p,0}_{\bar{\partial}}(X) = (\mathcal{H}^p(X) \otimes \mathbb{C}) \cap \Omega^{p,0}(X)$. Any anything in $\mathcal{H}^p(X) \otimes \mathbb{C}$ is d-closed but not d-exact. □

Corollary 12.13. *The odd degree Betti numbers $b_{2k+1}(X)$ are even.*

Proof. $b_{2k+1} = \sum_{p+q} h^{p,q}$ and $h^{p,q} = h^{q,p}$. □

Corollary 12.14. *The Hopf manifold (example sheet) given by $(\mathbb{C}^n \setminus \{0\})/(z \sim 2z) \cong S^{2n-1} \times S^1$ has $b_1 = 1$ (for $n > 1$), hence cannot possibly be Kähler.*

Corollary 12.15 ($\partial\bar{\partial}$-Lemma)**.** *Suppose X is a compact Kähler manifold and $\alpha \in \Omega^{p,q}(X)$ has $\mathrm{d}\alpha = 0$. Then the followings are equivalent:*
(i) $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^{p+q-1}(X)$.
(ii) $\alpha = \partial\beta$ for some $\beta \in \Omega^{p-1,q}(X)$.
(iii) $\alpha = \bar{\partial}\beta$ for some $\beta \in \Omega^{p,q-1}(X)$.
(iv) $\alpha = \partial\bar{\partial}\beta$ for some $\beta \in \Omega^{p-1,q-1}(X)$.

Proof. (iv) is of course the strongest of the four statements. We also see that if any of the four statements is true then α would be L^2-orthogonal to $\mathcal{H}^{p,q}(X)$. So it suffices to show that if $\alpha \perp \mathcal{H}^{p,q}(X)$ has $\mathrm{d}\alpha = 0$ (hence $\partial\alpha = \bar{\partial}\alpha = 0$) then α is in the image of $\partial\bar{\partial}$. We also get $\bar{\alpha} \perp \mathcal{H}^{q,p}(X)$.
By Theorem 12.11, $\bar{\alpha} = \bar{\partial}\bar{\beta}$ and $\alpha = \partial\beta$ for some β.
We can write $\beta = \gamma_0 + \bar{\partial}\gamma_1 + \bar{\partial}^*\gamma_2$ with $\partial\gamma_0 = 0$ by Theorem 12.6. So $\alpha = \partial\bar{\partial}\gamma_1 + \partial\bar{\partial}^*\gamma_2 = -\bar{\partial}\partial\gamma_1 - \bar{\partial}^*\partial\gamma_2$. But $\bar{\partial}\alpha = 0$, so $\bar{\partial}\bar{\partial}^*\partial\gamma_2 = 0$. Therefore $0 = \langle \bar{\partial}\bar{\partial}^*\partial\gamma_2, \partial\gamma_2\rangle_{X,h} = \|\bar{\partial}^*\partial\gamma_2\|^2_{X,h}$. So $\alpha = \partial\bar{\partial}\gamma_1$. □

Remark. 1. This is still true for some non-compact Kähler manifolds, such as open polydiscs with the standard Euclidean metric (using the usual Poincaré lemma and the $\bar{\partial}$-Poincaré lemma).
2. A holomorphic p-form on a compact Kähler manifold is never d-exact, hence never ∂-exact by the corollary (taking $q = 0$).
3. There is an equivalent variant of this, known as the dd^{c}-lemma.
4. If $[\xi] = [\tilde{\xi}] \in H^2_{\mathrm{dR}}(X)$ with $\xi, \tilde{\xi}$ d-closed real $(1,1)$-forms. Then $\tilde{\xi} = \xi + i\partial\bar{\partial}f$ for some $f \in C^\infty(X, \mathbb{R})$, unique up to adding a constant.

13 Another View on Kähler Manifolds

Let (X, h) be a Hermitian manifold and g the corresponding J-invariant Riemannian metric. There is a unique connection ∇ on TX (the Levi-Civita connection) such that $\mathrm{d}\langle u, v\rangle_g = \langle \nabla u, v\rangle_g + \langle u, \nabla v\rangle_g$ and $T(u, v) = \nabla_u v - \nabla_v u - [u, v] = 0$. The induced connection on $\bigwedge^2 T^*X$ is given by $\mathrm{d}(\alpha(u, v)) = (\nabla\alpha)(u, v) + \alpha(\nabla u, v) + \alpha(u, \nabla v)$ for all $\alpha \in \Omega^2(X)$. One can show that $\mathrm{d}\alpha(u, v, w) = (\nabla_u\alpha)(v, w) - (\nabla_v\alpha)(u, w) + (\nabla_w\alpha)(u, v)$. In particular, if $\nabla\alpha = 0$, then $\mathrm{d}\alpha = 0$. In fact, the composition of $\nabla : \Gamma(\bigwedge^2 T^*X) \to \Gamma(\bigwedge^2 T^*X \otimes T^*X)$ and the anti-symmetrisation $\Gamma(\bigwedge^2 T^*X \otimes T^*X) \to \Gamma(\bigwedge^3 T^*X)$ is exactly d.
Now let $\gamma : TX \to T^{1,0}X$ be such that u is sent to $(1/2)(u - iJu)$ (so $\gamma^{-1} = 2\,\mathrm{Re}$). Let $\mathrm{d}_{\mathcal{A}}$ be the unitary connection on $T^{1,0}X$, i.e. $\mathrm{d}\langle u, v\rangle_h = \langle \mathrm{d}_{\mathcal{A}}u, v\rangle_h + \langle u, \mathrm{d}_{\mathcal{A}}v\rangle_h$. This induces a connection $\nabla_{\mathcal{A}}$ on TX via $\nabla_{\mathcal{A}}u = 2\,\mathrm{Re}\,\mathrm{d}_{\mathcal{A}}(\gamma(u))$ for $u \in TX$,

which satisfies the first defining conditions for the Levi-Civita connection. Observe that $\nabla_{\mathcal{A}} J = J \nabla_{\mathcal{A}}$ since $\gamma(Ju) = i\gamma(u)$ and $\mathrm{d}_{\mathcal{A}}$ commutes with (multiplication by) i.

Proposition 13.1. *Suppose that $\nabla_{\mathcal{A}}$ is torsion-free and satisfies $T(u,v) = 0$ (i.e. $\nabla_{\mathcal{A}} = \nabla$). Then:*
(i) $\mathrm{d}_{\mathcal{A}}$ is the Chern connection for $T^{1,0}X$.
(ii) h is a Kähler metric on X.

Sketch of proof for (ii). We have

$$\begin{aligned}(\nabla\omega)(u,v) &= \mathrm{d}(\omega(u,v)) - \omega(\nabla u, v) - \omega(u, \nabla v) \\ &= \mathrm{d}(g(Ju,v)) - g(\nabla Ju, v) - g(Ju, \nabla v) = 0\end{aligned}$$

So $\nabla\omega = 0$, hence $\mathrm{d}\omega = 0$. □