Algebraic Topology *

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This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Algebraic Topology* in Michaelmas 2022. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

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0 Homotopies

When we say "space", we'll always mean a topological space; When we say "map", we'll always mean a continuous map. We write $\mathrm{Map}(X, Y)$ to denote the collection of maps between spaces X, Y.
Examples of spaces include $I = [0, 1], I^n = [0, 1]^n, D^n = \{v \in \mathbb{R}^n : \|v\| \leq 1\}, S^{n-1} = \{v \in \mathbb{R}^n : \|v\| = 1\}$. We of course know that $D^n \cong I^n, S^{n-1} \subset D^n$ and $D^n/S^{n-1} \cong S^n$ (via the action $(r, \phi) \mapsto (2r - 1, \phi)$, with the former in spherical and latter in cylindrical coordinates)).

Definition 0.1. For $f_0, f_1 : X \to Y$, we say f_0 is homotopic to f_1 if there is some continuous $H : X \times I \to Y$ with $H(-, 0) = f_0, H(-, 1) = f_1$. We write $f_0 \simeq_H f_1$ or $f_0 \simeq f_1$ to denote this.

We often write $f_t = H(-, t)$ to represent the intuition that f_0 "deforms" to f_1 via H. It's clear that homotopy is an equivalence relation on $\mathrm{Map}(X, Y)$.

Example 0.1. 1. The identity map $1_{\mathbb{R}^n}$ is homotopic to the constant map $0_{\mathbb{R}^n}$ via $f_t(x) = tx$.
2. Is the antipodal map $A_n : S^n \to S^n$ homotopic to the identity 1_{S^n}? When $n = 1$, they are via $f_t(z) = e^{2\pi it}z$. But they are not when $n = 2$. We'll develop the tools required to prove this.

Definition 0.2. $[X, Y] = \mathrm{Map}(X, Y)/\simeq$ is the set consisting of homotopy classes of maps between X and Y.

Homotopy classes behave nicely, as seen from the following easy lemma.

Lemma 0.1. *If $f_0, f_1 : X \to Y$ are homotopic, $g_0, g_1 : Y \to Z$ are homotopic, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.*

Example 0.2. If $f : X \to \mathbb{R}^n$, then $f = 1_{\mathbb{R}^n} \circ f \simeq 0_{\mathbb{R}^n} \circ f = 0$. So $\#[X, \mathbb{R}^n] = 1$.

Definition 0.3. A space Y is contractible if there is a fixed constant map c such that $1_Y \simeq c$ where 1_Y is the identity map on Y.

The idea in the previous example then shows that

Proposition 0.2. *Y is contractible iff $\#[X, Y] = 1$ for all X.*

Definition 0.4. Spaces X and Y are homotopy equivalent, written $X \simeq Y$, if there are $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq 1_Y, g \circ f \simeq 1_X$.

Example 0.3. 1. $\mathbb{R}^n \simeq \{0\}$ via the obvious maps. Indeed, Y is contractible iff it is homotopy equivalent to a one-point space.
2. $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ via $p : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, v \mapsto v/\|v\|$ and $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}, v \mapsto v$.

Some of the basic questions that algebraic topologists want to address are whether two given spaces are homotopic, and what $[X, Y]$ is. Not gonna to solve either of those at this point. Let's look at more definitions.

Definition 0.5. A pair of space is a pair (X, A) with X a space and $A \subset X$. A map between pairs of spaces $(X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$. Maps of pairs $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic if their underlying maps are homotopic via a map of pairs $H : (X \times I, A \times I) \to (Y, B)$.

Like our previous definitions, these all behave well under composition. We write $[(X, A), (Y, B)]$ for homotopy equivalence classes of maps of pairs.

Definition 0.6. If X is a space and $p \in X$, the n^{th} homotopy group $\pi_n(X, p)$ is the group with underlying set $[(I^n, \partial I^n), (X, p)] = [(D^n, S^{n-1}), (X, p)] = [(S^n, *), (X, p)]$.

The group operation is defined as follows: For $\phi, \psi : (I^n, \partial I^n) \to (X, p)$, we define $[\phi][\psi] = [\phi + \psi]$ where $\phi + \psi$ is the map constructed by cutting the hypercube I^n in half, scale both ϕ, ψ by half and mush them together (beware the gluing lemma). $\pi_n(X, p)$ has an identity given by the constant map at p and inverse given by composing it with $r : I^n \to I^n, (t_1, \dots, t_n) \mapsto (t_1, \dots, t_{n-1}, 1 - t_n)$. Notably, for $n \geq 2$ the group $\pi_n(X, p)$ must be abelian by "spinning" the half-hypercubes.
A nice thing about these homotopy groups is that they are functorial. A map $f : (X, p) \to (Y, q)$ induces $f_* : \pi_n(X, p) \to \pi_n(Y, q)$ via $f_*[\phi] = [f \circ \phi]$. This operation is (contravariant) functorial in the sense that $(f \circ g)_* = f_* \circ g_*$. If $f_0, f_1 : (X, p) \to (Y, q)$ are homotopic, then it's clear that $(f_0)_* = (f_1)_*$.

Example 0.4. $\pi_1(S^1, *) \cong \mathbb{Z}, \pi_1(S^n, *) = 0$ when $n \geq 2$. But $\pi_m(S^n, *)$ is complicated in general, e.g. we have the table

m	1	2	3	4	5	6
$\pi_m(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$

1 Singular Homology

1.1 Chain Complexes

Definition 1.1. The n-simplex Δ^n is the topological space $\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_i t_i = 1\}$.

So a 1-simplex is simply an interval, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc..
For $I \subset \{0, \dots, n\}$, we can associate with I a "face" $f_I = \{t \in \Delta^n : t_i = 0 \text{ if } i \notin I\}$ of the n-simplex. We'll write $I = i_0 \cdots i_k$ if $I = \{i_0, i_1, \dots, i_k\}$ and $i_0 < i_1 < \cdots < i_k$. To go with faces we have face maps, which are homeomorphisms $F_I : \Delta^{|I|-1} \to f_I \subset \Delta^n$ sending t to x where $x_i = 0$ if $i \notin I$ and $x_i = t_j$ if $i = i_j$.
Putting simplices aside for a minute, let's think about chain complexes. Let R be a commutative ring.

Definition 1.2. A chain complex $(C_\bullet, d)$ over R consists of:
(i) R-modules $C_i, i \in \mathbb{Z}$.
(ii) R-linear maps $d_i : C_i \to C_{i-1}$ such that $d_i \circ d_{i+1} = 0$ for all i.

We usually write $C = \bigoplus_i C_i$ to denoteb the asscociated graded module and $d = \bigoplus_i d_i : C \to C$. So the condition $\forall i, d_i \circ d_{i+1} = 0$ can be written as $d \circ d = 0$, or $d^2 = 0$.
We by definition have $\operatorname{Im} d_{i+1} \subset \ker d_i$ for all i.

Definition 1.3. For a chain complex $(C_\bullet, d)$, its i-th homology is the R-module $H_i(C_\bullet) = \ker d_i / \operatorname{Im} d_{i+1}$.

And we write $H_*(C_\bullet) = \bigoplus_i H_i(C_\bullet)$.
The d, d_i's are called the differentials (or boundary maps) of the chain complex. Elements of $\ker d$ are called closed and known as cycles and elements of $\operatorname{Im} d$ are called exact and known as boundary. For $dx = 0$, we write $[x]$ for its image in $H_*(C_\bullet)$.

Definition 1.4. The chain complex of the n-simplex $(S_\bullet(\Delta^n), d)$ is given by $S_k(\Delta^n) = \langle f_I : I \subset \{0, \ldots, n\}, |I| = k\rangle$ (the free $\mathbb{Z}$-module generated by the n-dimensional faces of Δ_n) for $0 \le k \le n$ and $S_k(\Delta^n) = 0$ for $k < 0, k > n$. The boundary map is given by

$$d(f_I) = \sum_{j=0}^{k} (-1)^j f_{I \setminus \{i_j\}}$$

where $I = i_0 \cdots i_k$.

Example 1.1. For $n = 2$, we have $d(f_{012}) = f_{12} - f_{02} + f_{01}$ and it's easily checked that $d^2(f_{012}) = (f_2 - f_1) - (f_2 - f_0) + (f_1 - f_0) = 0$.

To check $d^2 = 0$ in general, it suffices to show $d^2(f_I) = 0$ for all I. Indeed, for $I = i_0 \cdots i_k$

$$d^2(f_I) = d\left(\sum_{j=0}^{k} (-1)^j f_{I\setminus\{i_j\}}\right) = \sum_{j \ne j'} n_{j<j'} f_{I \setminus \{i_j, i_{j'}\}}$$

and $n_{jj'} = (-1)^j(-1)^{j'-1} + (-1)^{j'}(-1)^j = 0$.

Example 1.2. Again take $n = 2$. We have $\ker d_2 = 0$, so $H_2(S_\bullet(\Delta^2)) = 0$. It's also easy to check that $\ker d_1 = \operatorname{Im} d_2 = \langle f_0, f_1, f_2\rangle$, so $H_1(S_\bullet(\Delta^2)) = 0$. We have $\ker d_0 = \langle f_0, f_1, f_2\rangle$ and $\operatorname{Im} d_1 = \{\sum_i a_i f_i : \sum_i a_i = 0\}$. So $H_0(S_\bullet(\Delta^2)) \cong \mathbb{Z}$.

In fact, we have in general

$$H_i(S_\bullet(\Delta^n)) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.5. The reduced chain complex associated to Δ^n $(\tilde{S}_\bullet(\Delta^n), d)$ is the chain complex with $\tilde{S}_k(\Delta^n) = S_k(\Delta^n)$ for $k \ne -1$ and $\tilde{S}_{-1}(\Delta^n) = \langle f_\varnothing \rangle$ and the same differential applies as before, only with $I = \varnothing$ allowed (and with $df_\varnothing = 0$).

Then it's easy to show that $H_*(\tilde{S}_\bullet(\Delta^n)) = 0$.

Definition 1.6. For a space X, its singular chain complex is $(C_\bullet(X), d)$ where $C_k(X) = \langle \sigma : \Delta^k \to X \rangle$ is the free $\mathbb{Z}$-module generated by all continuous $\sigma : \Delta^k \to X$.

So the elements of $C_k(X)$ ("singular chains") are formal sums $\sum_i a_i\sigma_i$ with $a_i \in \mathbb{Z}$ and $\sigma_i : \Delta^k \to X$ continuous. The boundary map is given by

$$d\sigma = \sum_{j=0}^{k} (-1)^j \sigma \circ F_{\hat{j}}, F_{\hat{j}} = F_{\{0,\dots,n\}\setminus\{j\}}$$

extended by linearity. This is chosen such that the map $\phi_\sigma : S_*(\Delta^k) \to C_*(X), f_I \mapsto \sigma \circ F_I$ satisfies $d \circ \phi_\sigma = \phi_\sigma \circ d$. In particular, $\sigma = \phi_\sigma(f_{0\cdots n})$ and $0 = d^2\sigma = d^2(\phi_\sigma(f_{0\cdots n})) = \phi_\sigma(d^2(f_{0\cdots n})) = \phi_\sigma(0) = 0$.

Definition 1.7. $H_i(X) = H_i(C_\bullet(X))$ is the i^{th} singular homology group of X.

This is obviously a topological invariant.

Example 1.3. If $X = \{*\}$ is a one-point space, then $C_k(\{*\}) = \{\sigma_k\}$ where $\sigma_K : \Delta^k \to \{*\}$ is the unique map and $d\sigma_k = \sigma_{k-1}$ if k is even and positive and 0 if k is odd. So $\ker d = \langle \sigma_0, \sigma_1, \sigma_3, \sigma_5, \dots \rangle, \operatorname{Im} d = \langle \sigma_1, \sigma_3, \sigma_5, \dots \rangle$, so $H_*(X) = \langle [\sigma_0] \rangle \cong \mathbb{Z}$. That is,

$$H_i(\{*\}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.8. The reduced singular chain is defined the same way as before: We set $\tilde{C}_k(X) = C_k(X)$ for $k \neq -1$ and $\tilde{C}_{-1}(X) = \langle \sigma_\varnothing \rangle$, and the differentials are the only things which make sense.

Example 1.4. We have $\tilde{H}_i(\{*\}) = 0$ for all i.

Example 1.5. $\Delta^0 \cong \{*\}$, so $\operatorname{Map}(\Delta^0, X)$ are just constants.
$\Delta^1 \cong I = [0,1]$, so $\operatorname{Map}(\Delta^1, X)$ are the set of paths $\gamma : [0,1] \to X$, with $d\gamma = \sigma_{\gamma(1)} - \sigma_{\gamma(0)}$. When $X = S^1$, $\gamma : [0,1] \to S^1, t \mapsto e^{2\pi i t}$ is a cycle in $C_1(X)$. For $\gamma_\pm : t \mapsto e^{\pm\pi i t}$, we have $d\gamma_\pm = \sigma_{-1} - \sigma_1$, so $\gamma_+ - \gamma_-$ is again a cycle in $C_1(X)$. We claim that $[\gamma] = [\gamma_+ - \gamma_-]$. Indeed, let's consider $\tau : \Delta^2 \to S^1$ via $\tau(p) = e^{2\pi i\phi(p)}$ where $\phi : \Delta^2 \to \Delta^1 = [0,1]$ is the affine linear map with $f_0 \mapsto 0, f_1 \mapsto 1, f_2 \mapsto 1/2$. Then $d\tau = \tau \circ F_{\hat{0}} - \tau \circ F_{\hat{1}} + \tau \circ F_{\hat{2}} = \gamma_- - \gamma_+ + \gamma$. So indeed $[\gamma] = [\gamma_+ - \gamma_-]$.

Proposition 1.1. *If X is path-connected, then $H_0(X) = \langle \sigma_P \rangle \cong \mathbb{Z}$ for any $p \in X$.*

Proof. $C_{-1}(X) = 0$, so $\ker d_0 = C_0(X) = \langle \sigma_p : p \in X \rangle$, whereas $\operatorname{Im} d_1 = \langle d\gamma : \gamma : I \to X \rangle = \langle \sigma_p - \sigma_{p'} : p, p' \text{ joined by a path} \rangle = \langle \sigma_p - \sigma_{p'} : p, p' \in X \rangle$, hence the result. $\square$

Definition 1.9. Suppose $(C_\bullet, d)$ is a chain complex of R-modules. A subcomplex of $(C_\bullet, d)$ consists of submodules $A_i \leq C_i$ such that $d(A_i) \subset A_{i-1}$. This gives another chain complex $(A_\bullet, d)$ with the restriction of the differentials.

For a subcomplex $A_\bullet$ of $C_\bullet$, we have a chain complex $C_\bullet/A_\bullet$ consisting of modules C_i/A_i with differentials being the factors of the original differential through the quotients. This is called the quotient complex.

Example 1.6. If $A \subset X$, the $C_\bullet(A)$ is a subcomplex of $C_\bullet(X)$.

Definition 1.10. If (X, A) is a pair of spaces, $C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$ is called the singular chain complex of the pair.

Definition 1.11. If $(C_{\bullet,\alpha}, d_\alpha), \alpha \in A$ are chain complexes, the direct sum of them is the chain complex $(\bigoplus_\alpha C_{\bullet,\alpha}, \bigoplus_\alpha d_\alpha)$, which one can check is indeed a chain complex.

It's easy to see that $H_*(\bigoplus_\alpha C_{\bullet,\alpha}) \cong \bigoplus_\alpha H_*(C_{\bullet,\alpha})$.

Proposition 1.2. *If $X_\alpha, \alpha \in A$ are the path components of X, then*

$$H_*(X) = \bigoplus_{\alpha \in A} H_*(X_\alpha)$$

Proof. Since Δ^k is connected, $\mathrm{Map}(\Delta^k, X) = \coprod_\alpha \mathrm{Map}(\Delta^k, X_\alpha)$. As this decomposition respects d, the chain complex of X splits into the direct sum $C_\bullet(X) = \bigoplus_\alpha C_\bullet(X_\alpha)$, which gives the result. $\square$

1.2 Functoriality

Definition 1.12. A (locally small) category is the data of a collection of objects, as well as a set of morphisms $\mathrm{Hom}(A, B)$ for every pair of objects A, B (we write $f : A \to B$ for $f \in \mathrm{Hom}(A, B)$), equipped with the composition rule $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C), (f, g) \mapsto f \circ g$ such that:
1. $(f \circ g) \circ h = f \circ (g \circ h)$.
2. For each object A, there is an morphism $1_A \in \mathrm{Hom}(A, A)$ such that for every $f : A \to B$, we have $f \circ 1_A = 1_B \circ f = f$.

Example 1.7. We have the category of R-modules whose morphisms are R-linear maps, the category of topological spaces whose morphisms are continuous maps, the category of pairs of spaces whose morphisms are maps of pairs.

Definition 1.13. For categories $\mathcal{C}_1, \mathcal{C}_2$, a functor $F : C_1 \to C_2$ assigns an object FA in C_2 given any object A in C_1, and assigns a morphism $Ff : FA \to FB$ in C_2 given any morphism $f : A \to B$ in C_1, in such a way that $F1_A = 1_{FA}, F(f \circ g) = (Ff) \circ (Fg)$.

Definition 1.14. For chain complexes $(C_\bullet, d), (C'_\bullet, d')$ are chain complexes of R-modules. A chain map $f_\bullet : (C_\bullet, d) \to (C'_\bullet, d')$ is the data of R-linear maps $f_i : C_i \to C'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$. That is, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & C'_i \arrow[r, "d'_i"'] & C'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Lemma 1.3. *1. If $(C_\bullet, d)$ is a chain complex, the identity 1_C is of course a chain map.*
2. If f, g are chain maps such that the codomain of g is the domain of f, then $f \circ g$ is also a chain map.

We consequently have a category of chain complexes whose morphisms are chain maps. Observe that if $f_\bullet : (C_\bullet, d) \to (C'_\bullet, d')$ is a chain map, then $dx = 0$ implies $d'f(x) = f(dx) = 0$, so $f(\ker d) \subset \ker d'$. Also $f(dx) = d'f(x)$, so $f(\operatorname{Im} d) \subset \operatorname{Im} d'$. Therefore $f_\bullet$ induces a map $f_* : H_*(C_\bullet) \to H_*(C'_\bullet)$ via $f_*([x]) = [f(x)]$.

Lemma 1.4. $(1_C)_* = 1_{H_*(C_\bullet)}$, $(g \circ f)_* = g_* \circ f_*$.

Proof. Just check. ☐

So taking for each i, H_i defines a functor from the category of chain complexes of R-modules to the category of R-modules, and H_* defines a functor from the same starting category to the category of graded R-modules.

Definition 1.15. If $f : X \to Y$ is a contiuous map, the chain map associated to it is $f_\bullet : C_\bullet(X) \to C_\bullet(Y)$ via $\sigma \mapsto f \circ \sigma$.

Lemma 1.5. *$f_\bullet$ is indeed a chain map.*

Proof. How many times do I have to write "check"? ☐

Lemma 1.6. $(1_X)_\bullet = 1_{C_\bullet(X)}$ *and* $(f \circ g)_\bullet = f_\bullet \circ g_\bullet$.

And therefore there is a functor from the category of topological spaces to the category of chain complexes of $\mathbb{Z}$-modules which sends $X \to C_\bullet(X)$. Compositions of functors is of course again a functor, so we eventually get a functor $X \mapsto H_i(X)$ from the category of topological spaces to the category of $\mathbb{Z}$-modules.
Suppose $f : (X, A) \to (Y, B)$ is a map of pairs, $f_\bullet : C_\bullet(X) \to C_\bullet(Y)$ has $f_\bullet(C_\bullet(A)) \subset C_\bullet(B)$, thus descends to a map $f_\bullet : C_\bullet(X, A) \to C_\bullet(Y, B)$, functorial as usual. Taking homology givess us our favourite homology functors $(X, A) \mapsto H_i(X, A)$.

1.3 Homotopy Invariance

Definition 1.16. Suppose $g_0, g_1 : C \to C'$ are chain maps of R-modules. We say g_0 is chain homotopic to g_1 (written $g_0 \sim g_1$) if there are R-linear maps $h_i : C_i \to C'_{i+1}$ ("chain homotopy") such that $d'h + hd = g_1 - g_0$.

Lemma 1.7. *Chain homotopy is an equivalence relation.*

Proof. Apparently not enough number of times. ☐

Definition 1.17. Chain complexes C, C' are chain homotopy equivalent, written $C \sim C'$, if there exists chain maps $f : C \to C', g : C' \to C$. such that $fg \sim 1_{C'}, gf \sim 1_C$.

Proposition 1.8. *If $g_0, g_1 : C \to C'$ are chain maps with $g_0 \sim g_1$, then $(g_0)_* = (g_1)_*$.*

Proof. Suppose h is the chain homotopy. If $x \in C$ has $dx = 0$, then $(g_0)_*[x] - (g_1)_*[x] = [g_1(x) - g_0(x)] = [dh(x) + hd(x)] = [dhx] = 0$. ☐

Corollary 1.9. *If $C \sim C'$, then $H_*(C) \cong H_*(C')$.*

Why do we care about such a weird definition that is chain homotopy? Suppose $f_0, f_1 : X \to Y, f_0 \simeq f_1$ via $H : X \times [0,1] \to Y$. If $\sigma : \Delta^k \to X$ and $g_0 = f_{0*}, g_1 = f_{1*}$, then $h(\sigma)$ which is "$H(\sigma \times [0,1])$" should satisfy $dh + hd = g_0 - g_1$. Let's try to make sense of it rigorously.
Recall that if $\sigma : \Delta^k \to X$, there is a chain map $S_\bullet(\Delta^k) \to C_\bullet(X), f_I \mapsto \sigma \circ F_I$. Consider $\iota_0, \iota_1 : \Delta^n \to \Delta^n \times [0,1]$ via $x \mapsto (x, 0), x \mapsto (x, 1)$ respectively, which induces $\phi_{\iota_0}, \phi_{\iota_1} : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times [0,1])$.

Definition 1.18. If $X \subset \mathbb{R}^n$ is convex and $v_0, \dots, v_k \in X$, we define a k-simplex on X by $[v_0 \cdots v_k] : \Delta^k \to X, (t_i) \mapsto \sum_i t_i v_i$ (the "linear simplex" determined by $v_0, \dots, v_k$).

Then $[v_0 \cdots v_k] \circ F_{\hat{j}} = [v_0 \cdots \hat{v}_j \cdots v_k]$, therefore the boundary map has the nice formula $d[v_0 \cdots v_k] = \sum_j (-1)^j [v_0 \cdots \hat{v}_j \cdots v_k]$. For $f_i \in \Delta^n$ is a vertex, we write $i = f_i \times 0 \in \Delta^n \times [0,1], i' = f_i \times 1 \in \Delta^n \times [0,1]$.

Definition 1.19. The universal chain homotopy is the map $U_n : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times [0,1])$ given by

$$U_n f_{i_0 \cdots i_k} = \sum_{j=0}^{k} (-1)^j [i_0 \cdots i_j i'_j i'_{j+1} \cdots i'_k]$$

Proposition 1.10. $dU_n + U_n d = \phi_{\iota_1} - \phi_{\iota_0}$.

Proof. We have

$$\begin{aligned}(dU_n + U_n d)(f_I) = &\sum_{j<j'} m_{jj'} [i_0 \cdots \hat{i}_j \cdots i_{j'} i'_{j'} \cdots i'_k] \\ &+ \sum_{j'<j} n_{jj'} [i_0 \cdots i_{j'} i'_{j'} \cdots \hat{i}'_j \cdots i'_k] \\ &+ \sum_{j=0}^{k-1} r_j [i_0 \cdots i_j i'_{j+1} \cdots i'_k] \\ &+ a[i_0 \cdots i_k] + b[i'_0 \cdots i'_k]\end{aligned}$$

We have

$$\begin{aligned} m_{jj'} &= (-1)^j (-1)^{j'-1} + (-1)^{j'} (-1)^j = 0 \\ n_{jj'} &= (-1)^j (-1)^{j'} + (-1)^{j'} (-1)^{j+1} = 0 \\ r_j &= (-1)^j (-1)^{j+1} + (-1)^{j+1} (-1)^{j+1} = 0 \end{aligned}$$

On the other hand, $a = (-1)^k (-1)^{k+1} = -1$ and $b = (-1)^k (-1)^k = 1$. Hence the result. ☐

So, y'know, let's study $S_\bullet(\Delta^n)$.
For a face $I = i_0 \cdots i_k$, we have a chain map $\phi_I : S_\bullet(\Delta^k) \to S_\bullet(\Delta^n)$ given by $\phi_I(f_{j_0 \cdots j_l}) = f_{i_{j_0} \cdots i_{j_l}}$. Write $\phi_{\hat{j}} = \phi_{\{0,\dots,n\} \setminus \{j\}} : S_\bullet(\Delta^{n-1}) \to S_\bullet(\Delta^n)$. Let $f^n_{\text{top}} = f_{0 \cdots n} \in S_n(\Delta^n)$, then $df^n_{\text{top}} = \sum_j (-1)^j \phi_{\hat{j}} f^{n-1}_{\text{top}}$.

Lemma 1.11 (Naturality). *Let $\bar{F}_I : \Delta^k \times [0,1] \to \Delta^n \times [0,1], (x,t) \mapsto (F_I(x), t)$. Then the diagram*

$$\begin{tikzcd} S_\bullet(\Delta^k) \arrow[r, "\phi_I"] \arrow[d, "U_k"'] & S_\bullet(\Delta^n) \arrow[d, "U_n"] \\ C_\bullet(\Delta^k \times [0,1]) \arrow[r, "(\bar{F}_I)_\bullet"'] & C_\bullet(\Delta^n \times [0,1]) \end{tikzcd}$$

commutes.

Proof. This follows from the fact that $(F_I)_\bullet([j_0 \cdots j_l]) = [i_{j_1} \cdots i_{j_l}]$. $\square$

Suppose now that $f_0, f_1 : X \to Y$ are homotopic via $H : X \times [0,1] \to Y$. Given $\sigma : \Delta^n \to X$, we define $H_\sigma : \Delta^n \times I \to Y, (x,t) \mapsto H(\sigma(x), t)$. Observe that $H_{\sigma \circ F_I} = H_\sigma \circ \bar{F}_I$, so we are inspired to consider $h : C_\bullet(X) \to C_\bullet(Y)$ by $h(\sigma) = (H_\sigma)_\bullet(U_n(f^n_{\text{top}}))$ at $\sigma : \Delta^n \to X$.

Theorem 1.12. *$dh + hd = (f_0)_\bullet - (f_1)_\bullet$. So $(f_0)_\bullet$ is chain homotopic to $(f_1)_\bullet$.*

Proof.

$$\begin{aligned} hd(\sigma) = h\left(\sum_j \sigma \circ F_{\hat{j}}\right) &= \sum_j (-1)^j (H_{\sigma \circ F_{\hat{j}}})_\bullet(U_{n-1}(f^{n-1}_{\text{top}})) \\ &= \sum_j (-1)^j (H_\sigma)_\bullet \circ (\bar{F}_{\hat{j}})_\bullet(U_{n-1}(f^{n-1}_{\text{top}})) \\ &= \sum_j (-1)^j (H_\sigma)_\bullet(U_n(\phi_{\hat{j}}(f^{n-1}_{\text{top}}))) = (H_\sigma)_\bullet(U_n d(f^n_{\text{top}})) \end{aligned}$$

On the other hand, $dh(\sigma) = d((H_\sigma)_\bullet(U_n(f^n_{\text{top}}))) = (H_\sigma)_\bullet(dU_n(f^n_{\text{top}}))$, so

$$\begin{aligned} (dh + hd)(\sigma) &= (H_\sigma)_\bullet((dU_n + U_n d)(f^n_{\text{top}})) = (H_\sigma)_\bullet((\phi_{\iota_1} - \phi_{\iota_0})(f^n_{\text{top}})) \\ &= (f_1)_\bullet(\sigma) - (f_0)_\bullet(\sigma) \end{aligned}$$

as desired. $\square$

Corollary 1.13. *If $f_0, f_1 : X \to Y$ are homotopic, then $(f_0)_* = (f_1)_*$.*

Corollary 1.14. *If $f : X \to Y$ and $g : Y \to X$ give a homotopy equivalence between X and Y, then f_*, g_* are mutual inverses.*

Corollary 1.15. *If X is contractible, then*

$$H_k(X) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

1.4 Subdivision

Sike, we are actually gonna talk about homological algebra first.

Definition 1.20. A sequence of R-modules

$$\cdots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \cdots$$

is exact at A_i (or at i) if $\operatorname{Im} f_{i+1} = \ker f_i$.
We say the sequence is exact if it is exact at all i.

So an exact sequence is a chain complex with vanishing homology.

Example 1.8. 1. If

$$0 \longrightarrow A \longrightarrow 0$$

is exact at A iff $A = 0$.
2. If

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

is exact iff f is an isomorphism.
3. An sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact (and called a short exact sequence) if f is injective, g is surjective and $B/f(A) \cong C$ via g.

Definition 1.21. Suppose A, B, C are chain complexes. A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence of chain complexes if f, g are chain maps and the induced sequences

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$$

are short exact sequences for all i.

Example 1.9. If $A \subset X$, then there is a short exact sequence

$$0 \longrightarrow C_\bullet(A) \xrightarrow{f} C_\bullet(X) \xrightarrow{g} C_\bullet(X, A) \longrightarrow 0$$

The key thing about short exact sequences of complexes is the snake lemma.

Lemma 1.16 (Snake Lemma)**.** *If*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of chain complexes, then there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & H_{n+1}(A_\bullet) \arrow[r, "f_*"] & H_{n+1}(B_\bullet) \arrow[r, "g_*"] & H_{n+1}(C_\bullet) \arrow[dll, "\delta"] & \\
& H_n(A_\bullet) \arrow[r, "f_*"'] & H_n(B_\bullet) \arrow[r, "g_*"'] & H_n(C_\bullet) \arrow[r, "\delta"'] & \cdots
\end{tikzcd}$$

Proof. The map δ (the "connecting homomorphism") is defined as follows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Start with $c \in C_n$ with $dc = 0$, since f_n is surjective we can find some $b \in B_n$ such that $g_n(b) = c$. Now $g(db) = d(gb) = dc = 0$, so $db \in \ker g_{i-1} = \operatorname{Im} f_{i-1}$, hence we have some $a \in A_{i-1}$ such that $f_{i-1}(a) = db$. As $f(da) = d(fa) = d(db) = 0$, we have $da = 0$, so we can identify $\delta[c] = [a]$.
A tedious calculation reveals that δ is well-defined and does produce the claimed long exact sequence. □

Corollary 1.17. *If (X, A) is a pair of spaces, then we have a long exact sequence*

$$\cdots \longrightarrow H_{n+1}(A) \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow \cdots$$

Example 1.10. The long exact sequence of the pair $(X, \{p\})$ for $p \in X$ looks like

$$\cdots \longrightarrow 0 \longrightarrow H_2(X) \longrightarrow H_2(X, \{p\}) \longrightarrow 0 \longrightarrow H_1(X) \longrightarrow H_1(X, \{p\}) \longrightarrow \mathbb{Z} \longrightarrow H_0(X) \longrightarrow H_0(X, \{p\}) \longrightarrow 0$$

which first of all means that $H_i(X) \cong H_i(X, \{p\})$ for $i \geq 2$. Note that $f_*([\sigma_p]) = [\sigma_p] \neq 0$ (where $[\sigma_p]$ is the generator of $H_0(\{p\})$), so $f_* : H_0(\{p\}) \to H_0(X)$ is injective, and therefore $H_1(X) \cong H_1(X, \{p\})$ as well. Since $H_0(X) = \mathbb{Z}^N$ (where N is the number of path components of X), we have $H_0(X) = H_0(X, \{p\}) \oplus H_0(\{p\}) = H_0(X, \{p\}) \oplus \mathbb{Z}$.
Indeed, $\tilde{H}_i(X) \cong H_i(X, p)$. The chain complex $\tilde{C}_\bullet(X, p) = \tilde{C}_\bullet(X)/\tilde{C}_\bullet(\{p\}) \cong C_\bullet(X)/C_\bullet(\{p\}) = C_\bullet(X, p)$ has homology $\tilde{H}_*(X, p) = H_*(X, p)$. The vanishing of $\tilde{H}_*$ combined with the snake lemma then gives $\tilde{H}_i(X) \cong \tilde{H}_i(X, p) \cong H_i(X, p)$.

Let's now turn to subdivision. Suppose $\mathcal{U} = \{U_\alpha, \alpha \in A\}$ is an open covering of X. We define $C_k^{\mathcal{U}}(X) = \langle \sigma : \Delta^k \to X : \exists \alpha \in A, \operatorname{Im} \sigma \subset U_\alpha \rangle \subset C_k(X)$. This gives rise to a subcomplex $C_\bullet^{\mathcal{U}}(X)$ of $C_\bullet(X)$.
Let i be the inclusion map.

Lemma 1.18 (Subdivision Lemma). *$i_* : H_*^{\mathcal{U}}(X) = H_*(C_\bullet^{\mathcal{U}}(X)) \to H_*(X)$ is an isomorphism.*

Proof. "Too painful to write down", as it were.
For a sketch, we consider the natural maps $B_n : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n)$ and $h_n : S_\bullet(\Delta^n) \to C_\bullet(\Delta^n)$ where B_n is given by barycentric subdivision and h_n is a chain homotopy between B_n and $\phi_{\mathrm{id}_{\Delta^n}}$.
We then use B_n and h_n to define a chain homotopy between a globalised barycentric subdivision map $B : C_\bullet(X) \to C_\bullet(X)$ and $\mathrm{id}_{C_\bullet(X)}$. For $c \in C_k(X)$, this subdivison process means that for big k we have $B^k c \in C_k^{\mathcal{U}}(X)$. $\square$

This lemma allows us to compute homology very easily using the Mayer-Vietoris sequence. Suppose $U_1, U_2 \subset X$ are such that $\mathcal{U} = \{U_1, U_2\}$ form an open cover of X. We have the inclusion maps

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, "i_1"] \arrow[d, "i_2"'] & U_1 \arrow[d, "j_1"] \\ U_2 \arrow[r, "j_2"'] & X \end{tikzcd}$$

Proposition 1.19. *There is a short exact sequence*

$$0 \longrightarrow C_\bullet(U_1 \cap U_2) \xrightarrow{i} C_\bullet(U_1) \oplus C_\bullet(U_2) \xrightarrow{j} C_\bullet^{\mathcal{U}}(X) \longrightarrow 0$$

where

$$i = \begin{pmatrix} (i_1)_\bullet \\ (i_2)_\bullet \end{pmatrix}, j = ((j_1)_\bullet, -(j_2)_\bullet)$$

Proof. It's clear that both $(i_1)_\bullet, (i_2)_\bullet$ are injective, so i is injective.
Since $j_1 \circ i_1 = j_2 \circ i_2$, we have $ji = 0$. Suppose $j(a,b) = 0$ where $a = \sum_i a_i \sigma_i, b = \sum_j b_j \tau_j, \operatorname{Im} \sigma_i \subset U_1, \operatorname{Im} \tau_j \subset U_2$ with a_i, b_j nonzero and $(\sigma_i)_i$ pairwise distinct, $(\tau_j)_j$ pairwise distinct. Then we have $\sum_i a_i \sigma_i = \sum_j b_j \tau_j$ in X, which can only happen if, after rearrangement, $a_i = b_i$ and $\sigma_i = \tau_i$ (and hence has to land in $U_1 \cap U_2$). Therefore $(a,b) \in \operatorname{Im} i$.
The surjectivity of j follows from the construction of $C_\bullet^{\mathcal{U}}(X)$. $\square$

Massaging the -1 entry, we similarly have an exact sequence

$$0 \longrightarrow \tilde{C}_\bullet(U_1 \cap U_2) \xrightarrow{i} \tilde{C}_\bullet(U_1) \oplus \tilde{C}_\bullet(U_2) \xrightarrow{j} \tilde{C}_\bullet^{\mathcal{U}}(X) \longrightarrow 0$$

Corollary 1.20 (Mayer-Vietoris Sequence). *There is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & H_{n+1}(U_1 \cap U_2) \arrow[r, "i_*"] & H_{n+1}(U_1) \oplus H_{n+1}(U_2) \arrow[r, "j_*"] & H_{n+1}(X) \arrow[dll, "\delta"] & \\ & H_n(U_1 \cap U_2) \arrow[r, "i_*"'] & H_n(U_1) \oplus H_n(U_2) \arrow[r, "j_*"'] & H_n(X) \arrow[r, "\delta"'] & \cdots \end{tikzcd}$$

Similarly, we have such a sequence for reduced homology as well.

Proposition 1.21.

$$\tilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & \textit{if } i = n \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let's prove by induction on n. For $n = 0$, $S^0 = \{\pm 1\}$, so

$$H_i(S^0) = H_i(\{1\}) \oplus H_i(\{-1\}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}, \tilde{H}_i(S^0) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

In general, we consider $U_\pm = S^n \setminus \{(\pm 1, 0, \cdots, 0)\} \cong (D^n)^\circ$ by stereographic projection. Their intersection is $U_+ \cap U_- \cong (0,1) \times S^{n-1} \simeq S^{n-1}$. The Mayer-Vietoris sequence for reduced homology yields the existence of

$$\begin{tikzcd} \cdots \arrow[r] & \tilde{H}_{i+1}(S^{n-1}) \arrow[r] & 0 \arrow[r] & \tilde{H}_{i+1}(S^n) \arrow[dll, rounded corners] & \\ & \tilde{H}_i(S^{n-1}) \arrow[r] & 0 \arrow[r] & \tilde{H}_i(S^n) \arrow[r] & \cdots \end{tikzcd}$$

which gives the result inductively. $\square$

We prefer to generate $H^n(S^n)$ by $[S^n]$. If we consider $p : U_+ \cap U_- \to S^{n-1}, (x_1, \ldots, x_{n+1}) \mapsto (x_2, \ldots, x_{n+1})$. Then we have $p_* d[S_n] = [S_{n-1}]$.

Lemma 1.22. *Suppose we have a commutative diagram of chain complexes*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "f_A"] & B \arrow[r, "g"] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

with exact rows, then we have a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_i(B) \arrow[r] \arrow[d, "(f_B)_*"] & H_i(C) \arrow[r, "\delta"] \arrow[d, "(f_C)_*"] & H_{i-1}(A) \arrow[r] \arrow[d, "(f_A)_*"] & \cdots \\ \cdots \arrow[r] & H_i(B') \arrow[r] & H_i(C') \arrow[r, "\delta"] & H_{i-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

where the rows are the long exact sequences of homology.

Proof. Given $[c] \in H_i(C)$, pick $a \in A_{i-1}, b \in B_i$ with b mapping to c and a mapping to db. Then $\delta[c] = [a]$. Let $a' = f_A(a), b' = f_B(b), c' = f_C(c)$. Then $g'(b') = \pi' f_B(b) = f_C \pi(b) = f_C c = c'$ and $f'(a') = f' f_A(a) = f_B(fa) = f_B(db) = db'$. So $\delta[c'] = [a']$. The rest is easy. $\square$

Example 1.11. Suppose $f : X \to Y$ with $Y = U_1 \cup U_2$ with $U_i \subset Y$ open. Let $V_i = f^{-1}(U_i)$, then $X = V_1 \cup V_2$. Then $f_\bullet$ gives the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(V_1 \cap V_2) \arrow[r] \arrow[d] & C_\bullet(V_1) \oplus C_\bullet(V_2) \arrow[r] \arrow[d] & C_\bullet^{\{V_i\}}(X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_\bullet(U_1 \cap U_2) \arrow[r] & C_\bullet(U_1) \oplus C_\bullet(U_2) \arrow[r] & C_\bullet^{\{U_i\}}(Y) \arrow[r] & 0 \end{tikzcd}$$

which induces the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_i(V_1) \oplus H_i(V_2) \arrow[r] \arrow[d] & H_i(X) \arrow[r, "\delta"] \arrow[d] & H_{i-1}(V_1 \cap V_2) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_i(U_1) \oplus H_i(U_2) \arrow[r] & H_i(Y) \arrow[r, "\delta"] & H_{i-1}(U_1 \cap U_2) \arrow[r] & \cdots \end{tikzcd}$$

as in the lemma. The result for reduced homology is similar.

Proposition 1.23. *Let $r_n : S^n \to S^n, (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})$ be the antipodal map and decompose $S^n = U_+ \cup U_-$ as before. Then $(r_n)_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ sends $[S^n]$ to $-[S^n]$.*

Proof. For $n = 0$, we have $(r_0)_*[S^0] = [\sigma_{-1} - \sigma_1] = -[S^0]$ since $r_0(\pm 1) = \mp 1$. In general, r_n induces the diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_i(S^n) \arrow[r, "\delta"] \arrow[d] & \tilde{H}_{i-1}(U_+ \cap U_-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{H}_i(S^n) \arrow[r, "\delta"'] & \tilde{H}_{i-1}(U_+ \cap U_-) \arrow[r] & 0
\end{tikzcd}$$

The homotopy equivalence $p : U_+ \cap U_- \to S^{n-1}, (x_1, \ldots, x_{n+1}) \mapsto (x_2, \ldots, x_{n+1})$ has $p \circ r_n = r_{n-1} \circ p$, so we can in fact extend our diagram to

$$\begin{tikzcd}
\tilde{H}_i(S^n) \arrow[r, "\delta"] \arrow[d, "(r_n)_*"] & \tilde{H}_{i-1}(U_+ \cap U_-) \arrow[r, "p_*"] \arrow[d, "(r_n)_*"] & \tilde{H}_{i-1}(S^{n-1}) \arrow[d, "(r_{n-1})_*"] \\
\tilde{H}_i(S^n) \arrow[r, "\delta"'] & \tilde{H}_{i-1}(U_+ \cap U_-) \arrow[r, "p_*"'] & \tilde{H}_{i-1}(S^{n-1})
\end{tikzcd}$$

which implies $(r_n)_*[S^n] = -[S^n]$ since $(r_{n-1})_*[S^{n-1}] = -[S^{n-1}]$. $\square$

Corollary 1.24. *If $v \in S^n$, let $r_v : S^n \to S^n$ be the reflection across the plane perpendicular to v, then $(r_v)_*[S^n] = -[S^n]$.*

Proof. As S^n is path connected, r_v is homotopic to $r_{e_{n+1}} = r_n$. $\square$

1.5 Excisions and Collapsing a Pair

Definition 1.22. Suppose $A \subset Z$ (with $i : (A, A) \to (Z, A)$ the inclusion). A is a deformation retract of Z if there is a map $p : (Z, A) \to (A, A)$ such that $p \circ i = 1_{(A,A)}$ and $i \circ p \sim 1_{(Z,A)}$.

Consequently, Z is homotopic to A.

Example 1.12. $Y \times 0$ is a deformation retract of $Y \times (D^n)^\circ$.

Definition 1.23. A pair (X, A) is a good pair (Olympiads flashbacks) if there is some $U \subset X$ open, $A \subset U$ and A is a deformation retract of U.

Example 1.13. 1. If $X = S^2$ and A is the union of the north and south poles, then by taking the union of small disks around the poles, we see that (X, A) is a good pair.
2. Take $Y = T^2 = S^1 \times S^1$ and $B = S^1 \times 1 \subset Y$, then B is a deformation retract of the "band" around the "ring" that is B. So (Y, B) is a good pair.

More generally, if M is a manifold and N a submanifold of M, then (M, N) is a good pair.

Example 1.14 (Non-example)**.** $(\mathbb{R}, \mathbb{Q})$ is not a good pair.

Theorem 1.25. *Suppose (X, A) is a good pair and $\pi : (X, A) \to (X/A, A/A)$. Then $\pi_* : H_*(X, A) \to H_*(X/A, A/A) \cong \tilde{H}(X/A)$ is an isomorphism.*

Example 1.15. Suppose $X = S^2$ and A the union of the north and south poles. Then $Z = X/A$ looks like a pinched torus. Let's compute $\tilde{H}_*(Z) \cong H_*(X, A)$ using the long exact sequence for (X, A). We know that $\tilde{H}_0(A) \cong \mathbb{Z}$ and $\tilde{H}_i(A) = 0$ for $i \neq 0$. We also know that $\tilde{H}_2(X) \cong \mathbb{Z}$ and $\tilde{H}_i(X) = 0$ for $i \neq 2$. We therefore have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \tilde{H}_2(X) \arrow[r] & \tilde{H}_2(X,A) \arrow[dll] \\
 & 0 \arrow[r] & 0 \arrow[r] & \tilde{H}_1(X,A) \arrow[dll] \\
 & \tilde{H}_0(A) \arrow[r] & 0 \arrow[r] & \tilde{H}_0(X,A) \arrow[r] & 0
\end{tikzcd}$$

and hence

$$H_*(Z) = H_*(X, A) = \tilde{H}_*(X, A) \cong \begin{cases} \mathbb{Z} & \text{for } i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 1.16. Suppose $Y = S^1 \times S^1, B = S^1 \times \{1\}$. Denote as i_1, i_2 the inclusion of $S^1 = S^1 \times \{1\} = \{1\} \times S^1$ into $S^1 \times S^1$, and π_1, π_2 the projections. Then $\pi_1 \circ i_1 = \pi_2 \circ i_2 = 1_{S^1}$, so $(\pi_k)_* \circ (i_k)_* = 1_{H_*(S^1)}$, forcing $(i_k)_*$ to be injective.
We have $Y/B \cong Z$ where Z is as in the previous example. So we can compute $H_*(T^2)$ using $H_*(Y^2, B) \cong H_*(Z)$. Accio

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \tilde{H}_2(T^2) \arrow[r] & \tilde{H}_2(T^2,B) \arrow[dll] \\
 & \tilde{H}_1(B) \arrow[r] & \tilde{H}_1(T^2) \arrow[r] & \tilde{H}_1(T^2,B) \arrow[dll] \\
 & \tilde{H}_0(B) \arrow[r] & \tilde{H}_0(T^2) \arrow[r] & \tilde{H}_0(T^2,B) \arrow[r] & 0
\end{tikzcd}$$

The injectivity of $(i_1)_*$ forces $\delta : \tilde{H}_2(T^2, B) \to \tilde{H}_1(B)$ to be the zero map. We therefore have $\tilde{H}_2(T^2) \cong \mathbb{Z}$ and an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_1(T^2) \longrightarrow \mathbb{Z} \longrightarrow 0$$

which forces $\tilde{H}_1(T^2) = \mathbb{Z}^2$.

Let's do some more homological algebra (yay?).

Lemma 1.26 (Five Lemma)**.** *Suppose we have a commutative diagram of R-modules*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

with exact rows. If $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.

Proof. If you are confused by the colors, know it's Sae's fault. $\square$

Suppose $\mathcal{U} = \{U_j\}_{j\in J}$ is an open cover of X. If $A \subset X$, then $\mathcal{U}^A = \{U_j \cap A\}_{j\in J}$ is an open cover of A and $C_\bullet^{\mathcal{U}_A}(A) \subset C_\bullet^{\mathcal{U}}(X)$. Define $C_\bullet^{\mathcal{U}}(X, A) = C_\bullet^{\mathcal{U}}(X)/C_\bullet^{\mathcal{U}_A}(A)$. Then the map $i : C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ induces a map $i : C_\bullet^{\mathcal{U}}(X, A) \to C_\bullet(X, A)$.

Lemma 1.27. *$i_* : H_*^{\mathcal{U}}(X, A) \to H_*(X, A)$ is an isomorphism.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & C_\bullet^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C_\bullet^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_\bullet(A) \arrow[r] & C_\bullet(X) \arrow[r] & C_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This gives the diagram

$$\begin{tikzcd}
H_i^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & H_i^{\mathcal{U}}(X) \arrow[r] \arrow[d] & H_i^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & H_{i-1}^{\mathcal{U}_A}(A) \arrow[r] \arrow[d] & H_{i-1}^{\mathcal{U}}(X) \arrow[d] \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X, A) \arrow[r] & H_{i-1}(A) \arrow[r] & H_{i-1}(X)
\end{tikzcd}$$

with exact rows. We are then done by the preceding lemma. $\square$

Theorem 1.28 (Excision)**.** *Suppose $B \subset A \subset X$ and $\bar{B} \subset A^\circ$ and let $j : (X - B, A - B) \to (X, A)$ be the inclusion. Then $j_* : H_*(X - B, A - B) \to H_*(X, A)$ is an isomorphism.*

Proof. $\{A^\circ, X - \bar{B}\}$ is an open cover of X. For an open cover $\mathcal{U}$ of X and $\sigma : \Delta^k \to X$, we write $\sigma \lhd \mathcal{U}$ if $\operatorname{Im}\sigma \subset U$ for some $U \subset \mathcal{U}$. If $\mathcal{U} = \{A^\circ, X - \bar{B}\}$, then we have $C_\bullet^{\mathcal{U}}(X) = \langle \sigma : \sigma \lhd \mathcal{U}\rangle = C_\bullet^{\mathcal{U}}(X - B) \oplus M_B$ where $M_B = \langle \sigma : \operatorname{Im}\sigma \subset A, \operatorname{Im}\sigma \cap B \neq \varnothing\rangle$. Similarly $C_\bullet^{\mathcal{U}_A}(A) = C_\bullet^{\mathcal{U}_A}(A - B) \oplus M_B$.
For chain complexes $C' \subset C$ and M, the natural map $C/C' \to (C \oplus M)/(C \oplus M)$ is always an isomorphism. So $j_\bullet : C_\bullet^{\mathcal{U}}(X - B)/C_\bullet^{\mathcal{U}_A}(A - B) \to C_\bullet^{\mathcal{U}}(X)/C_\bullet^{\mathcal{U}_A}(A)$ is an isomorphism. Consequently, j_* is an isomorphism $H_*^{\mathcal{U}}(X - B, A - B) \to H_*^{\mathcal{U}}(X, A)$.
There is a commutative square

$$\begin{tikzcd}
H_i^{\mathcal{U}}(X - B, A - B) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H_i^{\mathcal{U}}(X, A) \arrow[d, "i_*"] \\
H_i(X - B, A - B) \arrow[r, "j_*"'] & H_i(X - A)
\end{tikzcd}$$

Since the vertical maps are isomorphisms and we just proved that the top map too is an isomorphism, we conclude that the bottom map is an isomorphism. $\square$

Proposition 1.29 (Long Exact Sequence of a Triplet)**.** *Suppose $Z \subset Y \subset X$, then there is a long exact sequence*

$$\cdots \longrightarrow H_i(Y, Z) \xrightarrow{(j_1)_*} H_i(X, Z) \xrightarrow{(j_2)_*} H_i(X, Y) \xrightarrow{\delta} H_{i-1}(Y, Z) \longrightarrow \cdots$$

where $j_1 : (Y, Z) \to (X, Z)$ and $j_2 : (X, Z) \to (X, Y)$ are inclusions.

Proof. This is the result of the short exact sequence

$$0 \longrightarrow C_\bullet(Y,Z) \longrightarrow C_\bullet(X,Z) \longrightarrow C_\bullet(X,Y) \longrightarrow 0$$

noting $C_\bullet(Y,Z) = C_\bullet(Y)/C_\bullet(Z)$, etc.. □

Recall that $A \subset U$ is a deformation retract of U if there exists $p : U \to A$ such that $p \circ i = \mathrm{id}_A$ and $i \circ p \sim \mathrm{id}_{(U,A)}$ as maps of pairs, where $i : A \to U$ is the inclusion.

Lemma 1.30. *If A is a deformation retract of U and $U \subset X$, then $j_* : H_*(X,A) \to H_*(X,U)$ is an isomorphism, where $j : (X,A) \to (X,U)$ is the inclusion.*

Proof. The long exact sequence for (U,A) reads

$$H_i(A) \xrightarrow{i_*} H_i(U) \longrightarrow H_i(U,A) \longrightarrow H_{i-1}(A) \xrightarrow{i_*} H_{i-1}(U)$$

i_* are isomorphisms, since i induces a homotopy equivalence, therefore $H_*(U,A)$ vanishes. Then the long exact sequence for $A \subset U \subset X$

$$H_i(U,A) \longrightarrow H_i(X,A) \xrightarrow{j_*} H_i(X,U) \longrightarrow H_{i-1}(U,A)$$

shows that j_* is an isomorphism. □

Theorem 1.31. *Suppose (X,A) is a good pair (i.e. there is some open $U \subset A$ such that (U,A) is a deformation retract and $\bar{A} \subset U$), $\pi : (X,A) \to (X/A, A/A)$ is the quotient map, then $\pi_* : H_*(X,A) \to H_*(X/A, A/A)$ is an isomorphism.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
H_i(X-A, U-A) \arrow[r, "j_*"] \arrow[d, "(\pi_1)_*"] & H_i(X,U) \arrow[d, "(\pi_2)_*"] & H_i(X,A) \arrow[l, "i_*"'] \arrow[d, "(\pi_3)_*"] \\
H_i(X/A - A/A, U/A - A/A) \arrow[r, "j_*"'] & H_i(X/A, U/A) & H_i(X/A, A/A) \arrow[l, "i_*"]
\end{tikzcd}$$

j_* are isomorphisms by Theorem 1.28, and i_* are isomorphisms by the preceding lemma. $(\pi_1)_*$ is an isomorphism since it's induced by a homeomorphism. So $(\pi_2)_*$ and $(\pi_3)_*$ are isomorphisms. □

Definition 1.24. A space X is an n-manifold if it is metrizable and every $x \in X$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

Example 1.17. $S^n, T^n = (S^1)^n$ are n-manifolds.

Proposition 1.32. *If X is an n-manifold and $x \in X$, then*

$$H_i(X, X - \{x\}) \cong \begin{cases} \mathbb{Z} & \textit{if } i = n \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Choose $U_x \subset X$ open neighbourhood of x homeomorphic to $\mathbb{R}^n$. WLOG the homeomorphism sends x to $0 \in \mathbb{R}^n$. We have $D^n \subset \mathbb{R}^n \cong U_x \subset X$, so by Theorem 1.28 we have $H_*(X, X-\{x\}) \cong H_*(D^n, D^n - \{0\}) \cong H_*(D^n, S^{n-1})$. Recall that $\tilde{H}_*(D^n) = 0$ and the long exact sequence for (D^n, S^{n-1}) shows that $\tilde{H}_i(D^n, S^{n-1}) \cong \tilde{H}_{i-1}(S^{n-1}) \cong \mathbb{Z}$ when $i = n$ and 0 otherwise. □

Corollary 1.33. *If M, N are m- and n-manifolds respectively and $M \cong N$, then $m = n$.*

2 Cellular Homology

2.1 Degree of Maps

Recall $H_n(S^n) \cong \mathbb{Z}$ for $n > 0$ is generated by the fundamental class $[S^n]$. So for $f : S^n \to S^n$, $f_*[S^n] = k[S^n]$ for some $k \in \mathbb{Z}$.

Definition 2.1. k is called the degree $\deg f$ of f.

The following properties are immediate:

Proposition 2.1. *(i)* $\deg 1_{S^n} = 1$.
(ii) If $f_0, f_1 : S^n \to S^n$ *are homotopic, then* $\deg f_0 = \deg f_1$.
(iii) For $f, g : S^n \to S^n$*, we have* $\deg(f \circ g) = (\deg f)(\deg g)$.
(iv) If $f : S^n \to S^n$ *is a homeomorphism, then* $\deg f = \pm 1$. *We say it is orientation-preserving if* $\deg f = 1$*, orientation-reversing if* $\deg f = -1$.
(v) If $r_v : S^n \to S^n$ *is the reflection across* $v^\perp$*, then* $\deg r_v = -1$.
(vi) If $A : S^n \to S^n$ *is the antipodal map, then* $A = r_{e_1} \circ \cdots \circ r_{e_{n-1}}$*, so* $\deg A = (-1)^{n+1}$. *In particular,* A *is not homotopic to* 1_{S^n} *for even* n.

For $p \in X$, we sometimes write $X - p = X - \{p\} = X \setminus \{p\}$ if there is no confusion.
If $p \in S^n$, $S^n \setminus \{p\} \cong (D^n)^\circ$ is contractible, so $\pi_* : H_n(S^n) \to H_n(S^n, S^n - p)$ is an isomorphism for $n > 1$. Define $[S^n, S^n - p]$ by $\pi_*[S^n] = [S^n, S^n - p]$ (the fundamental class of the pair $(S^n, S^n - p)$). If $I \subset S^n$ is open and contains p, we let $B = S^n \setminus U$. Then B is closed, so by Theorem 1.28 $j_* : H_n(U, U - p) \to H_n(S^n, S^n - p) \cong \mathbb{Z}$ is an isomorphism. Define $[U, U - p]$ by $j_*[U, U - p] = [S^n, S^n - p]$.
Observe that if $p \in U' \subset U$, we have a commutative diagram

$$\begin{tikzcd} H_n(U, U-p) \arrow[r, "{[U,U-p] \mapsto [S^n,S^n-p]}"] & H_n(S^n, S^n - p) \\ H_n(U', U'-p) \arrow[u, "{[U',U'-p] \mapsto [U,U-p]}"] \arrow[ur, "{[U',U'-p] \mapsto [S^n,S^n-p]}"'] & \end{tikzcd}$$

Suppose $f : S^n \to S^n$ and $f^{-1}(p) = \{q_1, \ldots, q_r\}$ is finite. As S^n is Hausdorff, there are pairwise disjoint opens $U_j \ni q_j$. Let $f_i : (U_i, U_i - q_i) \to (S^n, S^n - p)$ be the inclusion, so $f_*[U_i, U_i - q_i] = k_i[S^n, S^n - p]$.

Definition 2.2. k_i is called the local degree of f at q_i.

Lemma 2.2. *k_i does not depend on the choice of U_i.*

Proof. Suppose $q_i \in U_i' \subset U_i$, then the commutative diagram

$$\begin{tikzcd} H_n(U_i, U_i - q_i) \arrow[r, "f_*"] & H_n(S^n, S^n - q_i) \\ H_n(U_i', U_i' - q_i) \arrow[u, "i_*"] \arrow[ur, "f'_*"'] & \end{tikzcd}$$

shows that $\deg_{q_i} f_i = \deg_{q_i} f_i'$. In general, if U_i, U_i' both contains q_i, then $U_i \cap U_i'$ is open, contains q_i, and contained in both U_i and U_i', so we conclude by the case above. $\square$

Let $V = \coprod_i U_i \subset S^n$ which is open. By Theorem 1.28, the map $j_* : H_n(V, V - f^{-1}(p)) \to H_n(S_n, S_n - f^{-1}(p))$. On the other hand, $H_n(V, V - f^{-1}(p)) \cong \bigoplus_i H_n(U_i, U_i - q_i) \cong \mathbb{Z}^r$ with a basis given by $[U_i, U_i - q_i]$.

Lemma 2.3. *The map $H_n(S^n) \to H_n(S^n, S^n - f^{-1}(p)) \cong \bigoplus_i H_n(U_i, U_i - q_i)$ is given by $[S^n] \mapsto \sum_i [U_i, U_i - q_i]$.*

Proof. There's a commutative diagram

$$\begin{tikzcd}
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] & H_n(S^n, S^n - q_j) \\
H_n(V, V - f^{-1}(p)) \arrow[r] \arrow[u] & H_n(V, V - q_j) \arrow[u]
\end{tikzcd}$$

Note that $H_n(V, V - q_j) \cong H_n(U_j, U_j - q_j)$. The vertical maps (i.e. the j^*'s) are isomorphisms, so the diagram still commutes if we reverse those arrows. Let's have more fun with commutative diagrams

$$\begin{tikzcd}
H_n(S^n) \arrow[d] \arrow[dr] \arrow[ddrr, bend left, "\alpha"] \arrow[ddd, bend right, "\beta"'] & & \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r] \arrow[d] & H_n(S^n, S^n - q_j) \arrow[d] & \\
H_n(V, V - f^{-1}(p)) \arrow[r, "\pi"] \arrow[d, "\cong"] & H_n(V, V - q_j) \arrow[r, "\cong"] & H_n(U_j, U_j - q_j) \\
\bigoplus_i H_n(U_j, U_j - q_j) \arrow[urr] & &
\end{tikzcd}$$

We have $\alpha[S^n] = [U_j, U_j - q_j]$, so $\pi_j \circ \beta[S^n] = [U_j, U_j - q_j]$, hence $\beta[S^n] = \sum_j [U_j, U_j - q_j]$. $\square$

Theorem 2.4. *Suppose $f : S^n \to S^n$ and $f^{-1}(p) = \{q_1, \ldots, q_r\}$ as above. Then $\deg f = \sum_i \deg_{q_i} f$.*

Proof. Guess what?

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n - f^{-1}(p)) \arrow[r, "f_*"] \arrow[d] & H_n(S^n, S^n - p) \\
\bigoplus_j H_n(U_j, U_j - q_j) \arrow[ur] &
\end{tikzcd}$$

Let $\alpha : H_n(S^n) \to H_n(S^n, S^n - p)$ be the composition going through $H_n(S^n)$ and γ the composition going through $\bigoplus_j H_n(U_j, U_j - q_j)$. Then $\alpha[S^n] = (\deg f)[S^n, S^n - p]$ whereas $\gamma[S^n] = \sum_i \deg_{q_i} f_i [S^n, S^n - p]$, which is what we wanted. $\square$

Example 2.1. For $f : S^1 \to S^1, z \mapsto z^n$, then $f^{-1}(1) = \{1, \omega, \dots, \omega^{n-1}\}$ where $\omega = e^{2\pi i/n}$. Then there is a homeomorphism $\phi_k : S^1 \to S^1, z \mapsto \omega^k z$ homotopic to the identity, which shows that $\deg_{\omega^i} f = \deg_1 f = 1$, therefore $\deg f = n$.

Let's have some informal discussions. If $f : S^n \to X$ is a map, then $f_*[S^n]$ is something in $H_n(X)$, which is well-defined up to homotopy. This can be used to define a homomorphism $\Phi : \pi_n(X, *) \to H_n(X), f \mapsto f_*[S^n]$, known as the Hurewitz homomorphism. In general, this map is quite far from an isomorphism.

Example 2.2. $H_2(T^2) \cong \mathbb{Z}$, but any $f : S^2 \to T^2$ factors through $\hat{f} : S^n \to \mathbb{R}^2, \pi : \mathbb{R}^2 \to T^2$, so $f_*[S^n] = \pi_* \hat{f}_*[S^2] = \pi_*(0) = 0$ since $H^2(\mathbb{R}^2) = 0$.

If M is a closed compact connected n-manifold, we'll show that $H_n(M) \cong \mathbb{Z} = \langle [M] \rangle$ such that $H_n(M) \to H_n(M, M - *) \cong \mathbb{Z}$ takes $[M]$ to a generator. So if $f : M \to X$ is a map, we can extract from it an element $f_*[M] \in H_n(X)$. If W is a compact $(n+1)$-manifold with $\partial W = \coprod_k M_k$, then the inclusion $i : \partial W \to W$, then $i_*(\sum_k [M_k]) = 0$. So $f_*(\sum_k i_*[M_k]) = 0$, which is however still not a satisfactory model for H_n, but at least a bit better.

2.2 The Cellular Chain Complex

Definition 2.3. Suppose $B \subset Y$ is a subspace and $f : B \to X$, then we write $X \cup_f Y = (X \sqcup Y)/\sim$ where $\sim$ is the smallest equivalence relation such that $b \sim f(b)$ for all $b \in B$. This is the space obtained by gluing Y to X along f.
If $(Y, B) \cong (D^k, S^{k-1})$, we say $X \cup_f D^k$ is the space obtained by attaching a k-cell to X.

Definition 2.4. A finite cell complex of dimension n is a space X equipped with closed subsets $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n$ such that, for each k, X_k is obtained by attaching finitely many k-cells to X_{k-1}, i.e. $X_k = X_{k-1} \cup_F \coprod_\alpha D^k$ for some $F : \coprod_\alpha S^{k-1} \to X_{k-1}$.

X_k is known as the k-skeleton of X.

Remark. If we drop the finiteness condition (so we have an infinite chain $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$), then we get the definition of a CW-complex. $X = \bigcup_k X_k$ can be re-topologised by setting $U \subset X$ open iff $U \cap X_k$ open in X_k for all k.

Example 2.3. 1. If X is a graph with v vertices and e edges, then X is a finite cell complex with v 0-cells and e 1-cells.
2. If X is a finite cell complex with 1 0-cell and 1 k-cell, then $X \cong D^k/S^{k-1} \cong S^k$.
3. If X is a simplicial complex, then its underlying space $|X|$ is a finite chain complex with 1 k-cell for each k-dimensional face of X.
4. T^2 can be viewed as a finite cell complex with 1 0-cell, 2 1-cells and 1 2-cell since we can obtain it by gluing the sides of a square.

Definition 2.5. If (X_i, x_i) are pointed spaces, their vee product is

$$\bigvee_{i \in I} (X_i, x_i) = \left(\coprod_{i \in I} X_i \right) / \left(\coprod_{i \in I} \{x_i\} \right)$$

When it doesn't cause ambiguity, we often drop the x_i's and simply write $\bigvee_i X_i$ (or, when we have finitely many spaces $X_1, \dots, X_r$, $X_1 \vee \cdots \vee X_r$).

Example 2.4. If X is a finite cell complex with 1 0-cell and r k-cells, then $X \cong \bigvee^r S^k$.

Definition 2.6. An n-dimensional complex projective space $\mathbb{CP}^n$ is the topological space $(\mathbb{C}^{n+1} - 0)/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by scalar multiplication.

A point in $\mathbb{CP}^n$ can be specified by a homogeneous coordinates $[z_0 : \cdots : z_n]$ where $(z_0, \ldots, z_n) \neq 0$ is specified up to multiplication by a nonzero scalar. In light of the isomorphism $C^\times \cong \mathbb{R}_{>0} \times S^1$, we know that $(\mathbb{C}^{n+1} - 0)/\mathbb{R}_{>0} \cong S^{2n+1}$. So $\mathbb{CP}^n \cong S^{2n+1}/S^1$ where S^1 again acts by multiplication. This shows that $\mathbb{CP}^n$ is compact Hausdorff.

Definition 2.7. The Hopf map $p_n : S^{2n+1} \to \mathbb{CP}^n$ is the projection $S^{2n+1} \to S^{2n+1}/S^1 \cong \mathbb{CP}^n$.

Proposition 2.5. *$\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup_{p_{n-1}} D^{2n}$ where $p_{n-1} : S^{2n-1} \to \mathbb{CP}^{n-1}$ is the Hopf map.*

Proof. Consider $i_1 : \mathbb{CP}^{n-1} \to \mathbb{CP}^n, [z] \mapsto [z : 0]$ and $i_2 : D^n = \{z \in \mathbb{C}^n : |z| \leq 1\} \to \mathbb{CP}^n, z \mapsto [z : \sqrt{1 - |z|^2}]$. So $i_1|_{S^{2n-1}} = p_{n-1}$ and i_1, i_2 glue to give $i : \mathbb{CP}^{n-1} \cup_{p_{n-1}} D^{2n} \to \mathbb{CP}^n$.
Now i is a bijection: Indeed, its (set-theoretic) inverse is given by $[z_0 : \cdots : z_n] \mapsto (z_0, \ldots, z_{n-1})$ if $z_n \neq 0$, where the representative z is chosen so that it has unit length, and $[z_0 : \cdots : z_{n-1} : 0] \to [z_0 : \cdots : z_{n-1}] \in \mathbb{CP}^{n-1}$. □

Consequently, we inductively find that $\mathbb{CP}^n$ is a finite cell complex with exactly one cell of dimension $2i$ for each $0 \leq i \leq n$.

Example 2.5. $\mathbb{CP}^1 \cong S^2$ is the Riemann sphere.

Similarly, the real projective space $\mathbb{RP}^n = (\mathbb{R}^{n+1} - 0)/\mathbb{R}^\times \cong S^n/(\mathbb{Z}/2\mathbb{Z})$ (where $\mathbb{Z}/2\mathbb{Z}$ acts as the antipodal map) has $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_{p_{n-1}} D^n$, so it is a finite cell complex with exactly one cell of dimension i for each $0 \leq i \leq n$.

Proposition 2.6.

$$H_i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We'll prove this by induction. The base case is clear.
In general, consider the long exact sequence of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$. The quotient $\mathbb{CP}^n/\mathbb{CP}^{n-1}$ is a finite cell complex with one 0-cell (image of $\mathbb{CP}^{n-1}$) and one $2n$-cell (image of D^{2n}), hence homeomorphic to S^{2n}. Therefore we have $H_i(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \cong \tilde{H}_i(S^{2n})$, which is $\mathbb{Z}$ if $i = 2n$ and 0 otherwise. Let's now look at the long exact sequence of $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$.

$$H_{i+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \xrightarrow{\delta} H_i(\mathbb{CP}^n) \to H_i(\mathbb{CP}^n) \to H_i(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \xrightarrow{\delta} H_{i-1}(\mathbb{CP}^n)$$

If x generates $H_{2n}(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \cong \mathbb{Z}$, then $\delta x \in H_{2n-1}(\mathbb{CP}^{n-1}) = 0$ by induction. So $\delta = 0$ and the sequence splits to give $H_*(\mathbb{CP}^n) = H_*(\mathbb{CP}^{n-1}) \oplus \tilde{H}_*(S^{2n})$, which (inductively) is what we wanted. □

Observe that in the long exact sequence of the pair (D^k, S^{k-1}), the map $H_k(D^k, S^{k-1}) \cong H_{k-1}(S^{k-1})$ is an isomorphism. We write $[D^k, S^{k-1}]$ to denote the preimage of $[S^{k-1}]$ under this isomorphism.
Suppose X is a finite cell complex, A_k the set of k-cells of X and $f_\alpha : S^{k-1} \to X_{k-1}$ gives the gluing maps

$$X_k = X_{k-1} \cup_{\coprod_{\alpha \in A_k} f_\alpha} \left(\coprod_{\alpha \in A_k} D^k \right)$$

Consider

$$U_{k-1} = X_{k-1} \cup_{\coprod_{\alpha \in A_k} f_\alpha} \left(\coprod_{\alpha \in A_k} D^k - 0 \right)$$

As S^{k-1} is a deformation retract of $D^k - 0$, X_{k-1} is a deformation retract of U_{k-1} and therefore (X_k, X_{k-1}) is a good pair with $X_k/X_{k-1} \cong \bigvee_{\alpha \in A_k} S^k$. So $H_k(X_k, X_{k-1}) \cong H_k(\bigvee_{\alpha \in A_k} S^k) = \langle e_\alpha : \alpha \in A_k \rangle$ where $e_\alpha = (i_\alpha)_*[D^k, S^{k-1}]$. Consider $p_\beta : \bigvee_{\alpha \in A_k} S^k \to \bigvee_{\alpha \in A_k} S^k / \bigvee_{\alpha \neq \beta} S^k \cong S^k$, then $(p_\beta)_*(e_\alpha) = [S^k]$ when $\alpha = \beta$ and zero otherwise. Let $\delta_k : H_k(X_k, X_{k-1}) \to H_{k-1}(X_{k-1}, X_{k-2})$ be the boundary map for the triple (X_k, X_{k-1}, X_{k-2}).

Lemma 2.7. *$d_k = (\pi_{k-1})_* \circ \delta_k$ where $\delta_k : H_{k-1}(X_k, X_{k-1}) \to H_{k-1}(X_{k-1})$ is the boundary map in the long exact sequence of the pair (X_k, X_{k-1}), and $\pi_{k-1} : (X_{k-1}, \varnothing) \to (X_{k-1}, X_{k-2})$.*

Proof. $d_k[c] = [dc]$ can be thought of an element of $H_{k-1}(X_{k-1,X_{k-2}})$ and $\delta_k[c]$ an element of $H_{k-1}(X_{k-1})$. □

Corollary 2.8. *$d_k \circ d_{k+1} = 0$.*

Proof. $d_k \circ d_{k+1} = (\pi_{k-1})_* \circ \delta_k \circ (\pi_k)_* \circ \delta_{k+1}$. The claim follows from the fact that $\delta_k \circ (\pi_k)_* = 0$ since they are consecutive in the long exact sequence of the pair (X_k, X_{k-1}). □

Definition 2.8. If X is a finite cell complex, its cellular chain complex is $(C_\bullet^{\text{cell}}(X), d_\bullet^{\text{cell}})$ where $C_i^{\text{cell}}(X) = H_i(X_i, X_{i-1})$ and d_k^{cell} is as in the preceding corollary.

Theorem 2.9. *$H_*^{\text{cell}}(X) = H_*^{\text{cell}}(C_\bullet^{\text{cell}}(X))$ is isomorphic to $H_*(X)$.*

Proof. $C_k^{\text{cell}}(X) = H_k(X_k, X_{k-1}) \cong \langle e_\alpha : \alpha \in A_k \rangle$, and $d_k^{\text{cell}} : C_k^{\text{cell}}(X) \to C_{k-1}^{\text{cell}}(X)$ can only have the form $d_k^{\text{cell}}(e_\alpha) = \sum_{\beta \in A_{k-1}} n_{\alpha\beta} e_\beta$ for some integers $n_{\alpha\beta}$. We necessarily have $n_{\alpha\beta} = \deg(p_\beta \circ f_\alpha)$. Indeed,

$$\begin{aligned} d_k(e_\alpha) &= (\pi_{k-1})_* \circ \delta_k((i_\alpha)_*[D^k, S^{k-1}]) = (\pi_{k-1})_* \circ (i_\alpha)_*(\delta_k[D^k, S^{k-1}]) \\ &= (\pi_{k-1})_* \circ (i_\alpha)_*[S^{k-1}] = (f_\alpha)_*[S^{k-1}] \end{aligned}$$

So $n_{\alpha\beta}$ is the coefficient of $(f_*)_*[S^{k-1}]$ is the same as the coefficient of $[S^{k-1}]$ in $(p_\beta \circ f_\alpha)_*[S^{k-1}] = \deg(p_\beta \circ f_\alpha)$.
We delay the rest of the proof till further discussion. □

Example 2.6. 1. $\mathbb{CP}^n$ has one cell of dimension $2i$ for all $0 \leq i \leq n$. $C_\bullet^{\text{cell}}(\mathbb{CP}^n)$ then looks like (from degree $2n$) $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z}$. So $d^{\text{cell}} \equiv 0$, which means that $H_*(\mathbb{CP}^n) \cong H_*^{\text{cell}}(\mathbb{CP}^n)$ is isomorphic to $\mathbb{Z}$ for $n = 0, 2, \ldots, 2n$ and 0 otherwise.
2. $\mathbb{RP}^n$ has one cell of dimension k for all $0 \leq k \leq n$. $C_\bullet^{\text{cell}}(\mathbb{RP}^n)$ then looks like (from degree n) $\mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z}$, where $C_k^{\text{cell}}(\mathbb{RP}^n) = \langle e_k \rangle$. Then $d_k e_k = n_k e_{k-1}$ where n_{k-1} is the degree of $g_k : S^{k-1} \to \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$ (the first arrow is what we called p_k at some point). Note that $p_k = p_k \circ A$ where A is the antipodal map. So $\deg_{Aq} p_k = \deg_q(p_k) \deg A = (-1)^k \deg_q p_k$. Call $\deg_q p_k = \alpha$. Let U be a open neighbourhood of q such that $p_k|_U$ is a homeomorphism onto its image, so $\alpha = \pm 1$. Therefore

$$\deg g_k = \deg_q g_k + \deg_{Aq} g_k = \alpha + (-1)^k \alpha = \begin{cases} \pm 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Consequently, if n is even then

$$H_k(\mathbb{RP}^n) \cong H_k^{\text{cell}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } k = 1, 3, 5, \ldots, n-1 \\ \mathbb{Z} & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and if n is odd then

$$H_k(\mathbb{RP}^n) \cong H_k^{\text{cell}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{when } k = 1, 3, 5, \ldots, n-2 \\ \mathbb{Z} & \text{when } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.10. *Suppose X is a finite cell complex with one 0-cell and all other cells have dimension in $[m, M]$. Then $\tilde{H}_k(X) = 0$ for $k < m$ and $k > M$.*

Proof. Induction on $M - m$. For the base case, X has one 0-cell and all its other cells have dimension $m = M$. Then $X \cong \bigvee_\alpha S^n$, so $\tilde{H}_i(X) = 0$ for $i \neq m$.
As for the induction step, suppose we have proved the assertion for $M - m < k$. If X has cells of dimension in $[m, m+k]$, then X_{m+k-1} has cells of dimension in $[m, m+k-1]$. So the induction hypothesis applies to X_{m+k-1}, i.e. $\tilde{H}_i(X_{m+k-1})$ vanishes unless $i = m + k - 1$.
Consider the long exact sequence for the good pair (X, X_{m+k-1}) (recall that $X/X_{m+k-1} = \bigvee_{\alpha \in A} S^{m+k}$), which gives us $H_i(X, X_{m+k-1})$ vanishes unless $i = m + k$. Wield the long exact sequence again and you're there. □

Lemma 2.11. *If X is a finite cell complex, then (X, X_k) is a good pair.*

Proof. Done this before at some point. □

Corollary 2.12. *If X is a finite cell complex, then $H_k(X_{k+1}) \cong H_k(X)$.*

Proof. Look at the long exact sequence for (X, X_{k+1}):

$$H_{k+1}(X, X_{k+1}) \longrightarrow H_k(X_{k+1}) \longrightarrow H_k(X) \longrightarrow H_k(X, X_{k+1})$$

We know that $H(X, X_{k+1}) \cong \tilde{H}_k(X/X_{k+1})$ and X/X_{k+1} has one 0-cell and all other cells have dimension at leat $k+2$, so $H_k(X/X_{k+1}) = H_{k+1}(X/X_{k+1}) = 0$ which gives the result. □

Proof of Theorem 2.9. Consider the diagram

$$
\begin{tikzcd}
H_k(X_{k-1}) \arrow[d] & & \\
H_k(X_k) \arrow[r, "i"] \arrow[dr, "\pi_k"] & H_k(X_{k+1}) \arrow[r] & H_k(X_{k+1}, X_k) \\
H_{k+1}(X_{k+1}, X_k) \arrow[u, "\delta_{k-1}"] \arrow[r, "d_k"'] & H_k(X_k, X_{k-1}) \arrow[r, "d_{k-1}"] \arrow[d, "\delta_k"] & H_{k-1}(X_{k-1}, X_{k-2}) \\
H_{k-1}(X_{k-2}) \arrow[r] & H_{k-1}(X_{k-1}) \arrow[ur, "\pi_{k-1}"'] &
\end{tikzcd}
$$

where the coloured arrows indicate exact sequences. Note that $H_k(X_{k-1}) = H_k(X_{k-1}, X_k) = H_{k-1}(X_{k-2}) = 0$. So π_k, π_{k-1} are injections and i is a surjection. Consequently we have $\ker d_{k-1} = \ker \delta_k = \operatorname{Im} \pi_k \cong H_k(X_k)$. Under this isomorphism, $\operatorname{Im} d_{k+1}$ corresponds to $\operatorname{Im} \partial_{k+1}$, so $H_k^{\text{cell}}(X) = \ker d_k / \operatorname{Im} d_{k+1} \cong H_k(X_k)/\operatorname{Im} \partial_{k+1} \cong H_k(X_{k+1}) \cong H_k(X)$ by the preceding corollary. □

2.3 Homology with Coefficients

Definition 2.9. If M, N are R-modules, then $M \otimes_R N$ is the R-module generated by symbols $m \otimes n, m \in M, n \in N$ subject to the relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

We sometimes just write $\otimes$ in place of $\otimes_R$ when R is understood, or when $R = \mathbb{Z}$.

Proposition 2.13. *(i)* $M \otimes_R N \cong N \otimes_R M$ *via* $m \otimes n \mapsto n \otimes m$.
(ii) $R \otimes_R M \cong M$ *via* $r \otimes m \mapsto rm$.
(iii) $(M_1 \oplus M_2) \otimes_R M \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$. *In particular,* $R^m \otimes_R R^n \cong R^{mn}$.

Example 2.7. 1. For $n \geq 1$, we have $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$ since $q \otimes k = (q/n) \otimes (nk) = (q/n) \otimes 0 = 0$.
2. $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/\gcd(a,b)\mathbb{Z}$.

Remark. For an R-module M, $- \otimes_R M$ essentially defines an endofunctor on the category of R-modules, which sends N to $N \otimes_R M$ and sends an R-linear map $f : N_1 \to N_2$ sends to $f \otimes_R 1 : N_1 \otimes_R M \to N_2 \otimes_R M$.

If (C, d) is a chain complex of R-modules, then $(C \otimes_R M, d \otimes_R 1)$ is another chain complex of R-modules.

Lemma 2.14. *If* $f, g : C \to C'$ *are chain homotopic via a chain homotopy* h, *then* $f \otimes 1, g \otimes 1 : C \otimes M \to C' \otimes M$ *are chain homotopic via* $h \otimes 1$.

Proof. Check the definition. □

Definition 2.10. If G is a $\mathbb{Z}$-module (i.e. an abelian group), then $C_\bullet(X, G) = C_\bullet(X) \otimes_{\mathbb{Z}} G$ is the singular chain complex with coefficients in G. Its homology is denoted as $H_*(X; G)$.

Remark. If R' is a ring which is also an R-module (e.g. $R = \mathbb{Z}, R' = \mathbb{Q}, \mathbb{R}, \mathbb{Z}/n\mathbb{Z}$, etc.), then we have the functor $- \otimes R'$ from the category of R-modules to the category of R'-modules which sends an R-module N to $N \otimes_R R'$ (on which R' acts on the second factor). This induces a functor from the category of chain complexes of R-modules to the category of chain complexes of R'-modules.

Example 2.8. Take $C = C^{\text{cell}}_\bullet(\mathbb{RP}^3)$ which looks like (starting from the third grading) $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$, where the first and third arrows are zeros and the second arrow is multiplication by 2. Then $C \otimes \mathbb{Q}$ looks like $\mathbb{Q} \to \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q}$, where again the first and third maps are zero and the second map is multiplication by 2, which however is now an isomorphism. We can do this for other $\mathbb{Z}$-modules. Their homologies would be

	3	2	1	0
$H_*(C)$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
$H_*(C \otimes \mathbb{Q})$	$\mathbb{Q}$	0	0	$\mathbb{Q}$
$H_*(C \otimes \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

The last row shows that we do not necessarily have $H_*(C \otimes G) \cong H_*(C) \otimes G$.

A good thing about being able to take coefficents in a ring other than $\mathbb{Z}$ is that, then the ring is a field, the entries in the chian complex are vector spaces. And vector spaces are nice.
Suppose C is a chain complex of finite dimensional vector spaces over a field, all but finitely of whose chain groups vanish. Let $c_k = \dim C_k$ and $h_k = \dim H_k(C)$.

Definition 2.11. The Euler characteristic of the chain is $\chi(C) = \sum_k (-1)^k c_k$.

Theorem 2.15. $\chi(C) = \sum_k (-1)^k h_k$.

Proof. Let $z_k = \dim \ker d_k, b_k = \dim \operatorname{Im} d_k$, then $c_k = z_k + b_k$ by rank-nullity and $h_k = z_k - b_{k-1}$. Then $\chi(C) = \sum_k ((-1)^k z_k + \sum_k (-1)^k b_k) = \sum_k (-1)^k (z_k - b_{k+1}) = \sum_k (-1)^k h_k$. □

Definition 2.12. An ordinary homology theory with coefficients in an abelian group G is a functor H_* from the category of pairs of spaces to the category of graded $\mathbb{Z}$-modules satisfying the Eilenberg-Steenrod axioms:
(i) (Homotopy invariance) If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $(f_0)_* = (f_1)_*$.
(ii) (Long exact sequence of a pair) There is a long exact sequence of the form

$$\cdots \longrightarrow H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X, A) \longrightarrow H_{k-1}(A) \longrightarrow \cdots$$

where we identify $X = (X, \varnothing)$. Moreover, a map of pairs induce a map of their corresponding long exact sequences.
(iii) (Excision) If $\bar{B} \subset A^\circ$, then the inclusion map induces an isomorphism $H_*(X - B, A - B) \cong H_*(X, A)$.
(iv) (Dimension) $H_k(\{*\}) = G$ if $k = 0$ and vanishes otherwise.

Theorem 2.16. *If X is a finite cell complex and H_* is any functor satisfying the Eilenberg-Steenrod axioms, then $H_*(X) \cong H_*(C^{\text{cell}}_\bullet(X) \otimes G)$. In particular, $H_*(X; G)$ satsifies these axioms.*

Definition 2.13. If M is an R-module. A free resolution of M is a free chain complex $A_\bullet$ of R-modules such that:
(i) $A_k = 0$ for $k < 0$.
(ii) $H_0(A_\bullet) = M$ and $H_i(A_\bullet) = 0$ for $i \neq 0$.

Example 2.9. 1. Suppose R is a PID (e.g. $\mathbb{Z}$), then $0 \to R \to R \to \cdots$ is a free resolution of $R/(a)$, where the second arrow is multiplication by a.
2. If $R = \mathbb{C}[X, Y]$, then $R \to R \to R$, where the first arrow is multiplication by $(Y, -X)^\top$ and second by (X, Y), is a free resolution of $R/(X, Y)$.

Definition 2.14. If M, N are R-modules, $\mathrm{Tor}_i(M, N) = H_i(A_\bullet \otimes N)$, where $A_\bullet$ is a free resolution of M.

So $\mathrm{Tor}_i(M, N)$ measures the failure of the (generally false) equation $H_*(A \otimes N) = H_*(A_\bullet) \otimes N$.

Remark. It's a fact that any module M has a free resolution, unique up to chain homotopy equivalent. So $\mathrm{Tor}_i(M, N)$ is independent of the choice of $A_\bullet$.

Example 2.10. 1. $\mathrm{Tor}_0(M, N) \cong M \otimes N$.
2. Take $R = \mathbb{Z}$. Then for $a > 0$, $\mathrm{Tor}_i(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z}$ when $i = 0$ and zero otherwise. For $a, b > 0$, $\mathrm{Tor}_i(\mathbb{Z}/a, \mathbb{Z}/b) \cong \mathbb{Z}/\gcd(a, b)\mathbb{Z}$ when $i = 0, 1$ and zero otherwise. So $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ accounts for the extra $\mathbb{Z}/2\mathbb{Z}$ in $H_*(C_\bullet^{\mathrm{cell}}(\mathbb{RP}^2))$.

Definition 2.15. A chain complex C of R-modules is short injective if:
(i) $C_i = 0$ for $i \neq k, k+1$ and C_k, C_{k+1} are free R-modules.
(ii) $d : C_{k+1} \to C_k$ is injective.

So a short injective chain complex is a shifted free resolution of $H_k(C)$.

Theorem 2.17. *A free chain complex of R-modules, where R is a PID, is a direct sum of of short injective chain complexes.*

Lemma 2.18. *If R is a PID, then any submodule of a free R-module is free.*

Proof. Nah. $\square$

Proof of Theorem 2.17. We have a short exact sequence

$$0 \longrightarrow \ker d_k \longrightarrow C_k \longrightarrow \operatorname{Im} d_k \longrightarrow 0$$

As $\operatorname{Im} d_k \subset C_{k-1}$, it is free by the preceding lemma, so the sequence splits, i.e. $C_k \cong \ker d_k \oplus B_k$ and d_k maps B_k isomorphically onto $\operatorname{Im} d_k$. Since $d^2 = 0$, we have $\operatorname{Im} d_k \subset \ker d_{k-1}$. So C is the direct sum of $B_k \to \ker d_{k-1}$, each of which is short injective. $\square$

Corollary 2.19. *If two free chain complexes over a PID have the same homology, then they are chain homotopy equivalent.*

Corollary 2.20. *If C is a chain complex over a field, then C is chain homotopy equivalent to $(H_\bullet(C), 0)$.*

Corollary 2.21 (Universal Coefficient Theorem)**.** *If C is a free chain complex over a PID, then $H_k(C \otimes N) = \mathrm{Tor}_0(H_k(C), N) \oplus \mathrm{Tor}_1(H_{k-1}(C), N)$*

Proof. As C is a direct sum of short injective complexes, we can assume WLOG that C itself is short injective. But this case is clear. $\square$

So indeed $H_*(X; G)$ is determined by $H_*(X)$.

3 Cohomology and Products

3.1 Cohomology

Definition 3.1. If M, N are R-modules, then we set $\mathrm{Hom}_R(M, N)$ to be the R-module of R-linear maps $M \to N$.

Remark. 1. When R is understood, we sometimes just write Hom in place of Hom_R.
2. Suppose $f : M_1 \to M_2$ is R-linear, then it induces an R-linear map $f^* : \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$ via $\alpha \mapsto \alpha \circ f$. This respects composition, so $\mathrm{Hom}_R(-, N)$ is a contravariant endofunctor in the category of R-modules.

If (C, d) is a chain complex of R-modules and N an R-module, then we can construct a complex $(\mathrm{Hom}(C, N), d^*)$ where the k-th entry is $\mathrm{Hom}(C, N)^k = \mathrm{Hom}(C_k, N)$ and $d^* : \mathrm{Hom}(C_{k-1}, N) \to \mathrm{Hom}(C_k, N)$ is the induced map. We still have $(d^*)^2 = 0$, although the labelling goes the other way around compared to a chain complex. We say $\mathrm{Hom}(C, N)$ is a cochain complex, which is just a chain complex with the labelling reversed (i.e. with the differential raising the degree by 1). It's usual for us to put the labelling of a cochain complex as a superscript.
This gives a contravariant functor $\mathrm{Hom}(-, N)$ from the category of chain complexes of R-modules to the category of cochain complexes of R-modules, for a chain map $f : C \to C'$ induces a chain map $f^* : \mathrm{Hom}(C', N) \to \mathrm{Hom}(C, N)$ componentwise.
If (C^k, d^k) is a cochain complex, then its k-th cohomology is defined by the quotient $H^k(C) = \ker d^k / \operatorname{Im} d^{k-1}$.

Definition 3.2. If X is a space, its singular cochain complex with coefficients in G is $C^\bullet(X; G) = \mathrm{Hom}(C_\bullet(X), G)$.

$\alpha \in C^k(X; G)$ is called a (k-)cochain. It is uniquely specified by $\alpha(\sigma)$ for all $\sigma : \Delta^k \to X$. The dual differential d^k then has $(d^k\alpha)(\sigma) = \alpha(d_k\sigma)$, from where we have $d^{k+1} \circ d_k = 0$.

Definition 3.3. The k-th singular cohomology of X is

$$H^k(X) = H^k(C^\bullet(X; G)) = \frac{\ker d^{k+1}}{\operatorname{Im} d^k}$$

For $f : X \to Y$ we certainly get $f^\bullet : C^\bullet(Y; G) \to C^\bullet(X; G)$ defined via $f^\bullet(\alpha)(\sigma) = \alpha(f_\bullet(\sigma)) = \alpha(f \circ \sigma)$. This is a cochain map since we have

$$\begin{aligned} df^\bullet(\alpha)(\sigma) &= f^\bullet(\alpha)(d\sigma) = f^\bullet(\alpha)(\sum_j (-1)^j \sigma \circ F_j) \\ &= \alpha(\sum_j (-1)^j f \circ \sigma \circ F_j) = f^\bullet d(\alpha)(\sigma) \end{aligned}$$

So $f^\bullet$ is a chain map, hence induces $f^* : H^*(Y; G) \to H^*(X; G)$.

Remark. The definition clearly generalises to the cochain complex and cohomology of pairs of spaces.

Each H^k is then a contravariant functor from the category of pairs of spaces to the category of $\mathbb{Z}$-modules. This is the procedure of taking the singular chain complex, applying $\mathrm{Hom}(-, G)$ to get a cochain complex, and taking cohomology.

Definition 3.4. Suppose C, C' are cochain complexes and $f, g : C \to C'$ are cochain maps. We say f, g are cochain homotopic if $f - g = dh + hd$ for some $h : C^k \to (C')^{k-1}$..

The next two lemmas are clear.

Lemma 3.1. *If $f \sim g$, then $f^* = g^*$.*

Lemma 3.2. *If $f, g : C \to C'$ are chain maps and $f \sim g$ via h, then their duals $f^\vee, g^\vee : \mathrm{Hom}(C'; N) \to \mathrm{Hom}(C; N)$ are cochain homotopic via $h^\vee$.*

Most things from homology has an analogue in cohomology. We'll state and sketch some of them, but they are really just the same (or, as is in most of the time, implied by the results on homology). By the way, we sometimes omit G because why not.

Proposition 3.3. *1. If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $f_0^* = f_1^*$.*
2. We have a short exact sequence of cochain complexes

$$0 \longrightarrow C^\bullet(X, A) \longrightarrow C^\bullet(X) \longrightarrow C^\bullet(A) \longrightarrow 0$$

which induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & H^k(X, A) \arrow[r] & H^k(X) \arrow[r] & H^k(A) \arrow[dll, "\delta"] \\ & H^{k+1}(X, A) \arrow[r] & H^{k+1}(X) \arrow[r] & H^{k+1}(A) \arrow[r, "\delta"'] & \cdots \end{tikzcd}$$

3. (Excision) If $B \subset A \subset X$ and $\bar{B} \subset A^\circ$, then the inclusion $(X, A) \to (X - B, A - B)$ is an isomorphism.
4. $H^0(\{*\}; G) \cong G$ and $H^k(\{*\}; G) = 0$ for $k \neq 0$.

Remark. The proof of excision uses the fact that the chain groups are free.

Theorem 3.4. *Any functor from the category of finite cell complexes satisfying the proposition above is given by $H^*_{\mathrm{cell}}(X; G) = H^*(\mathrm{Hom}(C^\bullet_{\mathrm{cell}}(X); G))$.*

Theorem 3.5. *$H^*(X; G) \cong H^*_{\mathrm{cell}}(X; G)$ when X is a finite cell complex.*

Proof. $C_\bullet(X), C_\bullet^{\mathrm{cell}}(X)$ are homotopy equivalent free chain complexes over the PID $\mathbb{Z}$, so their duals are also homotopy equivalent. □

Example 3.1. We know that $C_\bullet^{\mathrm{cell}}(\mathbb{RP}^2)$ is given by (starting with the third grading) $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$ with the first and third maps zero and the second map multiplication by 2. So $C^\bullet_{\mathrm{cell}}$ is $\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}$ where the same description of the arrows applies. So $H^k_{\mathrm{cell}}(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, 3$, $\mathbb{Z}/2\mathbb{Z}$ for $k = 2$ and vanishes otherwise.

Definition 3.5. If M, N are R-modules, then $\mathrm{Ext}^i(M, N) = H^i(\mathrm{Hom}(A_\bullet, N))$, where $A_\bullet$ is a free resolution of M.

Again this does not depend on the choice of $A_\bullet$.

Example 3.2. For $n > 1$, $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$. $\mathrm{Ext}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ for $i = 0, 1$ and vanishes otherwise. So we have the shifting of the torsion part of the homology.

Theorem 3.6. *Suppose X is a finite cell complex. Then we have the splitting $H^k(X;G) = \mathrm{Ext}^0(H_k(X),G) \oplus \mathrm{Ext}^1(H_{k-1}(X),G)$.*

Proof. Split $C^{\mathrm{cell}}_\bullet(X)$ into direct sums of short injective complexes. $\square$

Example 3.3. If X is a finite cell complex, then $H_k(X) = \mathbb{Z}^{b_k} \oplus T_k$ for some finite group T_k. b_k is called the k-th Betti number of X. Then $H^k(X;\mathbb{Z}) = \mathbb{Z}^{b_k} \oplus T_{k-1}$.

Suppose C is a chain complex of R-modules and N is an R-module, then there is a bilinear pairing $\langle , \rangle : \mathrm{Hom}(C_k;N) \times C_k \to N$, $\langle \alpha, c\rangle = \alpha(c)$.

Lemma 3.7. *This descends to a bilinear pairing $H^k(\mathrm{Hom}(C,N)) \times H_k(C) \to N$.*

Proof. We have, for $d\alpha = dc = 0$, that $\langle \alpha + d\beta, c + db\rangle = (\alpha + d\beta)(c + db) = \alpha(c) + 0 + 0 + 0 = \alpha(c)$. $\square$

3.2 Cup Product

Let R be a commutative ring.

Definition 3.6. Suppose $\alpha \in C^k(X;R), \beta \in C^l(X;R)$. Their cup product is $\alpha \smile \beta \in C^{k+l}(X;R)$ given by $\alpha \smile \beta(\sigma) = \alpha(\sigma \circ F_{0\cdots k})\beta(\sigma \circ F_{k\cdots(k+l)})$

Lemma 3.8. *$\smile$ makes $C^*(X;R) = \bigoplus_k C^k(X;R)$ into a (usually noncommutative) graded ring with identity given by $1 \in C^0(X;R)$.*

Proof. Literal definition of trivial. $\square$

Lemma 3.9. *If $\alpha \in C^k(X;R)$ and $\beta \in C^l(X;R)$, then we have the Leibniz rule: $d(\alpha \smile \beta) = (d\alpha) \smile \beta + (-1)^k \alpha \smile (d\beta)$.*

Proof. Introducing

$$\begin{aligned} d(\alpha \smile \beta)(\sigma) = \alpha \smile \beta(d\sigma) = (\alpha \smile \beta)\left(\sum_j (-1)^j \sigma \circ F_{\hat{j}}\right) \\ = \sum_{j=0}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{\hat{j}} \circ F_{0\cdots k})\beta(\sigma \circ F_{\hat{j}} \circ F_{k\cdots(k+l)}) \\ = \sum_{j=0}^{k+1} (-1)^j (\sigma \circ F_{0\cdots\hat{j}\cdots k})\beta(\sigma \circ F_{(k+1)\cdots(k+l+1)}) \\ + \sum_{j=k}^{k+l+1} (-1)^j \alpha(\sigma \circ F_{0\cdots k})\beta(\sigma \circ F_{k\cdots\hat{j}\cdots(k+l+1)}) \\ = (d\alpha) \smile \beta(\sigma) + (-1)^k \alpha \smile (d\beta) \end{aligned}$$

Phew. $\square$

Corollary 3.10. *$\smile$ descends to a pairing $H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$. This makes $H^*(X;R)$ a ring with identity $1 = [1]$.*

Proof. If $d\alpha = 0, d\beta = 0$, then $d(\alpha \smile \beta) = (d\alpha) \smile \beta + (-1)^k \alpha \smile (d\beta) = 0$. Moreover, we have $(\alpha + d\gamma) \smile (\beta + d\delta) = \alpha \smile \beta + (d\gamma) \smile \beta + \alpha \smile (d\delta) + (d\gamma) \smile (d\delta) = \alpha \smile \beta + d(\gamma \smile \beta) \pm d(\alpha + d\gamma) \smile \delta$.
We also have $(d1)(\tau) = 1(\sigma_{\tau(1)} - \sigma_{\tau(0)}) = 1 - 1 = 0$ for any $\tau \in C_1(X)$. □

Proposition 3.11. *For $f : X \to Y$, $f^* : H^*(Y; R) \to H^*(X; R)$ is a ring homomorphism.*

Proof. Here we go again.

$$\begin{aligned} f^\bullet(\alpha \smile \beta)(\sigma) &= (\alpha \smile \beta)(f \circ \sigma) = \alpha(f \circ \sigma \circ F_{0\cdots k})\beta(f \circ \sigma \circ F_{k\cdots(k+l)}) \\ &= f^\bullet(\alpha)(\sigma \circ F_{0\cdots k}) f^\bullet(\beta)(\sigma \circ F_{k\cdots(k+l)}) = f^\bullet(\alpha) \smile f^\bullet(\beta)(\sigma) \end{aligned}$$

And we just descend this to cohomology. □

We have a chain map $r : C_\bullet(X) \to C_\bullet(X)$ defined as follows: Let $\rho_n : \Delta^n \to \Delta^n$ be the linear map $e_i \mapsto e_{n-i}$. Let $\epsilon(j) = j(j+1)/2$, then $\det \rho_j = (-1)^{\epsilon(j)}$. Define $r_j : C_j(X) \to C_j(X)$ by $r_j(\sigma) = (-1)^{\epsilon(j)} \sigma \circ \rho_j$.

Theorem 3.12. *r is a chain map homotopic to the identity.*

Corollary 3.13. *$H^*(X; R)$ is graded-commutative, in the sense that $a \smile b = (-1)^{|a||b|} b \smile a$, where $|c| = k$ if $c \in H^k(X; R)$.*

Proof. Dualising r gives $r : C^\bullet(X; R) \to C^\bullet(X; R)$ homotopy to the identity. Then $[r(\alpha)] = [\alpha]$ and $(-1)^{\epsilon(|\alpha|+|\beta|)} r(\alpha \smile \beta) = (-1)^{\epsilon(|\alpha|)}(-1)^{\epsilon(|\beta|)} r(\beta) \smile r(\alpha)$, so

$$\begin{aligned} [\alpha \smile \beta] &= [r(\alpha \smile \beta)] = (-1)^{\epsilon(|\alpha|+|\beta|)}(-1)^{\epsilon(|\alpha|)}(-1)^{\epsilon(|\beta|)}[r(\beta)] \smile [r(\alpha)] \\ &= (-1)^{|\alpha||\beta|}[\beta] \smile [\alpha] \end{aligned}$$

which is what we wanted. □

Proof of Theorem 3.12. Let's first show that r is a chain map. We have $\rho_n \circ F_{\hat{j}} = F_{\widehat{n-j}} \circ \rho_{n-1}$. Write $n = |\sigma|$, then

$$\begin{aligned} dr(\sigma) &= (-1)^{\epsilon(n)} \sum_j (-1)^j \sigma \circ \rho_n \circ F_{\hat{j}} = (-1)^{\epsilon(n)} \sum_j (-1)^j \sigma \circ F_{\widehat{n-j}} \circ \rho_{n-1} \\ &= (-1)^n (-1)^{\epsilon(n)} \sum_j (-1)^{n-j} \sigma \circ F_{\widehat{n-j}} \circ \rho_{n-1} = r(d\sigma) \end{aligned}$$

as desired. The relative version is analogous.
One can either find a chain homotopy explicitly, or doing something like follows: $C_\bullet(X)$ is free, so it suffices to show that $r_* = 1_{H_*(X)}$. For any $f : X \to Y$, we have $f_\bullet \circ r(\sigma) = (-1)^{\epsilon(|\sigma|)} f \circ \sigma \circ \rho_{|\sigma|} = r \circ f_\bullet$. Therefore the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d, "r"] & C_\bullet(X) \arrow[r] \arrow[d, "r"] & C_\bullet(X, A) \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & C_\bullet(A) \arrow[r] & C_\bullet(X) \arrow[r] & C_\bullet(X, A) \arrow[r] & 0 \end{tikzcd}$$

commutes.
Let $R_n(X, A)$ be the statement that $(r_*)_n = 1_{H_n(X,A)}$. Observe that if f_* :

$H_n(X,A) \to H_n(Y,B)$ is injective, then $R_n(Y,B) \implies R_n(X,A)$. And if $f_* : H_n(X,A) \to H_n(Y,B)$ is surjective, then $R_n(X,A) \implies R_n(Y,B)$.
Now, $R_0(X)$ always holds, since if $\sigma \in H_0(X)$ then $r(\sigma) = \sigma_0$. We then recall the square

$$\begin{tikzcd} H_n(D^n, S^{n-1}) \arrow[r, "\delta"] \arrow[d, "r_*"'] & H_{n-1}(S^{n-1}) \arrow[d, "r_*"] \\ H_n(D^n, S^{n-1}) \arrow[r, "\delta"'] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

which commutes by the above discussion. On the other hand, we have isomorphisms $H_n(D^n, S^{n-1}) \cong H_n(D^n/S^{n-1}, S^{n-1}/S^{n-1}) \cong H_n(S^n)$, so we have $R_n(D^n, S^{n-1}) \implies R_n(S^n)$. By induction, $R_n(D^n, S^{n-1}), R^n(S^n)$ are true for all n. Consequently, $R_n(\coprod D^n, \coprod S^{n-1})$ is true for any finite disjoint union.
We now claim that if X is a finite cell complex then $R_n(X)$ is true for all n. We'll show that $\forall n, R_n(X_k)$ holds for all k by induction. For $k = 0$, we have $R_0(X_0)$ holds and $R_n(X_0)$ is immediate for $n > 0$. Suppose $R_n(X_{k-1})$ holds for all n, we consider the long exact sequence of the pair (X_k, X_{k-1}):

$$0 \longrightarrow H_k(X_k) \longrightarrow H_k(X_k, X_{k-1}) \longrightarrow H_{k-1}(X_{k-1}) \longrightarrow H_{k-1}(X_k) \longrightarrow 0$$

since $H_k(X_{k-1}) = 0 = H_k(X_k, X_{k-1})$. Moreover, $H_i(X_{k-1}) \to H_i(X_k)$ is an isomorphism for all $i < k-1$.
Consider the map $F : H_*(\coprod D^k, S^{k-1}) \to H_*(X_k, X_{k-1})$ induced by the attaching map. We know that $R_n(X_k, X_{k-1})$ holds for all n implies that $R_k(X_k)$ holds, and we know by induction that $R_n(X_{k-1})$ holds for all n, therefore $R_i(X_k)$ holds for all $i < k$.
For arbitrary X and $x \in H_*(X)$, there is a finite cell complex Y and $f : Y \to X$ with $f_*(y) = x$ for some $y \in H_*(Y)$ (example sheet). Then $r_*(y) = r_*(f_*(x)) = f_*(r_*(x)) = f_*(x) = y$. $\square$

Recall that $C^\bullet(X,A) \subset C^\bullet(A)$ is given by $\{\alpha \in C^\bullet(X) : \operatorname{Im}\sigma \subset A \implies \alpha\sigma = 0\}$. So if $\alpha \in C^k(X,A), \beta \in C^l(X)$, then given that $\operatorname{Im}\sigma \subset A$ we must have $\operatorname{Im}\sigma \circ F_{0\cdots k} \subset A$. Consequently, we have

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma \circ F_{0\cdots k})\beta(\sigma \circ F_{k\cdots(k+l)}) = 0 \cdot \beta(\sigma \circ F_{k\cdots(k+l)}) = 0$$

So $\alpha \smile \beta \in C^\bullet(X,A)$.
This descends into a map $\smile: H^*(X,A) \times H^*(X) \to H^*(X,A)$. More generally, using subdivision we can get $\smile$ to become a map $H^*(X,A) \times H^*(X,B) \to H^*(X, A \cup B)$.

Example 3.4. 1. If X is path-connected, then $H^0(X) \cong \operatorname{Hom}(H_0(X), \mathbb{Z}) = \mathbb{Z}$ since $H_{-1}(X) = 0$. Indeed, $H^0(X)$ is generated by the class of 1, since if $\sigma_p \in C_0(X)$, then $\langle 1, [\sigma_p]\rangle = 1$, which means that 1 must be primitive.
2. $H_*(S^n)$ is free over $\mathbb{Z}$, so $H^*(S^n) = \operatorname{Hom}(H_*(S^n), \mathbb{Z})$. Let a be a generator for $H^n(S^n)$. We have $1 \smile 1 = 1, a \smile 1 = 1 \smile a = a$ and $a \smile a \in H^{2n}(S^n) = 0$, so $a \smile a = 0$. Therefore $H^*(S^n) = \mathbb{Z}[a]/(a^2)$ where $|a| = n$.
3. If X is path-connected and $p \in X$, the inclusion map induces an isomorphism $H_0(p) \to H_0(X)$, so $H^0(X) \to H^0(p)$ is an isomorphism as well since everything's free. So $H^*(X,p) = \ker(H^*(X) \to H^*(p)) = \bigoplus_{i>0} H^i(X)$. Note that the map is a ring homomorphism, not just a group homomorphism.

4. $H^*(X \sqcup Y) \cong H^*(X) \times H^*(Y)$ as rings. Indeed, from $C_\bullet(X \sqcup Y) = C_\bullet(X) \oplus C_\bullet(Y)$ we have

$$\begin{aligned} C^\bullet(X \sqcup Y) &= \mathrm{Hom}(C_\bullet(X) \oplus C^\bullet(Y), \mathbb{Z}) \\ &= \mathrm{Hom}(C_\bullet(X), \mathbb{Z}) \times \mathrm{Hom}(C_\bullet(Y), \mathbb{Z}) = C^\bullet(X) \times C^\bullet(Y) \end{aligned}$$

For an explicit identification, every $\alpha \in C^\bullet(X \sqcup Y)$ is essentially of the form $\alpha(\sigma) = \alpha_1(\sigma)$ if $\mathrm{Im}\,\sigma \subset X$ and $\alpha(\sigma) = \alpha_2(\sigma)$ if $\mathrm{Im}\,\sigma \subset Y$ for some $\alpha_1 \in C^\bullet(X), \alpha_2 \in C^\bullet(Y)$.
This satisfies $d(\alpha_1, \alpha_2) = (d\alpha_1, d\alpha_2)$, so it descends to an isomorphism on cohomology. On the other hand, we have $(\alpha_1, \alpha_2) \smile (\beta_1, \beta_2) = (\alpha_1 \smile \alpha_2, \beta_1 \smile \beta_2)$ simply by writing out definitions.
5. Let's compute $H^*((X,p) \vee (Y,q))$ for X, Y path-connected. $H^*(X,p) = \bigoplus_{i>0} H^i(X)$ is an ideal of $H^*(X)$. Let $\pi : (X \sqcup Y, p \sqcup q) \to (X,p) \vee (Y,q) = (X \vee Y, p)$ be the quotient map. By collapsing a pair, π^* induces an isomorphism $H^*(X \vee Y, p) \cong H^*(X \sqcup Y, p \sqcup q) = H^*(X,p) \oplus H^*(Y,q) \subset H^*(X) \oplus H^*(Y)$. So

$$H^i(X \vee Y) = \begin{cases} H^i(X) \oplus H^i(Y) & \text{for } i > 0 \\ \langle 1 \rangle \cong \mathbb{Z} & \text{for } i = 0 \end{cases}$$

The multiplication is given by $(a_1, a_2) \smile (b_1, b_2) = (a_1 \smile b_1, a_2 \smile b_2)$. For example, $H^*(S^2 \vee S^2 \vee S^4) = \langle 1, a, a', b \rangle$, where $a = ([S^2], 0, 0) \in H^2(S^2) \oplus H^2(S^2) \oplus H^2(S^4), a' = (0, [S^2], 0)$ and $b = (0, 0, [S^4]) \in H^4(S^2) \oplus H^4(S^2) \oplus H^4(S^4)$. We have $a \smile a' = 0$ and similarly for others, so there is no interesting cup products.

3.3 Exterior Products

Consider a pair (X, A) and a space Y. We have a map of pairs $\pi_1 : (X \times Y, A \times Y) \to (X \times A)$ and a map $\pi_2 : X \times Y \to Y$.

Definition 3.7. For $a \in H^k(X, A), b \in H^l(Y)$, then their exterior product is $a \times b = \pi_1^*(a) \smile \pi_2^*(b) \in H^{k+l}(X \times Y, A \times Y)$.

Remark. 1. $H^*(X, A) \times H^*(Y) \to H^*(X \times Y, A \times Y), (a, b) \mapsto a \times b$ is bilinear, so it extends to a map $\phi : H^*(X, A) \otimes H^*(Y) \to H^*(X \times Y, A \times Y), a \otimes b \mapsto a \times b$.
2. By graded-commutativity,

$$(a_1 \times b_1) \smile (a_2 \times b_2) = (-1)^{|b_1||a_2|}(a_1 \smile a_2) \times (b_1 \smile b_2)$$

Theorem 3.14. *If $H^*(Y; R)$ is free over R, then the map $\Phi : H^*(X, A; R) \otimes H^*(Y; R) \to H^*(X \times Y, A \times Y; R)$ is an isomorphism.*

We can use this to compute $H^*(X \times Y; R)$ from $H^*(X; R)$ and $H^*(Y; R)$ given the freeness of the latter. The theorem also gives us the ring structure on $H^*(X \times Y; R)$ basically for free.

Example 3.5. Consider $T^2 = S^1 \times S^1$. The theorem applies since $H^*(S^1)$ is free over $\mathbb{Z}$, so we get $H^*(T^2; \mathbb{Z})$ to be exactly what we know it is, except we now know that $H^2(T^2; \mathbb{Z}) = \langle [S^1] \times [S^1] \rangle, H^1(T^2; \mathbb{Z}) = \langle [S^1] \times 1, 1 \times [S^1] \rangle$.
Write $c = [S^1] \times [S^1], a = [S^1] \times 1, b = 1 \times [S^1]$. Then we have $a^2 = ([S^1] \times 1) \smile ([S^1] \times 1) = -([S_1]^2 \times 1) = 0$ and similarly $b^2 = 0, a \smile b = c, b \smile a = -c$. So

$H^*(T^2) = \bigwedge^*\langle\alpha_1, \alpha_2\rangle$ (wehre one can take $\alpha_1 = a, \alpha_2 = b$) satisfying $\alpha_i\alpha_j = -\alpha_j\alpha_i$.
More generally, $H^*(T^n) = H^*(S^1) \otimes \cdots \otimes H^*(S^1) \cong \bigwedge^*\langle\alpha_1, \dots, \alpha_n\rangle$.

Example 3.6. Consider $S^2 \times S^2$. $H^*(S^2)$ is free, so $H^*(S^2 \times S^2) = H^*(S^2) \otimes H^*(S^2)$. Write

$$H^i(S^2 \times S^2) = \begin{cases} \langle [S^2] \times [S^2] \rangle = \langle c \rangle & \text{if } i = 4 \\ \langle [S^2] \times 1, 1 \times [S^2] \rangle = \langle a, b \rangle & \text{if } i = 2 \\ \langle 1 \times 1 \rangle = \langle 1 \rangle & \text{if } i = 0 \end{cases}$$

Then $a^2 = b^2 = 0, a \smile b = b \smile a = c$ and hence $(a+b)^2 = 2c$. This is a different phenomenon as before: For any $\alpha \in H^1(T^2)$ we have $\alpha^2 = 0$ since $\alpha \smile \alpha = -\alpha \smile \alpha$.

Corollary 3.15. *$S^2 \times S^2$ is not homotopy equivalent to $S^2 \vee S^2 \vee S^4$, even though they have the same homology groups.*

Let's now prove Theorem 3.14 for finite cell complexes.
Fix Y. Consider contravariant functions $\overline{h}, \underline{h}$ from the category of pairs of spaces to graded $\mathbb{Z}$-modules. $\overline{h}(X, A) = H^*(X \times Y, A \times Y)$ and $f : (X, A) \to (X', A')$ induces $\overline{f^*} = (f \times \mathrm{id}_Y)^*$. $\underline{h}(X, A) = H^*(X, A) \otimes H^*(Y)$ and $f : (X, A) \to (X', A')$ induces $\underline{f^*} = f^* \otimes \mathrm{id}_Y$.
It's easy to check that both of them satisfy all the Eilenberg-Steenrod axioms except the dimension axiom (where we use the freeness of $H^*(Y)$ to get the long exact sequence of a pair for $\underline{h}$). Stuff like that are known as generalised cohomology theories. Let's show that this subset of axioms is already sufficient to show that $\overline{h}$ and $\underline{h}$ agree on (pairs of) finite cell complexes.

Lemma 3.16. *Φ commutes with the induced maps and boundary maps in the long exact sequences of pairs.*

Proof. Suppose $f : X_1 \to X_2$. Take $\overline{f^*}(\Phi(a \otimes b)) = F^*(a \times b)$ where $F = f \times \mathrm{id}_Y : X_1 \times Y \to X_2 \times Y$. This in turn equals $F^*(\pi_1^*(a) \smile \pi_2^*(b)) = F^*(\pi_1^*(a)) \smile F^*(\pi_2^*(b)) = (\pi_1 \circ F)^*(a) \smile (\pi_2 \circ F)^*(b) = \pi_1^* f^*(a) \smile \pi_2^*(b) = f^*(a) \times b = \Phi(f^*(a \otimes b))$. For boundary maps, see example sheet. $\square$

To finish the proof of Theorem 3.14, we do the following: Let $P(X, A)$ be the statement that $\Phi : \underline{h}(X, A) \to \overline{h}(X, A)$ is an isomorphism.
Step 1: $P(\{*\}), P(S^0)$ hold. Indeed, $\underline{h}(\{*\}) = H^*(\{*\}) \otimes H^*(Y) \cong \mathbb{Z} \otimes H^*(Y)$ and $\overline{h}(\{*\}) = H^*(\{*\} \times Y) \cong H^*(Y)$. Under these identifications, Φ brings $1 \otimes b$ to $1 \times b = \pi_1^*(1) \smile b = 1 \smile b = b$, which is an isomorphism.
As for S^0, this follows from the fact that $H^*(X \sqcup Y) = H^*(X) \times H^*(Y)$.
Step 2: If X_1 is homotopy equivalent to X_2 (say via f), then $P(X_1)$ and $P(X_2)$ are equivalent. To see this, note that the lemma gives a commutative square

$$\begin{array}{ccc} \underline{h}(X_2) & \xrightarrow{\underline{f^*}} & \underline{h}(X_1) \\ \Phi_2 \downarrow & & \downarrow \Phi_1 \\ \overline{h}(X_2) & \xrightarrow[\overline{f^*}]{} & \overline{h}(X_1) \end{array}$$

But we already know that $\underline{f^*}, \overline{f^*}$ are isomorphisms, so Φ_1 is an isomorphism iff Φ_2 is.
Step 3: If two of $P(X), P(A), P(X, A)$ hold, so does the third. This simply follows from the long exact sequences of pairs, the preceding lemma and five lemma.
Step 4: If (X, A) is a good pair, then $P(X, A)$ is equivalent to $P(X/A)$: By collapsing a pair, $P(X, A)$ holds iff $P(X/A, A/A)$ holds. But $P(A/A) = P(\{*\})$ always holds by Step 1, so $P(X/A, A/A)$ holds iff $P(X/A)$ holds.
Step 5: $P(S^n)$ and $P(D^n, S^{n-1})$ hold. This is done by induction on n. The base case is treated in Step 1. As D^n is homotopy equivalent to $\{*\}$, $P(D^n)$ holds by Step 2. So Step 3 implies that $P(S^{n-1})$ implies $P(D^n, S^{n-1})$, which in turn implies $P(S^n)$ by Step 4.
Step 6: $P(X)$ implies $P(X \cup_f D^n)$. This is just because $X \cup_f D^n / X$ is homotopic to S^n, so $P(X \cup_f D^n, X)$ holds.
Step 7: Induction using Step 6 allows one to conclude $P(X)$ for any finite cell complex X.

Example 3.7. Let Σ_2 be a surface of genus 2. Let's compute $H^*(\Sigma_2)$. Let A be a circle disconnecting Σ_2. Consider $\pi : \Sigma_2 \to \Sigma^2/A \cong T^2 \vee T^2$. Recall that $H_2(\Sigma_2) \cong \mathbb{Z}, H_2(T^2 \vee T^2) \cong H_2(T^2) \oplus H_2(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Here π_* sends 1 to $(1, 1)$. On H^1, π_* is an isomorphism of $\mathbb{Z}^4$.
Since $H_*(\Sigma_2), H_*(T^2 \vee T^2)$ are free over $\mathbb{Z}$, $H^*(\Sigma_2) = \mathrm{Hom}(H_*(\Sigma), \mathbb{Z})$ and same for $T^2 \vee T^2$, and π^* is dual to π_*.
So $\pi^* : H^2(T^2 \vee T^2) \to H^s(\Sigma_2)$ acts via $(1, 1)$. Let's write $H^2(T^2) \oplus H^2(T^2) = \langle c_1 \rangle \oplus \langle c_2 \rangle$ and $H^2(\Sigma_2) = \langle c \rangle$. For H^1 (on which π^* is an isomorphism), let's similarly write $H^1(T^2) \oplus H^1(T^2) = \langle a_1', b_1' \rangle \oplus \langle a_2', b_2' \rangle$ and $a_i = \pi^* a_i', b_j = \pi^* b_j'$. So $H^1(\Sigma_2) = \langle a_1, a_2, b_1, b_2 \rangle$.
Then $a_i \smile b_j = \pi^*(a_i') \smile \pi^*(b_j') = \pi^*(a_i' \smile b_j') = \pi^*(\delta_{ij} c_i) = \delta_{ij} c$. Similarly $a_i \smile a_j = b_i \smile b_j = 0$.
The same argument shows that $H^*(\Sigma_g) = \langle a_i, b_j \rangle_{i,j=1}^g$ with $a_i \smile b_j = \delta_{ij} c, a_i \smile a_j = b_i \smile b_j = 0$ where c generates $H^2(\Sigma_g)$

4 Vector Bundles

4.1 Definition and Examples

Definition 4.1. An n-dimensional real vector bundle (E, B, π) consists of two spaces E (the total space) and B (the base) as well as a map $\pi : E \to B$ with the following extra data:
(i) $\pi^{-1}(b)$ has the structure of a real n-dimensional vector space for each $b \in B$.
(ii) There is an open cover $\{U_\alpha\}_{\alpha \in A}$ of B and homeomorphisms $f_\alpha : \pi^{-1} U_\alpha \to U_\alpha \times \mathbb{R}^n$ such that:
(a) The diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "f_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{R}^n \arrow[d, "\pi_1"] \\ U_\alpha \arrow[r, "\mathrm{id}_{U_\alpha}"'] & U_\alpha \end{tikzcd}$$

commutes.
(b) $\pi_2 \circ f_\alpha : \pi^{-1}(b) \to \mathbb{R}^n$ is an isomorphism of vector spaces for all $v \in U_\alpha$.

f_α are called local trivialisations. By replacing every occurence of $\mathbb{R}$ with $\mathbb{C}$, we get the definition of a complex vector bundle. When no ambiguity can be cause, we may refer to (E, B, π) by (E, B) or just E.

Definition 4.2. A morphism $f : (E, B, \pi) \to (E', B', \pi')$ of vector bundles is a commuting square

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f_B"'] & B' \end{tikzcd}$$

such that $f_E|_{\pi^{-1}(b)} : \pi^{-1}(b) \to (\pi')^{-1}(f_B(b))$ is linear for each $b \in B$. And of course an isomorphism of vector bundles is a morphism with a two-sided inverse. E is a subbundle of E' if there is an injective morphism $(E, B) \to (E', B')$, i.e. each $f_E|_{\pi^{-1}(b)}$ is injective.

Definition 4.3. A section of $\pi : E \to B$ is a continuous map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$. A section s is nonvanishing if $s(b) \neq 0_b = 0_{\pi^{-1}(b)}$ for all b.

The zero section $s_0 : B \to E, b \mapsto 0_b$ is a section of $E \to B$, and is called the zero section. Note that s_0 is continuous since we can check continuity on a local trivialisation (i.e. that $f_\alpha \circ s$ is continuous).

Example 4.1. 1. $E = B \times \mathbb{R}^n$ is an n-dimensional vector bundle over B where $\pi : E \to B$ is the first projection. The trivialisation is simply given by the identity $E \to B \times \mathbb{R}^n$. This is known as the n-dimensional trivial bundle over B, and is sometimes denoted as $\underline{\mathbb{R}^n} = \underline{\mathbb{R}^n}_B$. A bundle $F \to B$ is trivial if there is an isomorphism $F \cong \underline{\mathbb{R}^n}_B$. Equivalently, F is trivial iff there are sections $s_i : B \to F$ such that $s_1(b), \dots, s_n(b)$ is a basis for $\pi^{-1}(b)$ for all $b \in B$.
2. Take $M = (0, 1) \times \mathbb{R}/\sim$ where $\sim$ is the smallest equivalence relation such that $(0, x) \sim (1, -x)$. We have a map $\pi : M \to S^1 = [0, 1]/(0 \sim 1)$, which is a 1-dimensional vector bundle, known as the Möbius bundle.
Since M is homeomorphic to the Möbius band, it is not isomorphic to $\underline{\mathbb{R}}_{S^1}$. We can also see this using sections: A section $s : S^1 \to M$ gives $f : [0, 1] \to \mathbb{R}$ with $f(0) = -f(1)$. In particular, f must vanish somewhere, so s is not nonvanishing.
3. On $B = \mathbb{RP}^n$, we have the tautological bundle $\mathcal{T}_{\mathbb{RP}^n} = \{([z], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in \langle z \rangle\}$ with π given by the first projection. Moreover, $\pi^{-1}([z]) = \langle z \rangle \cong \mathbb{R}$. And on the standard open cover $U_i = \{[z] \in \mathbb{RP}^n : z_i \neq 0\}$, we get local trivialisations $f_i : \pi^{-1}U_i \to U_i \times \mathbb{R}$ via $([z], v) \mapsto ([z], v_i)$.
One can check that $\mathcal{T}_{\mathbb{RP}^1}$ is isomorphic to the Möbius bundle under the identification $\mathbb{RP}^1 \cong S^1$. We can similarly define $\mathcal{T}_{\mathbb{CP}^n}$ which is a dimension 1 complex vector bundle over $\mathbb{CP}^n$.
4. $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : v \cdot x = 0\}$ with the first projection is known as the tangent bundle to S^n. Indeed we have $\pi^{-1}(x) = x^\perp \cong \mathbb{R}^n$ and on $U_i = \{x \in S^n : x_i \neq 0\}$ we have local trivialisation $f_i : \pi^{-1}U_i \to U_i \times \mathbb{R}^n, (x, v) \mapsto (x, \pi_{\hat{i}} v)$. TS^1 has a nonvanishing section $s(x, y) = ((x, y), (-y, x))$, so $TS^1 \cong \underline{\mathbb{R}}_{S^1}$. But TS^{2n} has no nonvanishing section (example sheet a long time ago), so TS^{2n} is not trivial for any $n \geq 1$.

Definition 4.4. If $\pi : E \to B$ is an n-dimensional real vector bundle and $g : B' \to B$ is continuous, then the pullback bundle is the vector bundle over

B' with total space $g^*(E) = \{(b', b, v) \in B' \times B \times E : g(b') = \pi(v) = b\}$ and $\pi_g : g^*E \to B'$ is given by the first projection. We have $\pi_g^{-1}(b') = \pi^{-1}(g(b'))$. If $f_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^n$ is a local trivialisation for E, we set $V_\alpha = g^{-1}U_\alpha$ and declare the local trivialisation $f'_\alpha : \pi_g^{-1}V_\alpha \to V_\alpha \times \mathbb{R}^n, (b', b, v) \mapsto (b', \pi_2(f_\alpha(v)))$.

Lemma 4.1. $(g \circ f)^*E = f^*(g^*E)$.

Definition 4.5. Suppose $A \subset B$. The restriction $E|_A$ of a vector bundle $E \to B$ to A is its pullback under the inclusion.

Definition 4.6. If $s : B \to E$ is a section, then its pullback is the section $g^*s : B' \to g^*E, b' \mapsto (b', f(b), s(f(b)))$ of g^*E, which is nonvanishing if s is.

Example 4.2. $\mathcal{T}_{\mathbb{RP}^n}|_{\mathbb{RP}^1} \cong \mathcal{T}_{\mathbb{RP}^1}$ has no nonvanishing section, so $\mathcal{T}_{\mathbb{RP}^n}$ cannot have any nonvanishing section either. Hence it too is nontrivial.

Definition 4.7. If $\pi : E \to B$ and $\pi' : E' \to B'$ are vector bundles of dimensions m, m', then their product is $\pi \times \pi' : E \times E' \to B \times B'$ which is a vector bundle of dimension $m + m'$.

Of course we have $(\pi \times \pi')^{-1}(b, b') = \pi^{-1}(b) \times \pi^{-1}(b')$, and if $f_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^m$ and $f'_\beta : (\pi')^{-1}U'_\beta \to U'_\beta \times \mathbb{R}^{m'}$ are trivialisations, we get $f_\alpha \times f'_\beta : (\pi \times \pi')^{-1}(U_\alpha \times U'_\beta) \to U_\alpha \times U'_\beta \times \mathbb{R}^{m+m'}$ which is a trivialisation.

Definition 4.8. The direct sum (or Whitney sum) of $E \to B, E' \to B$ is $\Delta^*(E \times E')$ where $\Delta : B \to B \times B$ is the diagonal $b \mapsto (b, b)$.

4.2 Partitions of Unity

Recall that for $\phi : B \to \mathbb{R}$, its support is $\operatorname{supp} \phi = \overline{\{b \in B : \phi(b) \neq 0\}}$.

Definition 4.9. Suppose $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open cover of B. A partition of unity subordinate to $\mathcal{U}$ is a family of functions $\phi_i : B \to \mathbb{R}$ for $i \in \mathbb{Z}_{\geq 0}$ such that:
(i) $0 \leq \phi_i(b) \leq 1$.
(ii) $\{i : \phi_i(b) \neq 0\}$ is finite for all $b \in B$.
(iii) $\operatorname{supp} \phi_i \subset U_{\alpha_i}$ for some $\alpha_i \in A$.
(iv) For all $b \in B$, $\sum_{i \geq 0} \phi_i(b) = 1$.
We say B admits partitions of unity if, for any open cover $\mathcal{U}$ of B, there is a partition of unity subordinate to $\mathcal{U}$.

Remark. If B is compact or metrizable, then it admits partitions of unity. More generally, B admits partitions of unity if it's paracompact and Hausdorff.

Theorem 4.2. *Suppose B admits partitions of unity and $pi : E \to B \times [0,1]$ is a vector bundle, then $E|_{B \times \{0\}} \cong E|_{B \times \{1\}}$.*

Lemma 4.3. *If $E|_{B \times [0,1/2]}$ and $E_{B \times [1/2,1]}$ are trivial, then E is trivial.*

Proof. Clear. □

Lemma 4.4. *Each $b \in B$ has an open neighbourhood U_b such that $E_{U_b \times [0,1]}$ is trivial.*

Proof. For each $(b,t) \in B\times[0,1]$, we can find an open neighbourhood U_t of b and I_t of $t \in I = [0,1]$ such that $E|_{U_t\times I_t}$ is trivial. $\{I_t : t \in I\}$ is an open cover of I and I is compact, so we can find $I_{t_0}, \dots, T_{t_n}$ which cover I. Then there exists $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $[s_i, s_{i+1}] \subset I_{t_k}$ for some k. In this case, $E|_{U_{t_k}\times[s_i,s_{i+1}]}$ is trivial. Take $U_b = \bigcap_{k=0}^n U_{t_k}$ is an open subset of B containing b. Then $E_{U_b\times[0,1]}$ is trivial by the preceding lemma (and induction). $\square$

Proof of Theorem 4.2. Let U_b be as in the preceding lemma and pick a partition of unity $\{\phi_i\}_{i=1}^\infty$ subordinate to $\{U_b : b \in B\}$. Suppose $\operatorname{Supp}\phi_i \subset U_{b_i}$. Write $\psi_k(b) = \sum_{i=1}^k \phi_i(b)$.
Let $g_k : B \to B \times I, b \mapsto (b, \psi_k(b))$ and define $E_k = g_k^*E = \{(b,(b,\psi_k(b)),v) : \pi(v) = (b,\psi_k(b))\}$. Let $f_i : \pi^{-1}(U_b\times I) \to U_b\times I\times\mathbb{R}^n$ be a trivialisation. Define $\beta_k : E_{k-1} \to E_k$ by

$$\beta_k((b, g_k(b), v)) = \begin{cases} (b, g_k(b), v) & \text{for } b \notin U_{b_k} \\ (b, f_k^{-1}(b, g_k(b), v')) & \text{for } b \in U_{b_k} \end{cases}$$

where $f_k(v) = (b, g_{k-1}(b), v')$. Then $\cdots\circ\beta_3\circ\beta_2\circ\beta_1$ is the desired isomorphism $E|_{B\times\{0\}} \to E_{B\times\{1\}}$. $\square$

Corollary 4.5. *Suppose $\pi : E \to B$ is a vector bundle and $g_0, g_1 : B' \to B$ are homotopic via $h : B'\times I \to B$. Suppose B' admits partitions of unity, then $g_0^*(E) = h^*(E)|_{B'\times\{0\}} \cong h^*(E)|_{B'\times\{1\}} = g_1^*(E)$.*

Corollary 4.6. *If B is contractible and admits partitions of unity, then every vector bundle $\pi : E \to B$ is trivial.*

We are not done with partitions of unity yet. Let's define Riemannian and Hermitian metric with them.

Definition 4.10. Suppose $\pi : E \to B$ is a real (resp. complex) vector bundle. A Riemannian (resp. Hermitian) metric on E is a continuous map $g : E\oplus E \to \mathbb{R}$ (resp. $E\oplus E \to \mathbb{C}$) such that $g|_{\pi^{-1}_{E\oplus E}(b)}$ is an inner product (resp. Hermitian inner product) on $\pi^{-1}_{E\oplus E}(b) = \pi^{-1}(b)\oplus\pi^{-1}(b)$.

Example 4.3. On $\mathcal{T}_{\mathbb{RP}^2} = \{([z], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in \langle z\rangle\}$ has a natural Riemannian metric given by $g(([z], v_1), ([z], v_2)) = \langle v_1, v_2\rangle_{\mathbb{R}^{n+1}}$. Similarly $\mathcal{T}_{\mathbb{CP}^n}$ has a natural Hermitian metric.

Definition 4.11. Suppose $\pi : E \to B$ is a vector bundle with Riemannian metric g. The unit sphere and unit disk bundles of E are

$$S_g(E) = \{v \in E : \langle v, v\rangle = 1\}, D_g(E) = \{v \in E : \langle v, v\rangle \le 1\}$$

π restricts to $S_g(E) \to B, D_g(E) \to B$ whose fibres are isomorphic to S^{n-1}, D^n, respectively.

Proposition 4.7. *If g, g' are Riemannian metrics on E, then we have commutative diagrams*

$$\begin{tikzcd} S_g(E) \arrow[r, "\cong"] \arrow[d] & S_{g'}(E) \arrow[dl] \\ B & \end{tikzcd} \quad \begin{tikzcd} D_g(E) \arrow[r, "\cong"] \arrow[d] & D_{g'}(E) \arrow[dl] \\ B & \end{tikzcd}$$

Proof. Exercise. $\square$

So we can drop g from our notation and just write $S(E)$ and $D(E)$.

Example 4.4. $S(\mathcal{T}_{\mathbb{RP}^n}) = \{([z], v) : \|v\|_{\mathbb{R}^{n=1}} = 1, v \in \langle z \rangle\} \cong S^n$ and $S(\mathcal{T}_{\mathbb{RP}^n}) \cong S^n \to \mathbb{RP}^n$ is just the natural projection (viewing $\mathbb{RP}^n$ as a quotient of S^n). Similarly, $S(\mathcal{T}_{\mathbb{CP}^n}) \cong S^{2n-1}$ and $S(\mathcal{T}_{\mathbb{CP}^n}) \cong S^{2n-1} \to \mathbb{CP}^n$ is nothing but the Hopf map.
If $\pi : E \to B$ is trivial, say it is trivialised by $f : E \to B \times \mathbb{R}^n$, then E has a Riemannian metric given by $g(v_1, v_2) = \langle \pi_2(f(v_1)), \pi_2(f(v_2)) \rangle$. Therefore $S(B \times \mathbb{R}^n) \cong B \times S^{n-1}$. In particular, $\mathcal{T}_{\mathbb{RP}^n}, \mathcal{T}_{\mathbb{CP}^n}$ are nontrivial since $\mathbb{RP}^n \times S^0 \ncong S^n, \mathbb{CP}^n \times S^1 \ncong S^{2n-1}$.

Proposition 4.8. *If B admits partitions of unity and $\pi : E \to B$ is a real vector bundle, then E has a Riemannian metric.*

Proof. B has an open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $E|_{U_\alpha}$ is trivial. In the previous example we've seen that $E|_{U_\alpha}$ has a Riemannian metric. Pick a partition of unity subordinate to $\mathcal{U}$ and take $g = \sum_i \phi_i g_{\alpha_i}$ where $\operatorname{supp} \phi_i \subset U_{\alpha_i}$. $\square$

4.3 Thom Isomorphism

Suppose $\pi : E \to B$ is an n-dimensional vector bundle. For $b \in B$, let $E_b = \pi^{-1}(b)$ be the fibre of E over b. Write $i_b : E_b \to E$ to be the inclusion. Let $s_0 : B \to E$ be the zero section. Define $E^\sharp : E \setminus \operatorname{Im} s_0, E_b^\sharp = E_b \setminus \{0_b\}$. Then $H^*(E_b, E_b^\sharp) \cong H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ which is $\mathbb{Z}$ at the n-th grading and zero elsewhere. This is free, so

$$H^i(E_b, E_b^\sharp; R) = \begin{cases} R & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

Definition 4.12. $u \in H^n(E, E^\sharp; R)$ is an R-Thom class (or R-orientation) for E if $i_b^* u$ generates $H^n(E_b, E_b^\sharp; R)$ for all $b \in B$.

From now on we'll always assume the coefficients of cohomology are in R.

Example 4.5. If $E = B \times \mathbb{R}^n$ is trivial, then $H^*(E, E^\sharp) = H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n \setminus \{0\})) \cong H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ since $H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is free. We therefore have an isomorphism $H^{k-n}(B) \to H^n(E, E^\#), a \mapsto a \times u$ whwere u generates $H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Note also that $H^0(B) = \prod_{B_i \in \pi_0(B)} H^0(B_i)$, so we can specify $r \in H^0(B)$ by a tuple $r = (r_1, \ldots, r_k)$ for $r_i \in H^0(B_i)$.
If $b \in B_i$, we have $i_b^*(r \times u) = r_i u \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. So $r \times u$ is a Thom class iff r_i generates $H^0(B_i)$ for all i. If $R = \mathbb{Z}/2\mathbb{Z}$, there is a unique Thom class. And if $R = \mathbb{Z}$, there are $2^{|\pi_0(B)|}$ Thom classes.

Suppose we have a continuous $f : B' \to B$, there is a morphism

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

where F is of course $(b', b, v) \mapsto v$. Since $F(\operatorname{Im} s_0') = \operatorname{Im} s_0$, it is a map of pairs $(f^*E, f^*E^\sharp) \to (E, E^\sharp)$.

Lemma 4.9. *If u is an R-Thom class for E, then F^*u is an R-Thom class for f^*E.*

Proof. There is a commuting square

$$\begin{tikzcd} f^*E \arrow[r, "F"] & E \\ (f^*E)_{b'} \arrow[u, "i_{b'}"] \arrow[r, "j=F|_{(f^*E)_b}"'] & E_{f(b')} \arrow[u, "i_{f(b')}"'] \end{tikzcd}$$

So j is an isomorphism and $i_{f(b')}^* u$ generates $H^n(E_{f(b')}, E_{f(b')}^*)$, so $i_{b'}^*(F^*u)$ generates $H^n((f^*E)_{b'}, (f^*E)_{b'}^\sharp)$. □

Lemma 4.10. *Suppose $B = B_1 \cup B_2, u \in H^n(E, E^\sharp)$. Let $i_k : B_k \to B$ be the inclusion. If i_1^*u, i_2^*u are Thom classes for $E|_{B_1}, E|_{B_2}$, then u is a Thom class for E.*

Proof. Exercise. □

Theorem 4.11 (Thom isomorphism)**.** *If $\pi : E \to B$ is an n-dimensional real vector bundle, then:*
(a) E has a unique $\mathbb{Z}/2\mathbb{Z}$-Thom class.
(b) If E has an R-Thom class u, then the map $\Phi : H^i(B; R) \to H^{i+n}(E, E^\sharp; R)$ given by $a \mapsto \pi^*(a) \smile u$ is an isomorphism.

Proof when B is compact. Step 1: The theorem holds when $E = B \times \mathbb{R}^n$ is trivial. We've already seen this in a previous example.
Step 2: Suppose $V_1, V_2 \subset B$ are open. Let $E_i = E|_{B_i}, E_\cap = E_{V_1 \cap V_2}, E_\cup = E|_{V_1 \cup V_2}$. We claim that if the theorem holds for E_1, E_2 and $E_\cap$, then it holds for $E_\cup$.
For (a), we look at the Mayer-Vietoris sequence (noting $H^{n-1}(E_\cap, E_\cap^\sharp) = 0$)

$$0 \longrightarrow H^n(E_\cup, E_\cup^\sharp) \xrightarrow{i=(i_1^*, i_2^*)^\top} H^n(E_1, E_1^\sharp) \oplus H^n(E_2, E_2^\sharp) \xrightarrow{j=j_1^*-j_2^*} H^n(E_\cap, E_\cap^*)$$

with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Let $u_i \in H^n(E_i, E_i^\sharp)$ be the unique $\mathbb{Z}/2\mathbb{Z}$-Thom classes. By Lemma 4.9, $j_i^*u_i$ is a $\mathbb{Z}/2\mathbb{Z}$-Thom class for $E_\cap$. By uniqueness, $j_1^*u_1 = j_2^*u_2 = u_\cap$ is the unique $\mathbb{Z}/2\mathbb{Z}$-Thom class for $E_\cap$. So $(u_1, u_2) \in \ker j = \operatorname{Im} i$, which means that $(u_1, u_2) = i(u_\cup)$ for some $u_\cup \in H^n(E_\cup, E_\cup^\sharp)$. Then $i_1^*(u_\cup) = u_i$, so by the preceding lemma $u_\cup$ is a $\mathbb{Z}/2\mathbb{Z}$-Thom class for $E_\cup$. It is moreover unique since if $u_\cup' \in H^n(E_\cup, E_\cup^\sharp)$ is a $\mathbb{Z}/2\mathbb{Z}$-Thom class, then $i(u_\cup') = (u_1, u_2)$ by uniqueness and Lemma 4.9, so $u_\cup' = u_\cup$ as i is injective.
For (b), recall the commutative diagram of Mayer-Vietoris sequences

$$\begin{tikzcd} H^*(V_1 \cup V_2) \arrow[r] \arrow[d, "\Phi_\cup"] & H^*(V_1) \oplus H^*(V_2) \arrow[r] \arrow[d, "\Phi_1 \oplus \Phi_2"] & H^*(V_1 \cap V_2) \arrow[d, "\Phi_\cap"] \\ H^*(E_\cup, E_\cup^\sharp) \arrow[r] & H^*(E_1, E_1^\sharp) \oplus H^*(E_2, E_2^\sharp) \arrow[r] & H^*(E_\cap, E_\cap^\sharp) \end{tikzcd}$$

By hypothesis $\Phi_1, \Phi_2, \Phi_\cap$ are all isomorphisms, so $\Phi_\cup$ is an isomorphism by five lemma.
Step 3: Cover B by local trivialisations and profit. □

4.4 Gysin Sequence

Suppose $\pi : E \to B$ has an R-Thom class u. Note that $E^\sharp = E \setminus \operatorname{Im} s_0 \sim S(E), v \mapsto v/\sqrt{g(v,v)}$. Consider the long exact sequence of the pair $(E, E^\sharp)$ given by

$$\begin{tikzcd}
H^i(E,E^\sharp) \arrow[r, "j^*"] & H^i(E) \arrow[r] \arrow[d, "s_0^*"] & H^i(E^\sharp) \arrow[r] \arrow[d, "\cong"] & H^{i+1}(E,E^\sharp) \\
H^{i-n}(B) \arrow[r, "\alpha"] \arrow[u, "\Phi"] & H^i(B) \arrow[r] & H^i(S(E)) \arrow[r] & H^{i-n+1}(B) \arrow[u, "\Phi"]
\end{tikzcd}$$

where $j : (E, \varnothing) \to (E, E^\sharp)$ is the inclusion of pairs. So s_0^* is an isomorphism, and indeed its inverse is given by π^* (since they give homotopy equivalences). Then $\alpha(a) = s_0^*(j^*(\Phi(a))) = s_0^* j^*(\pi^* a \smile u) = s_0^*(\pi^* a \smile j^* u) = (s_0^* \pi^* a) \smile (s_0^* j^* u) = a \smile s_0^* j^* u$.

Definition 4.13. If $\pi : E \to B$ is an R-oriented n-dimensional real vector bundle with Thom class u, then its Euler class is $e(E) = s_0^* j^* u \in H^n(B)$.

Theorem 4.12 (Gysin sequence)**.** *There is a long exact sequence of the form*

$$H^{i-n}(B) \xrightarrow{\alpha} H^i(B) \xrightarrow{\pi^*} H^i(S(E)) \longrightarrow H^{i-n+1}(B)$$

where $\alpha(a) = a \smile e(E)$.

Proposition 4.13. *Suppose E is as before.*
(i) For $f : B' \to B$, f^*E is oriented and $e(f^*E) = f^*(e(E))$.
(ii) If E is trivial and $n > 0$ then $e(E) = 0$.
(iii) $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$.
(iv) If E has a nonvanishing section, then $e(E) = 0$.

Proof. (i) Consider the diagram

$$\begin{tikzcd}
(B, \varnothing) \arrow[r, "s_0"] & (E, \varnothing) \arrow[r, "j"] & (E, E^\sharp) \\
(B', \varnothing) \arrow[r, "s_0'"] \arrow[u, "f"] & (f^*E, \varnothing) \arrow[r, "j'"] \arrow[u, "f_E"] & (f^*E, f^*E^\sharp) \arrow[u, "f_E"]
\end{tikzcd}$$

By an earlier lemma, $f_E^* u$ is an orientation on f^*E, so $e(f^*E) = (s_0')^*(j')^* f_E^* u = f^* s_0^* j^* u = f^*(e(E))$.
(ii) This is true if $B = \{*\}$ since $H^n(\{*\}) = 0$. In general, E is trivial iff $E = f^* E_*$ where E_* is the trivial rank n bundle on $\{*\}$. So $e(E) = f^*(e(E_*)) = f^* 0 = 0$.
(iii) Example sheet.
(iv) If s is a nonvanishing section then $E = \langle s \rangle \oplus \langle s \rangle^\perp$, then $e(E) = e(\langle s \rangle) \smile e(\langle s \rangle^\perp) = 0 \smile e(\langle s \rangle^\perp) = 0$ since $\langle s \rangle$ is trivial. $\square$

Theorem 4.14. $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$ *where x can be taken as $e(\mathcal{T}_{\mathbb{RP}^n}) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.*

Proof. We suppress the notation for the $\mathbb{Z}/2\mathbb{Z}$ coefficient because we are (I am) lazy.
$S(\mathcal{T}_{\mathbb{RP}^n}) = S^n$. The Gysin sequence becomes

$$H^{k-1}(\mathbb{RP}^n) \xrightarrow{\alpha} H^k(\mathbb{RP}^n) \longrightarrow H^k(S^n) \longrightarrow H^k(\mathbb{RP}^n)$$

This immediately tells us that $\alpha = - \smile x$ is an isomorphism for $1 \le k \le n$. So by induction $\langle x^k \rangle$ generates $H^k(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \le k \le n$, which gives us what we want. □

Similarly, the real vector bundle that underlies $\mathcal{T}_{\mathbb{CP}^n}$ is $\mathbb{Z}$-orientable (example sheet). Almost the same argument (using the fact that $S(\mathcal{T}_{\mathbb{CP}^n}) = S^{2n-1}$) shows that

Theorem 4.15. *$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ where x can be taken as $e(\mathcal{T}_{\mathbb{CP}^n}) \in H^2(\mathbb{CP}^n; \mathbb{Z})$.*

Corollary 4.16. *$\pi_3(S^2) \neq 0$.*

Proof. Suppose otherwise, then $\mathbb{CP}^2 \simeq S^2 \vee S^4$ since $\mathbb{CP}^2 = S^2 \cup_h D^4$ where $h : S^3 \to S^2$ is the Hopf map. But anything in $H^2(S^2 \vee S^4)$ squares to zero! □

Remark. 1. Every vector bundle is $\mathbb{Z}/2\mathbb{Z}$-orientable. It turns out that for $p \neq 2$, E is $\mathbb{Z}/p\mathbb{Z}$-orientable iff it is $\mathbb{Z}$-orientable. We often say a vector bundle is orientable if it is $\mathbb{Z}$-orientable.
2. $M = \mathcal{T}_{\mathbb{RP}^1}$ is not $\mathbb{Z}$-orientable because $H^*(M, M^\sharp) \cong H^*(D(M), S(M)) \cong H^*(\bar{M}, \partial\bar{M})$ where $\bar{M}$ is the usual closed Möbius band. But $H^2(\bar{M}, \partial\bar{M}) \cong \mathbb{Z}/2\mathbb{Z} \not\cong H^1(S^1)$, so Thom isomorphism fails for $\mathbb{Z}$-coefficients.
3. There is in general a homomorphism $\phi : \pi_1(B) \to \mathbb{Z}/2\mathbb{Z}$ by saying $\phi([\gamma]) = 0$ (for $\gamma : S^1 \to B$) iff $\gamma^* E$ is orientable (example sheet). Consequently, if $\pi_1(B) = \{1\}$ then any vector bundle over B is orientable.

5 Manifolds

5.1 Fundamental Class and Orientability

Definition 5.1. An n-manifold M is a second-countable Hausdorff topological space with an open cover $\{U_\alpha\}_{\alpha \in A}$ such that there are homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$.

The transition functions $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are homeomorphisms.

Definition 5.2. M is smooth if the transition functions can be chosen to be diffeomorphisms.

Any smooth manifold M has a tangent bundle $\pi : TM \to M$ which is an n-dimensional vector bundle.
For $A \subset M$ compact, we write $(M \mid A) = (M, M - A)$. For $B \subset A$ compact, we have the inclusion of pairs $i : (M \mid A) \to (M \mid B)$. Given $w \in H_*(M \mid A)$, we write $w|_B = i_* w$. If $x \in U_\alpha \cong \mathbb{R}^n$, excision gives $H_i(M \mid x) \cong H_i(U_\alpha \mid x) \cong H_i(\mathbb{R}^n, \phi_\alpha(x)) = H_i(\mathbb{R}^n, \mathbb{R}^n - \phi(x))$ which is $\mathbb{Z}$ when $i = n$ and 0 elsewhere. So

$$H_i(M \mid x; R) \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Definition 5.3. An R-fundamental class for $(M \mid A)$ is $w \in H_n(M \mid A; R)$ such that $w|_x$ generates $H_n(M|_x)$ for all $x \in M - A$.

This can be viewed as an analogue to Thom classes.

Theorem 5.1. *For $A \subset M$ compact, $(M \mid A)$ has a unique $\mathbb{Z}/2\mathbb{Z}$-fundamental class.*

The proof is similar to the proof of Theorem 4.11 We are most interested in the case where M is itself compact. In this occasion, a fundamental class for $(M \mid M)$ is denoted by $[M] \in H_n(M)$.

Definition 5.4. M is orientable if it has a $\mathbb{Z}$-fundamental class.

Proposition 5.2. *M is orientable iff TM is orientable.*

Definition 5.5. $N \subset M$ is a k-dimensional submanifold of an n-dimensional submanifold M if for every $x \in N$, there is a chart $\phi_x : U_x \to \mathbb{R}^n$ such that $\phi_x(U_\alpha \cap N) = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. If $N \subset M$ is a smooth submanifold (i.e. if ϕ_x can be taken to be part of the smooth structure), then $TN \subset TM|_N$.

Definition 5.6. $\nu_{M/N} = TN^\perp \subset TM|_N$ is the normal bundle of N in M.

So $TM|_N = \nu_{M/N} \oplus TN$.

Theorem 5.3 (Tubular Neighbourhood Theorem)**.** *If $N \subset M$ is a smooth closed submanifold, then there is an open $V \subset M$ (the "tubular neighbourhood of N") containing N with $(V, N) \cong (\nu_{M/N}, s_0(N))$.*

Lemma 5.4. *If $E = E_1 \oplus E_2$ is orientable, then E_1 is orientable iff E_2 is.*

Proof. Follows from something on example sheet. □

Proof of Proposition 5.2. Any $\gamma : S^1 \to M$ is an embedding give rise to a tubular neighbourhood $V(\gamma)$ of its. Then M is orientable iff $V(\gamma)$ is orientable for all γ iff $\nu_{M/\gamma}$ is orientable for all γ iff $TM|_\gamma$ is orientable for all γ iff TM is orientable. □

Corollary 5.5. *If M is orientable and $N \hookrightarrow M$ is a smooth closed submanifold, then N is orientable iff $\nu_{M/N}$ is orientable.*

5.2 Poincaré Duality

We now work with coefficients in a field $\mathbb{F}$ (and we'll also just omit mentioning it from now on). So H^k is just the dual of H_k in this case. Let ϕ be the isomorphism $\mathrm{Hom}_{\mathbb{F}}(H^k(M), \mathbb{F}) \to H_k(\mathbb{F})$, then $\langle a, \phi(\alpha)\rangle = \alpha(a)$.

Definition 5.7. For $a \in H^k(X)$, the cap product $- \frown a : H_{k+l}(X) \to H_l(X)$ is the dual of $a \smile - : H^l(X) \to H^{k+l}(X)$.

In other words, $\langle b, x \frown a\rangle = \langle a \smile b, x\rangle$.
Suppose M is an $\mathbb{F}$-oriented n-manifold with fundamental class $[M] \in H_n(M)$.

Definition 5.8. The intersection pairing $(,) : H^k(M) \times H^{n-k}(M) \to \mathbb{F}$ is the bilinear pairing given by $(a, b) = \langle a \smile b, [M]\rangle$.

We have $(b,a) = (-1)^{k(n-k)}(a,b)$. For $a \in H^k(M)$, $(a,-)$ is an element of the dual of $H^{n-k}(M)$, hence an element of $H_{n-k}(m)$.

Definition 5.9. The (algebraic) Poincaré dual of a is $\mathrm{PD}(a) = \phi((a,-)) = [M] \frown a$.

So $\langle b, \mathrm{PD}(a)\rangle = (a,b) = \langle a \smile b, [M]\rangle$.

Theorem 5.6. *If M is a connected n-manifold, then the map $H_n(M) \to H_n(M \mid x) \cong \mathbb{F}$ is injective. So if $[M]$ is ($\mathbb{F}$-)oriented, then $H_n(M) \cong \mathbb{F}$, which means that $H^n(M) \cong \mathbb{F} = \langle [M]^*\rangle$, where $\langle [M]^*, [M]\rangle = 1$.*

Assume $i : N \to M$ is a smooth oriented connected closed submanifold of dimension k. Let V be a tubular neighbourhood of N and $\nu = \nu_{M/N}$ the normal bundle, then we have the diagram

$$\begin{tikzcd}
(M,\varnothing) \arrow[r, "j"] \arrow[dr, "j_x"'] & (M \mid N) \arrow[d] & (V \mid N) \cong (\nu, \nu^\sharp) \arrow[l, "i"'] \\
 & (M \mid x) &
\end{tikzcd}$$

Since N is connected, $H^k(N) = \langle [N]^*\rangle \cong \mathbb{F}$. Consequently, Theorem 4.11 gives $H^n(\nu,\nu^\sharp) = \langle u \smile \pi^*[N]^*\rangle \cong \mathbb{F}$ where u is an orientation for $\nu_{M/N}$. Thus $H_n(\nu,\nu^\sharp) \cong \mathbb{F}$.
By Theorem 1.28, i_* gives an isomorpism $H_n(V,V^\sharp) \cong H_n(M \mid N) \cong \mathbb{F}$. On the other hand, $(j_x)_*$ too gives an isomorphism $H_n(M) \cong H_n(M \mid x) \cong \mathbb{F}$, so $j_* : H_n(M) \to H_n(M \mid N)$ too is an isomorphism.
So $i_*^{-1} j_*[M]$ generates $H_n(\nu,\nu^\sharp) \cong \mathbb{F}$, so $\kappa = \langle u \smile \pi^*[N]^*, i_*^{-1} j_*[M]\rangle \in \mathbb{F}^\times$.

Definition 5.10. The class $u_{M/N} = \kappa^{-1} u$ is known as the orientation on $\nu_{M/N}$ induced by $[N]$ and $[M]$.

So $\langle u_{M/N} \smile \pi^*[N]^*, i_*^{-1} j_*[M]\rangle = 1$.

Definition 5.11. $\mathrm{pd}(N) = j^*((i^*)^{-1} u_{M/N}) \in H^{n-k}(M)$ is the geometric Poincaré dual of N.

Proposition 5.7. *For $a \in H^k(M)$, $\langle \mathrm{pd}([N]) \smile a, [M]\rangle = \langle a, i_*[N]\rangle$.*

That is, $\mathrm{PD}(\mathrm{pd}(N)) = i_*[N]$.

Lemma 5.8. *Let $i : V \to M$ be the inclusion, then $i^*(a) = \langle a, i_*[N]\rangle \pi^*[N]^*$.*

Proof. $\pi : V \to N$ is a homotopy equivalence and $H^n(V)$ is generated by $\pi^*[N]^*$. So it suffices to check that $\langle i^* a, [N]\rangle = \langle \langle a, i_*[N]\rangle \pi^*[N]^*, [N]\rangle$ which let's just left as an exercise. □

Proof of Proposition 5.7. If $b \in H^l(M \mid N), j^*(b \smile a) = (j^*b) \smile a$, so by the preceding lemma,

$$\begin{aligned}
\langle \mathrm{pd}(N) \smile a, [M]\rangle &= \langle i_*^{-1}(u_{M/N}) \smile a, j_*[M]\rangle = \langle u_{M/N} \smile i^*a, i_*^{-1} j_*[M]\rangle \\
&= \langle u_{M/N} \smile \langle a, i_*[N]\rangle \pi^*[N]^*, i_*^{-1} j_*[M]\rangle = \langle a, i_*[N]\rangle \qquad \square
\end{aligned}$$

Eventually we'll want to show that PD is an isomorphism. This will be done by considering $\mathrm{pd}(\Delta) \in H^n(M \times M)$ where $\Delta : M \to M \times M$ is the diagonal map.

5.3 Homology of Products

We're working over the field, so most prayers are answered. In particular,

$$\begin{aligned} H_*(X \times Y) &\cong \mathrm{Hom}_{\mathbb{F}}(H^*(X \times Y), \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H^*(X) \otimes H^*(Y), \mathbb{F}) \\ &\cong \mathrm{Hom}_{\mathbb{F}}(H^*(X), \mathbb{F}) \otimes \mathrm{Hom}_{\mathbb{F}}(H^*(Y), \mathbb{F}) \cong H_*(X) \otimes H_*(Y) \end{aligned}$$

Denote by $\alpha \times \beta \in H_*(X \times Y)$ the element corresponding to $\alpha \otimes \beta \in H_*(X) \otimes H_*(Y)$ under this isomorphism. This can be alternatively characterised by the identity $\langle a \times b, \alpha \times \beta \rangle = \langle a, \alpha \rangle \langle b, \beta \rangle$.
Recall that $\langle b, z \frown a \rangle = \langle a \smile b, z \rangle$.

Lemma 5.9. $(z_1 \times z_2) \frown (a_1 \times a_2) = (-1)^{|a_2|(|z_1|-|a_1|)}(z_1 \frown a_1) \times (z_2 \frown a_2)$.

Proof. Applying $\langle b_1 \times b_2, - \rangle$ equates two sides of the equality. □

Lemma 5.10. *Suppose X is connected and $p \in X$ (so $H^0(X)$ is generated by $[p]$) and $a \in H^k(X), \alpha \in H_k(X)$. Then $\alpha \frown a = \langle a, \alpha \rangle [p]$.*

Proof. $\langle 1, \alpha \frown a \rangle = \langle a \smile 1, \alpha \rangle = \langle a, \alpha \rangle$ and $\langle 1, [p] \rangle = 1$. □

Lemma 5.11. *Let $\Delta : X \to X \times X$ be the diagonal and $a, b \in H^*(X)$, then $\Delta^*(a \times b) = a \smile b$.*

Proof. $\Delta^*(a \times b) = \Delta^*(\pi_1^*(a) \smile \pi_2^*(b)) = \Delta^*\pi_1^*(a) \smile \Delta^*\pi_2^*(b) = a \smile b$. □

Orient $M \times M$ by $[M \times M] = [M] \times [M]$ and let $\tilde{u} = \mathrm{pd}(\Delta) \in H^n(M \times M)$. Let $p \in M$.

Proposition 5.12. $\langle \tilde{u}, [M] \times [p] \rangle = (-1)^n$

Proof.

$$\begin{aligned} \langle \tilde{u} \smile (1 \times [M]^*), [M] \times [M] \rangle &= (-1)^n \langle (1 \times [M]^*) \smile \tilde{u}, [M] \times [M] \rangle \\ &= (-1)^n \langle \tilde{u}, ([M] \times [M]) \frown (1 \times [M]^*) \rangle \\ &= (-1)^n \langle \tilde{u}, ([M] \frown 1) \times ([M] \frown [M]^*) \rangle \\ &= (-1)^n \langle \tilde{u}, [M] \times [p] \rangle \end{aligned}$$

On the other hand, $\tilde{u} = \mathrm{pd}(\Delta)$, so

$$\begin{aligned} \langle \tilde{u} \smile (1 \times [M]^*), [M] \times [M] \rangle &= \langle 1 \times [M]^*, [\Delta] \rangle = \langle \pi_2^*[M]^*, \Delta_*[M] \rangle \\ &= \langle [M]^*, (\pi_2)_* \Delta_* [M] \rangle = \langle [M]^*, [M] \rangle = 1 \end{aligned}$$ □

Lemma 5.13. $\tilde{u} \smile (a \times b) = (-1)^{|a||b|} \tilde{u} \smile (b \times a)$.

Proof. Let V be a tubular neighbourhood for $\Delta \subset M \times M$. Denote the inclusions by

$$\begin{tikzcd} & M \arrow[dl, "j_\Delta"'] \arrow[d, "\Delta"] \\ V \arrow[r, "i'"'] \arrow[d, "j'"'] & M \times M \arrow[d, "j"] \\ (V \mid \Delta) \arrow[r, "i"'] & (M \times M \mid \Delta) \end{tikzcd}$$

Let $\pi : V \to \Delta$ be the projection, with V viewed as the normal bundle of Δ. Then π, j_Δ are homotopy inverses. We then have

$$\begin{aligned} u \smile (i')^*(a \times b) &= u \smile \pi^* j_\Delta^* (i')^*(a \times b) = u \smile \pi^* \Delta^*(a \times b) \\ &= u \smile \pi^*(a \smile b) = (-1)^{|a||b|} u \smile \pi^*(b \smile a) \\ &= (-1)^{|a||b|} u \smile (i')^*(b \times a) \end{aligned}$$

Applying $j^*(i^*)^{-1}$ to both sides gives the result. □

Proposition 5.14. $\langle \tilde{u}, \mathrm{PD}(a) \times y \rangle = (-1)^{n(n-|a|)} \langle a, y \rangle$ *for any* $a \in H^k(M), y \in H_k(M)$.

Proof. By the preceding lemma,

$$\begin{aligned} \langle \tilde{u}, \mathrm{PD}(a) \times y \rangle &= \langle \tilde{u}, ([M] \frown a) \times (y \frown 1) = (-1)^0 \langle \tilde{u}, ([M] \times y) \frown (a \times 1) \rangle \\ &= \langle (a \times 1) \smile \tilde{u}, [M] \times y \rangle = \langle (1 \times a) \smile \tilde{u}, [M] \times y \rangle \\ &= \langle \tilde{u}, ([M] \times y) \frown (1 \times a) \rangle = (-1)^{n|a|} \langle \tilde{u}, ([M] \frown 1) \times (y \frown a) \rangle \\ &= (-1)^{n|a|} \langle \tilde{u}, [M] \times [p] \rangle \langle a, y \rangle = (-1)^{n(n-|a|)} \langle a, y \rangle \end{aligned}$$

□

Theorem 5.15. PD *is an isomorphism.*

Proof. For $0 \neq a \in H^k(M)$, choose $y \in H_k(M)$ with $\langle a, y \rangle \neq 0$. Then the preceding proposition shows that $\mathrm{PD}(a) \times y \neq 0$, in particular $\mathrm{PD}(a) \neq 0$. Consequently PD is injective. But we also hvae $\dim H^*(M) = \dim H_*(M)$, so it must be an isomorphism. □

Corollary 5.16. $(,)$ *is nondegenerate.*

Suppose $\{a_i\}_i$ is a basis for $H^*(M)$. Let $\{b_i\}_i$ be the dual basis with respect to $(,)$. Then $\langle b_j, \mathrm{PD}(a_i) \rangle = (a_i, b_j) = \delta_{ij}$, so $\mathrm{PD}(a_i) = b_i^*$ where $\{b_i^*\}_i$ is the dual basis of $\{b_i\}_i$ with respect to $\langle , \rangle$. Similarly, $\langle a_i, \mathrm{PD}(b_j) \rangle = (b_j, a_i) = (-1)^{|a_i||b_j|} \delta_{ij}$, so $\mathrm{PD}(b_j) = (-1)^{|a_i||b_j|} a_i^*$.

Corollary 5.17. $\tilde{u} = \sum_i (-1)^i a_i \times b_i$.

Proof.

$$\begin{aligned} \langle \tilde{u}, a_i^* \times b_j^* \rangle &= (-1)^{|a_i|(n-|a_i|)} \langle \tilde{u}, \mathrm{PD}(b_i) \times \mathrm{PD}(a_j) \rangle \\ &= (-1)^{|a_i|(n-|a_i|) + n|a_i|} \langle b_i, \mathrm{PD}(a_j) \rangle \\ &= (-1)^{|a_i|} (a_j, b_i) = (-1)^{|a_i|} \delta_{ij} \end{aligned}$$

□

5.4 Intersection Pairing on Homology

Definition 5.12. Suppose $N_1, N_2 \hookrightarrow M$ are smooth submanifolds. We say $N_1 \pitchfork N_2$ ("N_1 is transverse to N_2") if $TN_1|_x + TN_2|_x = TM|_x$ for all $x \in N_1 \cap N_2$.

Proposition 5.18. *Suppose* $N_1 \pitchfork N_2$*, then:*
(i) $N_1 \cap N_2$ *is a smooth submanifold of dimension* $\dim N_1 + \dim N_2 - \dim M$.
(ii) $(T(N_1 \cap N_2))|_x = TN_1|_x \cap TN_2|_x$.
(iii) $\nu_{M/(N_1 \cap N_2)} = \nu_{M/N_1} \oplus \nu_{M/N_2}$.
(iv) $\mathrm{pd}(N_1 \cap N_2) = \mathrm{pd}(N_1) \smile \mathrm{pd}(N_2)$.

Definition 5.13. For smooth submanifolds N_1, N_2 of M, their intersection pairing is $[N_1] \cdot [N_2] = (\mathrm{pd}(N_1), \mathrm{pd}(N_2)) = \langle \mathrm{pd}(N_1) \smile \mathrm{pd}(N_2), [M] \rangle$.

When $N_1 \pitchfork N_2$, we have $[N_1] \cdot [N_2] = \langle \mathrm{pd}(N_1 \cap N_2), [M] \rangle$, which equals the number of points in $N_1 \cap N_2$, counted with intersection sign, if $\dim(N_1 \cap N_2) = 0$, and zero otherwise.
Let $j : N_1 \hookrightarrow M$ be the inclusion and $i = j|_{N_1 \cap N_2} : N_1 \cap N_2 \hookrightarrow N_2$.

Proposition 5.19. $j^*(\mathrm{pd}(N_2)) = \mathrm{pd}_{N_1}(N_1 \cap N_2)$.

Proof. $\nu_{N_1/(N_1 \cap N_2)} \cong i^* \nu_{M/N_2}$, so $u_{N_1/(N_1 \cap N_2)} = j^* u_{M/N_2}$. □

Proposition 5.20. *Suppose $\pi : E \to M$ is an oriented vector bundle and $s : M \to E$ is a section and $s \pitchfork s_0$, then $e(E) = \mathrm{pd}_M(s \cap s_0) = \mathrm{pd}_M(s^{-1}(0))$.*

Proof. $i_*^{-1} u_E = \mathrm{pd}_E(s_0) = \mathrm{pd}_E(s)$ since $s \sim s_0$. So $e(E) = s_0^*(i_*^{-1} u_E) = s_0^*(\mathrm{pd}_E(s)) = \mathrm{pd}_M(s_0 \cap s)$. □

Corollary 5.21. $\langle e(TM), [M] \rangle = \chi(M)$.

Proof. In $M \times M$, we have $\Delta^* \nu_{M \times M/\Delta} \cong TM$, so $\langle e(TM), [M] \rangle = [\Delta] \cdot [\Delta] = (\tilde{u}, \tilde{u}) = \chi(M)$. □