Algebraic Geometry *

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This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part III course *Algebraic Geometry* in Michaelmas 2022. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

Contents

*Based on the lectures under the same name taught by Dr. D. Ranganathan in Michaelmas 2022.

0 Introduction and Motivations

There are roughly four parts of the course: We'll first study the Zariski spectrum of a commutative ring, which constitutes the local models of schemes, our main objects of interest. While doing this, we'll also build the basics of sheaf theory. Next, we motivate and define schemes and morphisms between them. We'll then study properties of schemes and morphisms, which replace certain naïve topological notions like compactness, Hausdorffness, etc., and go through some basic constructions like Picard groups. The last part of the course will be a rather rapid introduction to sheaf cohomology.
The prerequisites of this course are basic undergraduate mathematics: groups, rings, modules, topological spaces, etc.. The main thing is commutative algebra, which happens to not be part of undergraduate mathematics (shocking!).
Why study schemes? Let's motivate this by discussing what's known as the Weil conjectures. We start with a homogeneous polynomial $F \in \mathbb{Z}[X_0, \dots, X_n]$. One can solve F in the complex projective space, which gives a projective hypersurface $X = \mathbb{V}(F) \subset \mathbb{P}^n_{\mathbb{C}}$. Let's assume that X is smooth, which in this case means that not all partial derivatives of F vanish at any point on X. X can of course be interpreted both as an algebraic variety or as a complex manifold. In the latter case, we have access to the standard tools in algebraic topology. In particular, we can compute its singular homology, and hence its topological Euler characteristic $\chi(X)$, its Betti numbers $b_i(X)$, and so on. For example, when $X = \mathbb{P}^1$, we have $\chi(X) = 2$. In general, $\chi(\mathbb{P}^k) = k + 1$.
On the other hand, we can fix a prime number p and solve F over $\bar{\mathbb{F}}_p$. Suppose p is chosen such that this is again smooth (easy exercise: show that such p exists). We can then consider N_m, the number of solutions to $F = 0$ over $\mathbb{F}_{p^m}$, and turn our attention to the generating function (the Hasse-Weil ζ function)

$$\zeta_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m T^m\right)$$

Theorem 0.1 (Grothendieck, part of Weil's conjectures). *ζ_X is a rational function*

$$\zeta_X(T) = \frac{P_0(T) \cdots P_{2n-2}(T)}{P_1(T) \cdots P_{2n-3}(T)}$$

for some polynomials P_i. Furthermore, $\deg P_i = b_i(X)$.

The theory of algebraic varieties does not permit one to make a connection as grand as this. Grothendieck had invented the theory of l-adic cohomology to prove this theorem, which rests on the theory of schemes.

1 Beyond Algebraic Varieties

1.1 Summary of Affine Varieties

The classical theory of algebraic varieties starts with an algebraically closed field k. One initially wants to study subsets of $\mathbb{A}_k^n$ of the form $\mathbb{V}(S)$ ("affine varieties"), the vanishing locus of a set S of polynomials in n variables. We easily have $\mathbb{V}(S) = \mathbb{V}((S)) = \mathbb{V}(\sqrt{(S)})$, where $\sqrt{}$ denotes the operation of taking the radical of an ideal.
One of the basic results about these is that the theory of affine varieties (up to isomorphism, to remove the data associated with the embedding) is exactly the same as the theory of finitely generated, nilpotent-free, k-algebras: Given an affine variety V, we get such a k-algebra by taking the coordinate ring $k[V] = k[X_1, \dots, X_n]/I$, where $I = \{f \in k[X_1, \dots, X_n] : f|_V = 0\}$ is the ideal of V (which is automatically radical). Conversely, given a finitely generated, nilpotent-free, k-algebra A, we can find (by finiteness) a surjective map $\phi : k[X_1, \dots, X_n] \to A$. We then get an affine variety $\mathbb{V}(\ker \phi) \subset \mathbb{A}_k^n$.
On an affine variety $V \subset \mathbb{A}_k^n$, it also has a topology whose closed sets are given by sets of the form $V \cap \mathbb{V}(S)$ for some $S \subset k[X_1, \dots, X_n]$. This is also very extrinsic, but we can make it intrinsic: We can just define $W \subset V$ to be closed iff $W = \mathbb{V}(J)$ for some $J \in k[V]$ (by viewing elements of $k[V]$ as functions $V \to k$), which clearly coincides with the previous definition.
One of the important results, that we should've mentioned already but for some reason didn't, is Hilbert's Nullstellensatz. It gives more insight into how $k[V]$ encodes geometric information about V. For $V = \mathbb{V}(I)$ and $p \in V$, there is a natural ring homomorphism $\mathrm{ev}_p : k[V] \to k, f \mapsto f(p)$. This gives us a maximal ideal $\mathfrak{m}_p \subset k[V]$. So points on V gives rise to maximal ideals in $k[V]$.

Theorem 1.1 (Hilbert's Nullstellensatz)**.** *There is a bijection between points in V and maximal ideals in $k[V]$.*

The identification of $f \in k[V]$ as a function $V \to k$ can be then packaged into the quotient map $k[V] \to k[V]/\mathfrak{m}_p$.
Given $V \subset \mathbb{A}_k^n$ and $W \subset \mathbb{A}_k^m$, recall that a morphism $V \to W$ is given by a tuple $\phi = (f_1, \dots, f_m)$ with $f_i \in k[V]$ such that $\phi(V) \subset W$. This is very extrinsic, but we can make it intrinsic as follows: Any such ϕ gives rise to a pullback map $\phi^* : k[W] \to k[V]$ via $g \mapsto g \circ \phi$. And it turns out that this is a bijection between the morphisms $V \to W$ and the k-algebra homomorphisms $k[W] \to k[V]$. So what we want as an "abstract" affine variety is exactly a finitely generated, nilpotent-free, k-algebra.
If one wants something topological, we take such a k-algebra R and consider the set X_R of maximal ideals in R. On this set, we can define a topology with closed sets given by $\mathbb{V}(S), S \subset R$ where evaluation of a function at $\mathfrak{m} \in X_R$ is given by the quotient map $R \to R/\mathfrak{m}$. This gives a topological space homeomorphic to the Zariski topology on any affine variety R corresponds to.
So affine varieties are nice this way. How about varieties in general? We certainly

want them to be things that "locally" looks like affine varieties, but what does it mean?
Another problem is how one might deal with varieties over non-algebraically close fields, but things like $\varnothing = \mathbb{V}(x^2+y^2+1) \subset \mathbb{R}^2$ demonstrates the failure of Hilbert's Nullstellensatz for these fields, which means that our classical theory is very insufficient.
The last problem is that, well, "finitely generated, nilpotent-free, k-algebra" is a very long name. Could there be a geometric picture X_R for general rings R? Part of the issue, which arises even in the study of affine varieties over $k = \bar{k}$, is the adjective "nilpotent-free". Let's look at how $C = \mathbb{V}(y - x^2)$ and $D = \mathbb{V}(y)$ intersect in $\mathbb{A}^2$. On one hand, it's just a point. Why is this a problem? If we consider the intersection of C with $D_\delta = \mathbb{V}(y+\delta)$ for $\delta \neq 0$, then we always see 2 points instead of 1. So $C \cap D$ should morally be two points squashed together, instead of just one single point, and we should have some ways to extract this information. Indeed, if we solve the equations, we get $(y, y - x^2) = (x^2)$. So in some sense, we "forgot" that this is supposed to be a "tangent point of multiplicity 2" by taking radical ideals.

1.2 The Spectrum of a Ring

The idea is this: Given an arbitrary ring A, we'll define a geometric object $\operatorname{Spec} A$, whose underlying set is the set of prime ideals of A. Why primes instead of maximal ideals? Basically just because the preimage of maximal ideals needs not be maximal, but preimages of primes are always primes (so $B \to A$ induces a map $\operatorname{Spec} A \to \operatorname{Spec} B$), and because many interesting rings have too few maximal ideals.
Let A be a commutative ring with identity. The first goal is to define a topological space $\operatorname{Spec} A$ associated with A on which A has a natural interpretation as the "ring of functions".

Definition 1.1. The (Zariski) spectrum $\operatorname{Spec} A$ of A, as a set, is the collection of prime ideals in A.

Intuition are often provided by considering the subset $\operatorname{mSpec} A$ of $\operatorname{Spec} A$ consisting of all maximal ideals, but as we've said this fails to be functorial. For spectrum defined with prime ideals, a ring homomorphism $\phi : B \to A$ gives rise to a map $\operatorname{Spec} A \to \operatorname{Spec} B, \mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$.
Another reason for this definition is the following: We want our geometric object X_A associated to A to have the property that every element in A should have a "natural" evaluation homomorphism ev_p at every point p in X, with values in fields. We don't need the field to be independent of the point because we only care about vanishing anyways, and we don't need the evaluation map to be surjective. But the kernel of every ring homomorphism to a field is a prime ideal! The sensible thing to do is then to take ev_p as the quotient map of a prime ideal (which goes into the fraction field of the quotient).
And indeed this is how it works: For $x \in A$, its "evaluation" at a prime ideal $\mathfrak{p}$ is the image of x in $A \to A/\mathfrak{p} \hookrightarrow \mathrm{FF}(A/\mathfrak{p})$.

Example 1.1. 1. For $A = \mathbb{Z}$, $\operatorname{Spec} \mathbb{Z}$ consists of (p) for p prime as well as (0). Say we take a "function" $123 \in \mathbb{Z}$, "evaluating" it at (2) gives me an element of $\mathbb{Z}/(2)$, namely $1 \bmod 2$, and "evaluating" it at (3) is $0 \pmod 3 \in \mathbb{Z}/(3)$.

Evaluating it at (0), on the other hand, gives me $123 \in \mathbb{Z} \hookrightarrow \mathbb{Q}$.
2. For $A = \mathbb{R}[X]$, $\operatorname{Spec} A$ consists of (0), $(X - a), a \in \mathbb{R}$ and ideals principally generated by irreducible quadratics. So $\operatorname{Spec} A$ can be viewed as $\mathbb{C}$ modulo complex conjugation (and evaluation works exactly that way!), together with the point (0).

When A is an integral domain, (0) is an element of $\operatorname{Spec} A$, which is called the generic point of $\operatorname{Spec} A$.
Let's now put a topology on $\operatorname{Spec} A$. In the classical theory, the closed sets are exactly the vanishing loci of functions. Let's do the same thing.

Definition 1.2. For $f \in A$, the vanishing loci of f is $\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \in \mathfrak{p}\}$. More generally, for a subset $S \subset A$, the vanishing loci of S is $\mathbb{V}(S) = \mathbb{V}((S)) = \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$

Proposition 1.2. *Sets of the form $\mathbb{V}(I)$ for I ideals of A satisfy the axioms for closed sets in a topological space.*

This gives rise to a topology (called the Zariski topology) on $\operatorname{Spec} A$.

Proof. $\varnothing, \operatorname{Spec} A$ are clearly of this form.
Suppsoe we have a collection of ideals $(I_\alpha)_{\alpha \in J}$, then

$$\bigcap_{\alpha \in J} \mathbb{V}(I_\alpha) = \mathbb{V}\left(\sum_{\alpha \in J} I_\alpha\right)$$

where $\sum_{\alpha \in J} I_\alpha$ is the ideal consisting of finite sums of elements in $I_\alpha, \alpha \in J$.
Lastly, for two ideals I, J, we have $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$. Indeed, we clearly have $\mathbb{V}(I) \cup \mathbb{V}(J) \subset \mathbb{V}(I \cap J)$. On the other hand, we have $IJ \subset I \cap J$ and if a prime ideal contains IJ then it must contain one of I or J. This gives the reverse containment. □

Example 1.2. 1. Let $k = \bar{k}$ and $A = k[X, Y]$. We have $\operatorname{Spec} A \supset \operatorname{mSpec} A = \{(X - a, Y - b) : a, b \in k\} = k^2$. The other points in $\operatorname{Spec} A$ are (0) are principal ideals corresponding to irreducible polynomials in A.
What is the closure $\overline{(0)}$ of (0)? Everything! That's why we call it the generic point. On the other hand, the closure of a maximal ideal is itself, whereas the closure of (f) for a nonzero irreducible $f \in A$ consists of $(f), (0)$ as well as the maximal ideals $(x - a, y - b)$ with $f(a, b) = 0$.
2. Consider the ring $A = \mathbb{C}[[X]]$ of formal power series with complex coefficients. Then $\operatorname{Spec} A$ is just $\{(0), (X)\}$, with (X) closed and with $\overline{(0)} = \operatorname{Spec} A$. On the other hand $\operatorname{mSpec} A = \{(X)\}$. Unlike the previous example, there is not really a way to recover $\operatorname{Spec} A$ from $\operatorname{mSpec} A$, as topological spaces.

We certainly need a little bit more than just a topological space, namely a reasonable function theory on it. We already think of A as the functions on $\operatorname{Spec} A$, but how about functions on its open sets?

1.3 Function Theory on Some Open Sets

Definition 1.3. For $f \in A$, the distinguished open corresponding to f is the open set $U_f = D(f) = (\operatorname{Spec} A) \setminus \mathbb{V}(f)$.

Lemma 1.3. *The distinguished opens form a base for the Zariski topology on* $\operatorname{Spec} A$.

Proof. Example sheet. □

Recall that for $f \in A$, the localisation of A at f is the ring $A_f = A[T]/(Tf - 1)$. There is an obvious map $A \to A_f$ (the "localisation map"), which may not be injective when f is a zerodivisor.

Lemma 1.4. *The subspace* $U_f \subset \operatorname{Spec} A$ *is naturally homeomorphic to* $\operatorname{Spec} A_f$.

Proof. The homeomorphism $\operatorname{Spec} A_f \to U_f$ is induced by the natural map $A \to A_f$. Indeed, the map gives a one-to-one correspondence between primes in A_f and primes in A which miss f. □

Example 1.3. 1. Take $A = \mathbb{C}[X]$ and $f = X$, then $A_f = \mathbb{C}[X, X^{-1}]$.
2. Take $A = \mathbb{C}[X, Y]$ and $f = XY$, then the closed points of U_f are exactly $\mathbb{C}^2$ with axes removed. More algebraically, $A_f = \mathbb{C}[X, Y][T]/(TXY - 1) \cong \mathbb{C}[X, X^{-1}, Y, Y^{-1}]$ and the natural map $A \to A_f$ is injective. If a prime ideal $\mathfrak{p}$ in $\mathbb{C}[X, Y]$ contains x or y, then the ideal generated by the image of $\mathfrak{p}$ is the unit ideal.

Consequently, the "correct" ring of functions on U_f should be A_f. One easily checks that A_f depends only on the open set U_f, but not on the choice of $f \in A$. Observe also that if $U_{f_1} \subset U_{f_2}$, then there is a natural homomorphism $A_{f_2} \to A_{f_1}$.
So we've associated a ring to each distinguished open set. To package these up, let's introduce the language of sheaves.

2 Sheaves

2.1 Presheaves and Sheaves

Fix a topological space X.

Definition 2.1. A presheaf $\mathcal{F}$ of abelian groups on X is the assignment of an abelian group $\mathcal{F}(U)$ to each open $U \subset X$, equipped with homomorphisms $\operatorname{res}^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ ("restriction maps") for each $U \subset V$ such that $\operatorname{res}^U_U = \operatorname{id}_{\mathcal{F}(U)}$ and for $U \subset V \subset W$ open, $\operatorname{res}^V_U \circ \operatorname{res}^W_V = \operatorname{res}^W_U$.

For $s \in \mathcal{F}(V)$, we sometimes write $s|_U$ to denote $\operatorname{res}^V_U(s)$.

Example 2.1. The rule which assigns open $U \subset X$ to the abelian group of continuous real-valued functions on U gives a presheaf with the obvious restriction maps.

Definition 2.2. A morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X is the data of morphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subset X$, compatible with restriction in the sense that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ {\scriptstyle \operatorname{res}^V_U}\downarrow & & \downarrow{\scriptstyle \operatorname{res}^V_U} \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

where the restriction maps are understood to be part of the corresponding presheaves.

Definition 2.3. A sheaf $\mathcal{F}$ on X is a presheaf on X satisfying the following conditions ("sheaf axioms"):
(S1) Suppose $U \subset X$ is open and $\{U_\alpha : \alpha \in A\}$ form an open cover of U. If $s \in \mathcal{F}(U)$ has $s|_{U_\alpha} = 0$ for all $\alpha \in A$, then $s = 0$.
(S2) Suppose $U \subset X$ is open and $\{U_\alpha : \alpha \in A\}$ form an open cover of U. If we have $s_\alpha \in \mathcal{F}(U_\alpha)$ such that for every $\alpha, \beta \in A$, $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there is some $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$.

Definition 2.4. A morphism of sheaves is just a morphism of the underlying presheaves.

Example 2.2. 1. The presheaf in Example 2.1 is a sheaf.
2. (non-example) Consider $X = \mathbb{C}$ with the Euclidean topology. We set $\mathcal{F}(U) = \{f : U \to \mathbb{C} \text{ bounded}\}$ with the obvious restriction maps. This is a presheaf, but this violates (S2), since gluing together bounded functions together can give unbounded functions.
3. (non-example) Fix an abelian group G and set $F(U) = G$ for all U. This is a presheaf with the restriction maps being the identity on G. This again isn't a sheaf since if there are disjoint open sets U_1, U_2 in X and $|G| > 1$, the sheaf axioms would force $F(U_1 \sqcup U_2)$ to be $G \times G$ which restrict to $\mathcal{F}(U_1)$ by projection.
4. Fix an abelian group G. The constant sheaf $\underline{G}$ with values in G is the sheaf $\mathcal{F}$ that assigns to an open set U the ring of all locally constant functions $U \to G$. And guess what, the restriction is again the one you're thinking about. In practice, if U is connected, then $\mathcal{F}(U) = G$. In general, $\mathcal{F}(U) = G^{\#\text{connected components of } X}$.
In particular, if we take $G = \mathbb{Z}$, then $\underline{\mathbb{Z}}(X) = \mathbb{Z}^{\#\text{connected components of } X} = H^0(X, \mathbb{Z})$, which is actually not a coincidence.
5. Let V be an irreducible (classical) affine variety over $k = \bar{k}$. We define a sheaf $\mathcal{O}_V$ on V as $\mathcal{O}_V(U) = \{f \in k(V) = \mathrm{FF}(k[V]) : \forall p \in U, f \text{ is regular at } p\}$. This is called the structure sheaf of V.

2.2 Basic Constructions

Fix a presheaf $\mathcal{F}$ on a topological space X.

Definition 2.5. A section of $\mathcal{F}$ over $U \subset X$ is an element of $\mathcal{F}(U)$. Elements of $\mathcal{F}(X)$ are known as global sections.

For a sheaf $\mathcal{F}$ on a space X and $U \subset X$ open, we also write $\Gamma(U, \mathcal{F})$ or $H^0(U, \mathcal{F})$ to denote $\mathcal{F}(U)$.

Definition 2.6. For $p \in X$, the stalk of $\mathcal{F}$ at p is the abelian group

$$\mathcal{F}_p = \{(s, U) : p \in U \subset X, s \in \mathcal{F}(U)\} / \sim$$

where $(s, U) \sim (s', U')$ iff there is an open neighbourhood $W \subset U \cap U'$ of p such that $s|_W = s'|_W$. The abelian group structure on $\mathcal{F}_p$ is given by $[(s, U)] + [(t, V)] = [(s|_{U \cap V} + t|_{U \cap V}, U \cap V)]$.
Elements of the stalk at p are called the germs at p.

It is easy to check that this is well-defined. Moreover, for every open $U \subset X$ and $p \in U$, we have a natural map $\mathcal{F}(U) \to \mathcal{F}_p, s \mapsto s_p = [(s, U)]$.

Example 2.3. Consider the structure sheaf $\mathcal{O}_V$ of the (classical) affine variety $V = \mathbb{A}^1_k, k = \bar{k}$. Let's calculate its stalk at 0. The neighbourhoods of 0 in $\mathbb{A}^1_k$ are $U_S = \mathbb{A}^1_k \setminus S$ where $S \subset \mathbb{A}^1 \setminus \{0\}$ finite. We have $\mathcal{O}_{\mathbb{A}^1_k}(U_S) = k[X][1/(X-x) : x \in S]$. And two of these sections are equivalent when and exactly when they are the same in the fraction field $k(X)$. So $\mathcal{F}_0 = \{f(X)/g(X) : g(0) \neq 0\}$.

Let $\mathcal{F}, \mathcal{G}$ be sheaves on X and $f : \mathcal{F} \to \mathcal{G}$ be a morphism. For every $p \in X$, we have an induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p, [(s, U)] \mapsto [(f(U)(s), U)]$.

Proposition 2.1. *f is an isomorphism if and only if f_p is an isomorphism for all $p \in X$.*

Proof. We'll show that $f(U)$ is an isomorphism for every open $U \subset X$. The sheaf morphism given by $g(U) = f(U)^{-1}$ would then give an inverse to f.
We first show injectivity. Suppose $s \in \ker f(U)$, then $s_p = 0$ for all $p \in U$ as $f_p(s_p) = 0$. So for every $p \in U$, there is some open $U_p \subset U$ containing p such that $s|_{U_p} = 0$, i.e. $s = 0$ by (S1).
As for surjectivity, suppose $t \in \mathcal{G}(U)$ and for every $p \in U$ we let $s_p = f_p^{-1}(t_p)$. Let (s_{U_p}, U_p) a representative of s_p. These glue (in the manner of (S2)) by the injectivity part, which gives some $s \in \mathcal{F}(U)$ with $f(U)(s) = t$. □

Remark. 1. The induced map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p, s \mapsto (s_p)_{p \in U}$ is injective.
2. Given $\phi, \psi : \mathcal{F} \to \mathcal{G}$. If $\phi_p = \psi_p$ for all $p \in X$, then $\phi = \psi$.

Definition 2.7 (Sheafification)**.** Let $\mathcal{F}$ be a presheaf on X. A morphism (the "sheafification") $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ is a sheafification if $\mathcal{F}^{\mathrm{sh}}$ is a sheaf and for any morphism $\phi : \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf, there is a unique morphism $\phi^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$ with $\phi = \phi^{\mathrm{sh}} \circ \mathrm{sh}$.

Remark. 1. Since we've defined it using a universal property, once we've shown the existence of sh, it is automatically unique up to unique isomorphism (exercise).
2. Sheafification is also functorial: Each morphism $\mathcal{F} \to \mathcal{G}$ of presheaves induces a canonical morphism $\mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$ commuting with sheafification.

Proposition 2.2. *Sheafification always exists.*

Proof. Let $\mathcal{F}$ be a presheaf on X. Define $\mathcal{F}^{\mathrm{sh}}(U)$ to be the set of tuples $(f_p)_{p \in U}, f_p \in \mathcal{F}_p$ such that for any p, there is an open $V_p \subset U$ containing p and a section $f_{V_p} \in \mathcal{F}(V_p)$ such that $s_q = f_q \in \mathcal{F}_q$ for any $q \in V_p$. Each $\mathcal{F}^{\mathrm{sh}}(U)$ inherits the structure of an abelian group from $\prod_{p \in U} \mathcal{F}_p$, and the restriction maps are clear. It also follows from construction that $\mathcal{F}$ is a sheaf.
We have a morphism $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}, \mathrm{sh}(U)(s) = (s_p)_{p \in U}$, which we claim to satisfy the universal property of sheafification. Indeed, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism with $\mathcal{G}$ a sheaf, then the only possible choice for ϕ^{sh} is by setting $\phi^{\mathrm{sh}}(U)((f_p)_{p \in U})$ as the section of $\mathcal{G}(U)$ obtained from gluing together $\phi(V_p)(f_{V_p}), p \in U$. □

Corollary 2.3. *If $\mathcal{F}$ is a presheaf on X, then $\mathrm{sh}_p : \mathcal{F}_p \to \mathcal{F}^{\mathrm{sh}}_p$ is an isomorphism for all $p \in X$.*

Remark. One can find a nonzero presheaf with zero sheafification (exercise).

2.3 Kernels, Image, and Cokernels

Definition 2.8. Let $\phi : \mathcal{F} \to \mathcal{G}$ be presheaves on X. The presheaf kernel (resp. presheaf image, presheaf cokernel) is the presheaf which assigns an open $U \subset X$ to the kernel (resp. image, cokernel) of $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ with the obvious restriction maps.

Remark. If ϕ is in fact a morphism of sheaves, then its presheaf kernel is a sheaf. However, the presheaf image and presheaf cokernel need not be sheaves.

Example 2.4. Let $X = \mathbb{C}$ equipped with Euclidean topology and $\mathcal{O}_X$ the sheaf whose sections on $U \subset X$ are holomorphic functions $U \to \mathbb{C}$ under addition. We have another sheaf $\mathcal{O}_X^\times$ whose sections on $U \subset X$ are nonvanishing holomorphic functions $U \to \mathbb{C}$ under multiplication.
We have a morphism $\mathcal{O}_X \to \mathcal{O}_X^\times$ by sending a section $f \in \mathcal{O}_X(U)$ to $\exp \circ f \in \mathcal{O}_X^\times(U)$. This is not surjective on $U_0 = \mathbb{C}^\times$ due to the nonexistence of global logarithm, but it is surjective on $U_+ = \mathbb{C} \setminus [0, \infty)$ and $U_- = \mathbb{C} \setminus (-\infty, 0]$ due to the existence of a branch of logarithm there. For example, z is in the presheaf image over $U_\pm$ but not in the presheaf image over U_0, which then means that the image presheaf fails (S2), i.e. is not a sheaf.
On the other hand, the kernel of this map is simply the constant sheaf $\underline{2\pi i\mathbb{Z}}$, which is a sheaf.

Definition 2.9. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X, the sheaf image $\operatorname{Im} \phi$ (resp. sheaf cokernel $\operatorname{coker} \phi$) of ϕ is the sheafification of the presheaf image (resp. presheaf cokernel).
We say that ϕ is surjective (resp. injective) if $\operatorname{Im} \phi = \mathcal{G}$ (resp. $\ker \phi = 0$).

Example 2.5. For $X = \mathbb{C}$, there is an exact sequence

$$0 \longrightarrow \underline{2\pi i\mathbb{Z}} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

which is called the exponential exact sequence.

Remark. 1. We say $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if there are inclusion maps $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ for each open U, compatible with restriction.
2. Given a subsheaf $\mathcal{F} \hookrightarrow \mathcal{G}$, the quotient of $\mathcal{G}$ by $\mathcal{F}$ is the sheaf cokernel of the inclusion map, i.e. the sheafification of $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.
3. If $\mathcal{F} \to \mathcal{G}$ is surjective, then on any particular U we may not have the surjectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$. But it will be surjective "locally" (see example sheet, where one shows that exactness can be checked at the level of stalks).

2.4 Moving between Spaces

Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

Definition 2.10 (Pushforward)**.** The pushforward $f_*\mathcal{F}$ of $\mathcal{F}$ along f is the presheaf on Y given by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$.

Proposition 2.4. *$f_*\mathcal{F}$ is a sheaf.*

Proof. Trivial (ask Dhruv if you don't believe it). □

Definition 2.11. The inverse image presheaf $(f^{-1}\mathcal{G})^{\text{pre}}$ of $\mathcal{G}$ along f is the presheaf on X given by

$$(f^{-1}\mathcal{G})^{\text{pre}}(V) = \varinjlim_{U\supset f(V)} \mathcal{G}(U) = \{(s_U, U) : f(V) \subset U \subset Y, s_U \in \mathcal{G}(U)\}/\sim$$

where $(s_U, U) \sim (s_{U'}, U')$ iff there is some open U'' such that $f(V) \subset U'' \subset U \cap U'$ and $s_U|_{U''} = s_{U'}|_{U''}$.
The inverse image sheaf $f^{-1}\mathcal{G}$ is the sheafification of $(f^{-1}\mathcal{G})^{\text{pre}}$.

Remark. Even when f is an open map, the inverse image presheaf may not be a sheaf. Indeed, if we take Y to be a nontrivial topological space and $X = Y \sqcup Y$ and f is $f : X \to Y$ is the identity map on each factor, which is clearly open. Take any nonzero sheaf $\mathcal{G}$ on Y. $\mathcal{F} = (f^{-1}\mathcal{G})^{\text{pre}}$ has value $\mathcal{G}(V)$ at $U = f^{-1}V$, yet $U = V \sqcup V$ so $\mathcal{F}^{\text{sh}}(U) \cong \mathcal{G}(V) \times \mathcal{G}(V)$ by the sheaf axioms.

3 Schemes

3.1 The Spectrum of a Ring, but for real

Let A be a ring. Recall that we have a topological space $\operatorname{Spec} A$.

Definition 3.1. A subset $S \subset A$ is multiplicative if $1 \in S$ and $a \in S, b \in S \implies ab \in S$. The localisation of A at S is the ring $S^{-1}A = \{((a, s) : a \in A, s \in S)\}/\sim$ where $(a, s) \sim (a', s')$ iff $s''(as' - a's)$ for some $s'' \in S$. The multiplication is $(a, s)(a', s') = (aa', ss')$ and the addition is $(a, s)(a', s') = (as' + a's, ss')$.

Example 3.1. 1. For $S = \{1, f, f^2, \ldots\}$ for $f \in A$, then $S^{-1}A = A_f = A[T]/(Tf - 1)$.
2. For a prime ideal $\mathfrak{p}$ of A, we can take $S = A \setminus \mathfrak{p}$ which is multiplicative. The localisation at S is called $A_{\mathfrak{p}} = S^{-1}A$.

Remark. When (f) is a prime ideal, $A_{(f)}$ and A_f very much do not mean the same thing. Don't ask me who invented this amazing notaton.

Example 3.2. 1. If we take $A = \mathbb{C}[X]$, and $\mathfrak{p} = (X)$, then $A_{\mathfrak{p}}$ consists of rational functions that are well-defined at 0.
2. If we take $A = \mathbb{Z}$ and p a prime number, then $A_{(p)}$ consists of fractions in $\mathbb{Q}$ with denominator not divisible by p.

Let's now define a sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} A$ with the following properties: If U_f is a distinguished open, then $\mathcal{O}_X(U_f) = A_f$ (in particular $\mathcal{O}_X(X) = A$). Moreover, if $\mathfrak{p} \leq A$ is a prime ideal, then the stalk at $\mathfrak{p}$ is $A_{\mathfrak{p}}$.
We basically know what $\mathcal{O}_X$ is, but for the sake of completeness we still have to construct it and show that it's a sheaf. So get ready for definitions.
Suppose we have a topological space X, a sheaf $\mathcal{F}$ of rings (i.e. a sheaf of abelian group except the restriction maps are ring homomorphisms) on X and a basis $\mathcal{B} = \{B_\alpha\}_{\alpha\in A}$. $\mathcal{F}$ allows the assignment to each $B \in \mathcal{B}$ a ring $\mathcal{F}(B)$, and to each $B \subset C, B, C \in \mathcal{B}$ a ring homomorphism $\text{res}^C_B : \mathcal{F}(C) \to \mathcal{F}(B)$. Can we recover $\mathcal{F}$ from this identification?

Definition 3.2. A presheaf on a base $\mathcal{B}$ is the data of an assignment of abelian groups (rings, etc.) $B \mapsto F(B)$ to each $B \in \mathcal{B}$, along with restriction maps res^C_B

for every $B \subset C, B, C \in \mathcal{B}$.
A sheaf on a base $\mathcal{B}$ is a presheaf on $\mathcal{B}$ satisfying:
(SB1) If $B = \bigcup_i B_i$ in $\mathcal{B}$ and $f \in F(B)$ has $\text{res}^B_{B_i}(f) = 0$ for all i, then $f = 0$.
(SB2) If $B = \bigcup_i B_i$ in $\mathcal{B}$ and $f_i \in F(B_i)$ are such that for every basis element $C \subset B_i \cap B_j$ we have $\text{res}^{B_i}_C(f_i) = \text{res}^{B_j}_C(f_j)$, then there is some $f \in B$ such that $\text{res}^B_{B_i}(f) = f_i$.

As always, we write $\text{res}^C_B(f) = f|_B$.

Proposition 3.1. *If F is a sheaf on a base $\mathcal{B}$, then there is a unique sheaf $\mathcal{F}$ on X extending it, i.e. with $\mathcal{F}(B) = F(B)$ for every $B \in \mathcal{B}$ and the restriction maps between sections on basic opens are the same.*

Proof. Idk, example sheet? □

Recall that a basis for $\text{Spec}\, A$ is given by $U_f = \{\mathfrak{p} \in \text{Spec}\, A : f \notin \mathfrak{p}\} = \text{Spec}\, A \setminus \mathbb{V}(f)$, which is homeomorphic to $\text{Spec}\, A_f$. This is quite a nice basis, since $U_f \cap U_g = U_{fg}$.
On the other hand, f, f^N, fu for a unit u all give the same distinguished open. But if $U_f = U_g$, then $U_f = U_{fg}$ and the canonical (localisation) maps $A_f \to A_{fg}, A_g \to A_{fg}$ are both isomorphisms. So although U_f doesn't let us recover f, it does let us recover A_f up to canonical isomorphisms.

Proposition 3.2. *On $\text{Spec}\, A$, the assignment $U_f \mapsto A_f$ is a sheaf on the base $\{U_f : f \in A\}$ of distinguished opens, with restriction maps given by localisation.*

Proof. Since U_f is homeomorphic to $\text{Spec}\, A_f$ (and the restriction of the proposed sheaves on a basis coincides), we can replace A by A_f and hence assume WLOG that we are in the situation of an distinguished open cover $\mathcal{U} = \{U_{f_i} : i \in I\}$ of $X = \text{Spec}\, A$.
Note that $\text{Spec}\, A = \bigcup_{i \in I} U_{f_i}$ implies $\sum_i f_i A = A$, which means that there is a finite subset $J \subset I$ such that $\sum_{j \in J} f_j a_j = 1$ for some $a_j \in A$ (so in fact $\text{Spec}\, A = \bigcup_{j \in J} U_{f_j}$).
Let's first check (SB1). Suppose $s \in A$ is such that $s|_{U_{f_i}} = 0$ for all $i \in I$. By the discussion above, we can assume WLOG that I is finite. Then $f_i^{n_i} s = 0$ for some $n_i \geq 0$. Replace each f_i by $f_i^{n_i}$ (noting $U_{f_i} = U_{f_i^{n_i}}$) reduces the system to $f_i s = 0$. But since U_{f_i} covers X, there are some $a_i \in A$ such that $\sum_i a_i f_i = 1$, so $s = \sum_i a_i f_i s = 0$.
Moving on to (SB2). Suppose we have $s_i \in A_{f_i}$ and $s_i|_{U_{f_i f_j}} = s_j|_{U_{f_i f_j}}$ for all i, j. We want to glue them together.
First we assume that I is finite. Replacing f_i's by a power of it if necessary, we have $s_i = t_i / f_i$ in A_{f_i}. Then $(f_i f_j)^{n_{ij}} t_i f_j = (f_i f_j)^{n_{ij}} t_j f_i$ for some n_{ij}. As I is finite, we can replace all n_{ij} by the maximum n over them. We thus have $f_i^n f_j^{n+1} t_i = f_i^{n+1} f_j^n t_j$.
Choose $b_i \in A$ such that $1 = \sum_i b_i f_i^{n+1}$ which is possible as $\text{Spec}\, A = \bigcup_i U_{f_i^{n+1}}$. Take $s = \sum_i f_i^n b_i t_i$. As $f_j^n b_j|_{U_{f_i}} = f_i^{-n-1}(f_i^{n+1} f_j^n t_j) = f_i^{-n-1}(f_i^n f_j^{n+1} t_i) = f_j^{n+1} s_i$, we have $s|_{U_{f_i}} = s_i$.
For possibly infinite I, we pick a finite J such that $\text{Spec}\, A = \bigcup_{j \in J} U_{f_j}$. The above process yields some $s \in A$ such that $s|_{U_{f_j}} = s_j$ for all $j \in J$. For other $i \in I$, there is some $s' \in A$ such that $s'|_{U_{f_j} = s_j}$ for all $j \in J \cup \{i\}$. But by (SB1), we must have $s = s'$, so $s|_{U_{f_i}} = s_i$. □

Corollary 3.3. *There is a unique sheaf $\mathcal{O} = \mathcal{O}_{\operatorname{Spec} A}$ of rings on* $\operatorname{Spec} A$ *such that $\mathcal{O}(U_f) = A_f$ and the restriction maps between U_f's are localisations. Moreover, the stalk of $\mathcal{O}$ at $\mathfrak{p}$ is $A_{\mathfrak{p}}$.*

3.2 Definition and Examples of Schemes

Definition 3.3. A ringed space is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ on X. An isomorphism of ringed spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ is a homeomorphism $\pi : X \to Y$ along with an isomorphism of sheaves $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$.

So $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space, which we usually just denote as $\operatorname{Spec} A$. Ringed spaces of this form are called affine schemes.

Definition 3.4. A scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. for every $x \in X$, there is some open $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine scheme. Here, $\mathcal{O}_X|_U = i^{-1}\mathcal{O}_X$ where $i : U \to X$ is the inclusion.

Remark. The stalks of a scheme must be isomorphic to $A_{\mathfrak{p}}$ for some ring A and prime $\mathfrak{p} \leq A$, hence are local rings.

Example 3.3. 1. Let k be a field. The affine n-space over k is the affine scheme $\operatorname{Spec} k[T_1, \ldots, T_n]$. For an ideal $I \leq k[T_1, \ldots, T_n]$ and $A = k[T_1, \ldots, T_n]/I$, $\operatorname{Spec} A$ is homeomorphic to $\mathbb{V}(I) \subset \mathbb{A}^n_k$. This gives us a sheaf on $\mathbb{V}(I)$, which however depends on I and not just on $\mathbb{V}(I)$: If $n = 1$, then $I = (T_1), I' = (T_1^2)$ have the same vanishing locus (i.e. a point) and yet $\operatorname{Spec} k[T_1]/(T_1)$ and $\operatorname{Spec} k[T_1]/(T_1^2)$ have different rings of global sections.
2. A toric monoid P is the positive integer span of a finite subset $\{v_1, \ldots, v_k\} \subset \mathbb{Z}^n$ for some n which, guess what, is a monoid. The monoid ring (sometimes known as the group ring) associated to P is $\mathbb{Z}[P] = \{\sum_{u \in P} a_u \chi^u : a_u \in \mathbb{Z}, a_u = 0$ for all but finitely many $u\}$. This is a ring under the obvious operations. The associated affine scheme $\operatorname{Spec} \mathbb{Z}[P]$ is called the toric scheme associated to P.
If $P = \mathbb{N}^2$, then $\mathbb{Z}[\mathbb{N}^2] \cong \mathbb{Z}[X, Y]$ and we commonly denote its spectrum as $\mathbb{A}^n_{\mathbb{Z}}$.
If $P = \mathbb{Z}^2$, then $\mathbb{Z}[\mathbb{Z}^2] \cong \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$.
3. Let $R = \mathbb{Z}[X_1, \ldots, X_n]/(f)$ for some $f \in \mathbb{Z}[X_1, \ldots, X_n]$, then R is known as a hyeprsurface ring and $\operatorname{Spec} R$ a hypersurface (over $\mathbb{Z}$).
4. If G is a group acting on a ring A, the invariant ring A^G is the subring $\{a \in A : \forall g \in G, g \in G\}$ and $\operatorname{Spec} A^G$ "looks like" the "quotient of A by G".

Definition 3.5. Let X be a scheme and $U \subset X$ an open subset. We define a structure sheaf $\mathcal{O}_U$ on U by $\mathcal{O}_U = \mathcal{O}_X|_U = i^{-1}\mathcal{O}_X$ where $i : U \to X$ is the inclusion.

Indeed, for any $V \subset U$ open, V must be open in X and $\mathcal{O}_U(V) = \mathcal{O}_X(V)$.

Proposition 3.4. *$(U, \mathcal{O}_U)$ is a scheme.*

Definition 3.6. The scheme $(U, \mathcal{O}_U)$ is called the open subscheme of X associated to the open set $U \subset X$.

Proof. For $x \in U$, there is some open $W \subset X$ which is affine. $U \cap W$ is open in W and contains p, so there is a distinguished open V of W containing p and contained in $U \cap W$. Then $V \subset U$ is an affine scheme. $\square$

The open subscheme of $\operatorname{Spec} A$ associated to U_f is then just $\operatorname{Spec} A_f$. However, open subschemes of affine schemes are not necessarily affine in general.

Example 3.4. Take $X = \mathbb{A}_k^2 = \operatorname{Spec} k[X,Y]$. Let's assume $k = \bar{k}$ for good measure. Let $U = \mathbb{A}_k^2 \setminus \{(X,Y)\}$.
We claim that $(U, \mathcal{O}_U)$ is not affine. Note that if $(Y, \mathcal{O}_Y)$ is an affine scheme, then it is isomorphic to $\operatorname{Spec} \mathcal{O}_Y(Y)$. So let's calculate $\mathcal{O}_U(U)$. Observe that $U_X \cup U_Y = U$ and we know $U_X = \operatorname{Spec} k[X, X^{-1}, Y], U_Y = \operatorname{Spec} k[X, Y, Y^{-1}]$. By the sheaf axioms, we must then have $\mathcal{O}_U(U) = \mathcal{O}_X(U) = k[X, X^{-1}, Y] \cap k[X, Y, Y^{-1}] = k[X, Y]$.
But U isn't even homeomorphic to $\operatorname{Spec} k[X,Y] \cong \mathbb{A}_k^2$! Indeed, every proper ideal of $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$ has nonempty vanishing locus and yet $(X,Y) \subsetneq \mathcal{O}_U(U)$ has empty vanishing locus.

3.3 Gluing Sheaves and Gluing Schemes

Let X be a topological space with an open cover $\{U_\alpha\}_{\alpha \in A}$. Suppose we are given a sheaf (of abelian groups, rings, etc.) $\mathcal{F}_\alpha$ on each U_α and sheaf isomorphisms $\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$ such that $\phi_{\alpha\alpha} = \mathrm{id}_{\mathcal{F}_\alpha}$ and $\phi_{\alpha\gamma} = \phi_{\beta\gamma} \circ \phi_{\alpha\beta}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. The last condition is called the cocycle condition.
Let's glue them together. Consider the sheaf $\mathcal{F}$ on X as follows: Given $V \subset X$ an open set, we define $\mathcal{F}(V) = \{(s_\alpha) : s_\alpha \in \mathcal{F}_\alpha(V \cap U_\alpha), \phi_{\alpha\beta}(s_\alpha|_{V \cap U_\alpha \cap U_\beta}) = s_\beta|_{V \cap U_\alpha \cap U_\beta}\}$. The restriction maps is induced by the restriction maps on each $\mathcal{F}_\alpha$.

Proposition 3.5. *$\mathcal{F}$ is indeed a sheaf and $\mathcal{F}|_{U_\gamma} \cong \mathcal{F}_\gamma$.*

Proof. It's easy to check that $\mathcal{F}$ is a sheaf – we don't even need the cocycle condition. The isomorphism $\mathcal{F}_\gamma \to \mathcal{F}|_{U_\gamma}$ is constructed as follows: Given $V \subset U_\gamma$ for some γ and $s \in \mathcal{F}_\gamma(U)$. Define its image in $\mathcal{F}(V)$ to be $\phi_{\gamma\alpha} = (\phi_{\gamma\alpha}(s|_{V \cap U_\alpha}))_\alpha$. This is a well-defined member of $\mathcal{F}(V)$ by the cocycle condition. □

Suppose we have schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ with opens $U \subset X, V \subset Y$ together with a choice of isomorphism $(U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$. We glue them: On the level of topological spaces, we take $(X \sqcup Y)/(U \cong V)$. The cocycle condition on sheaves is automatic since there are only two open sets we are gluing together, so we get a structure sheaf. For more details, see example sheet.

Example 3.5 (Bug-Eyed Line and Bug-Eyed Plane)**.** Let k be a field, $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[u]$, both abstractly isomorphic to $\mathbb{A}_k^1$. We take the open sets $U = X \setminus \mathbb{V}(t) \cong \operatorname{Spec} k[t, t^{-1}]$ and $V = X \setminus \mathbb{V}(u) \cong \operatorname{Spec} k[u, u^{-1}]$. Identifying $k[u, u^{-1}] \cong k[t, t^{-1}], u \mapsto t$ gives us an isomorphism $U \to V$, so we can glue to form a new scheme B, which looks like a "line with doubled origin". Note that B is not affine, since (as one can check) $\mathcal{O}_B(B) \cong k[v]$ but B is not isomorphic to $\mathbb{A}_k^1$.
There are some variations on this theme. One can similarly make a bug-eyed plane P by gluing $X = \operatorname{Spec} k[x,y], Y = \operatorname{Spec} k[u,v]$ together along $U = X \setminus \{(x,y)\}$ and $V = Y \setminus \{(u,v)\}$. Let $p_1, p_2 \in P$ be the images of the origins of X, Y, respectively. Then $W_i = P \setminus \{p_i\} \cong \mathbb{A}_k^2, i = 1,2$ are open affines of P. Observe that $W_1 \cap W_2$ is isomorphic to $\mathbb{A}_k^2$ with the origin removed, which is not affine! So even finite intersections of affine opens need not be affine. Soon, we'll

see that the feature that affine opens intersect at affine opens is a consequence of a property known as separatedness, which these "bug-eyed stuff" do not satisfy.

Example 3.6 (Projective Line, a first definition)**.** We again take $X = \operatorname{Spec} k[t]$, $Y = \operatorname{Spec} k[u]$, $U = X \setminus \mathbb{V}(t) = \operatorname{Spec} k[t, t^{-1}]$, $V = Y \setminus \mathbb{V}(u) = \operatorname{Spec} k[u, u^{-1}]$. But this time we identify U, V by the ring isomorphism $k[u, u^{-1}] \cong k[t, t^{-1}]$ identifying u with t^{-1}. The result of the gluing is denoted $\mathbb{P}^1_k$.

Remark. We'll define projective spaces multiple times in this course. It's left to the reader to check that they give the same schemes.

Proposition 3.6. *$\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$. That is, the only globally-defined functions on $\mathbb{P}^1_k$ are constants.*

Proof. Cover $\mathbb{P}^1_k$ by the open sets $U_1 = \operatorname{Spec} k[t] \subset \mathbb{P}^1_k, U_2 = \operatorname{Spec} k[u] \subset \mathbb{P}^1_k$ arising from gluing. So $\mathcal{O}_{\mathbb{P}^1_k}(U_1) = k[t], \mathcal{O}_{\mathbb{P}^1_k}(U_2) = k[u]$, and on $U_1 \cap U_2$ we identified u with t^{-1}. So every element of $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k)$ must be constant, since every element of $k[t, t^{-1}]$ that are simultaneously polynomials in t and t^{-1} are the constants. $\square$

3.4 The Proj Construction

Spec is a construction which gives the scheme-theoretic enhancement of the theory of classical affine varieties when applied to a finitely-generated, nilpotent-fee, $\bar{k}$-algebras. Proj is gonna be a similar kind of construction, except now we want projective varieties from the data of a graded ring.

Definition 3.7. A $\mathbb{Z}$-grading on a commutative ring A is a choice of decomposition $A = \bigoplus_{i \in \mathbb{Z}} A_i$ into abelian groups $(A_i)_i$ such that for any $x \in A_j, y \in A_k$ we have $xy \in A_{j+k}$. A ring A together with a $\mathbb{Z}$-grading is called a graded ring. This combined data is sometimes denoted by $A_\bullet$.
An ideal $I \leq A$ is called homogeneous if $I = \bigoplus_i (I \cap A_i)$.

For $x \in A_i$, we write $\deg x = i$.

Example 3.7. We can take $A = k[X_0, \dots, X_n]$ and A_i the subgroup of degree i homogeneous polynomials.

Throughout the course, we'll only consider the situation where $A_\bullet$ is $\mathbb{Z}_{\geq 0}$-graded, i.e. $A_i = 0$ for any $i < 0$. Moreover, we'll always assume that A is generated by A_1 as an A_0-algebra.
For intuition, we recall that the core motivation for the geometric picture of $\mathbb{P}^n_k$ is supposed to be $\mathbb{A}^{n+1}_k \setminus \{\text{origin}\}/k^\times$. The fact that we only care about the $k^\times$-invariant geometry means that we need to focus on the $k^\times$-invariant ideals in $k[X_0, \dots, X_n]$. But of course these are just the homogeneous ideals.
Back to construction. Let $A_\bullet$ be a $\mathbb{Z}_{\geq 0}$-graded ring with all the assumptions we made. The canonical example of this is $A_\bullet = k[X_0, \dots, X_n]$ with the usual grading by homogeneous degree. We write $A_+ = \bigoplus_{i=1}^\infty A_i$ which is a prime ideal of A, known as the irrelevant ideal.

Definition 3.8. As a set, $\operatorname{Proj} A_\bullet$ is the set of homogeneous prime ideals of $A_\bullet$ which do not contain A_+.
We equip $\operatorname{Proj} A_\bullet$ with the topology whose closed sets are subsets of the form $\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} A_\bullet : \mathfrak{p} \supset I\}$ for homogenous ideals I.

Remark. 1. If $A_\bullet$ is finitely generated over A_0 by $a_1, \ldots, a_r \in A_1$, then we can present $A_\bullet$ as a quotient of $A_0[X_1, \ldots, X_r]$ with the usual grading.
2. Since $A_\bullet$ is generated by A_1, there is an open cover of $\operatorname{Proj} A_\bullet$ by sets of the form $U_f = (\operatorname{Proj} A_\bullet)_f = \operatorname{Proj} A_\bullet \setminus \mathbb{V}(f)$ for $f \in A_1$.

Let's now put a scheme structure on $\operatorname{Proj} A_\bullet$ by gluing (yay?). So we need natural open cover of $\operatorname{Proj} A_\bullet$ and a sheaf on every element of the cover (satisfying cocyle conditions and so on).

Proposition 3.7. *Fix $f \in A_1$.*
(i) There is a bijection between $U(f)$ and the set of homogeneous primes of $A_\bullet[1/f]$.
(ii) There is a bijection between homogenenous primes of $A_\bullet[1/f]$ and all primes in $(A_\bullet[1/f])_0$, the subring of degree 0 elements in $A_\bullet[1/f]$.

Remark. The ring $A_\bullet[1/f]$ is naturally $\mathbb{Z}$-graded by setting $\deg(f^{-1}) = -1$.

Proof. (i) Straightforward (use the graded version of the correspondence of primes along localisations).
(ii) We define the bijection as follows: For a homogenous prime $\mathfrak{Q}$ of $A_\bullet[1/f]$, we obtain $\mathfrak{q} = \mathfrak{Q} \cap (A_\bullet[1/f])_0$ which is automatically a prime in $(A_\bullet[1/f])_0$. To see it is a bijection, we use the following fact: Suppose $I \subset A_\bullet$ is a homogenous ideal, generated by homogenous elements $\{g_\alpha\}_\alpha$, then $(IA_\bullet[1/f]) \cap (A_\bullet[1/f])_0$ is generated by $\{g_\alpha/f^{\deg g_\alpha}\}_\alpha$. So the map we defined has to be injective. More importantly we know that by picking generators for $\mathfrak{p} \leq (A_\bullet[1/f])_0$ and clearing denominators, we can product a set of generators of an ideal $\mathfrak{P}$ in $A_\bullet$. Some algebra reveals that $\mathfrak{P}$ is prime and $\mathfrak{P} \cap (A_\bullet[1/f])_0 = \mathfrak{p}$. □

Example 3.8. Take $A_\bullet = k[X_0, X_1]$ and $f = x_1$, then $A_\bullet[1/f] = k[x_0, x_1, x_1^{-1}]$ whose degree 0 elements are polynomials in x_0/x_1, i.e. $(A_\bullet[1/f])_0 = k[x_0/x_1]$. This corresponds to our intuition that the projective space should also be covered by "standard" open sets isomorphic to the affine space of the same dimension.

Remark. If one analyses the bijection just a little bit, one sees that the correspondence between U_f and $\operatorname{Spec}((A_\bullet[1/f])_0)$ respects ideal inclusion and is a homeomorphism.

So $\operatorname{Proj} A_\bullet$ is a topological space with an open cover $\{U_f\}_{f \in A_1}$ with each U_f homeomorphic to an affine scheme $\operatorname{Spec}((A_\bullet[1/f])_0)$. This gives us a sheaf $\mathcal{O}_{U_f}$ on each U_f. Note that for any $f, g \in A_1$,

$$\begin{aligned}(\operatorname{Proj} A_\bullet)_f \cap (\operatorname{Proj} A_\bullet)_g &= \operatorname{Spec}((A_\bullet[1/f])_0[f/g]) = \operatorname{Spec}((A_\bullet[1/f, 1/g])_0) \\ &= \operatorname{Spec}((A_\bullet[1/g])_0[g/f])\end{aligned}$$

These $\mathcal{O}_{U_f}$ agree on intersections. Similarly, we have the natural identification

$$(\operatorname{Proj} A_\bullet)_f \cap (\operatorname{Proj} A_\bullet)_g \cap (\operatorname{Proj} A_\bullet)_h = \operatorname{Spec}((A_\bullet[1/f, 1/g, 1/h])_0)$$

which implies the cocycle condition. We therefore can glue them together to obtain a sheaf on $\operatorname{Proj} A_\bullet$, which makes it a scheme since each U_f would be affine.

Remark. One can generalise this to $\mathbb{Z}_{\geq 0}$-graded rings which are not generated in degree 1. In fact, we barely used this assumption.

4 Morphisms

We know how to make an affine scheme out of a ring. What can we make out of a ring homomorphism? For example, quotient homomorphisms and localisation maps have obvious geometric meanings in the classical setting (the inclusion of a closed subvariety for the former, the inclusion of a distinguished open set for the latter). If you think about it, a ring homomorphism $A \to B$ also gives B the structure of an A-module, which also seems geometrically interesting.
We'll develop the language of morphisms for schemes, which captures these algebraic data in a geometric picture.

4.1 Locally Ringed Spaces and Morphisms

Definition 4.1. A morphism of ringed spaces $f = (f^{\text{top}}, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f^{\text{top}} : X \to Y$ with a morphism of sheaves $f^\sharp : \mathcal{O}_Y \to f_*^{\text{top}}\mathcal{O}_X$.

Remark. 1. Sometimes we'll write f in place of f^{top} for brevity.
2. We can of course replace the data of a sheaf morphism $f^\sharp$ by the data of a sheaf morphism $f^\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ due to the adjunction of f_* and f^{-1} (example sheet).

Can we take a morphism of schemes to be a morphism of ringed spaces? Not quite, since they can be pathological and not really suitable for geometry. We'll remove this pathology by introducing the notion of a locally ringed space.
Recall that a scheme $(X, \mathcal{O}_X)$ has the property that the stalk $\mathcal{O}_{X,p}$ at every $p \in X$ is a local ring, i.e. possesses a unique maximal ideal $\mathfrak{m}_p$, since they are isomorphic to the localisation of a ring at a prime.

Definition 4.2. A ringed space is called a locally ringed space if every stalk is a local ring.

So every scheme is a locally ringed space. For a locally ringed space $(X, \mathcal{O}_X)$, we write $\mathcal{O}_{X,p}$ to denote the stalk at $p \in X$ and $\mathfrak{m}_p$ the unique maximal ideal of it.

Definition 4.3. A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for every $p \in X$, the preimage of $\mathfrak{m}_p$ under the induced map $f_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is $\mathfrak{m}_{f(p)}$.
A morphism of schemes is a morphism of locally ringed spaces.

Remark. We clearly have a natural map $(f_*\mathcal{O}_X)_p \to \mathcal{O}_{X,p}$, precomposing this with $f_p^\sharp$ gives the induced map $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$.

Why do we need this extra condition, geometrically? If $f : X \to Y$ is a morphism of locally ringed spaces and $s \in \mathcal{O}_{Y,f(p)}$ is invertible ("not vanishing"), then $f_p^\sharp(s) \in \mathcal{O}_{X,p}$ ("pullback along f") is also invertible. So the condition means that vanishing is preserved after precomposition, which is the least we would want.

Theorem 4.1. *Morphisms* $\operatorname{Spec} B \to \operatorname{Spec} A$ *are in bijection with homomorphisms* $A \to B$.

Remark. This is not true if we had taken morphism of schemes to be morphisms of ringed spaces.

Proof. We'll induce a scheme morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ when given $A \to B$, and show that every scheme morphism arises this way.
Suppose $\phi : A \to B$ is a ring homomorphism, we consider the scheme morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ which on the level of topological spaces is just $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$. We then have $f^{-1}(\mathbb{V}(I)) = \mathbb{V}(\phi(I))$, which shows that f is indeed continuous.
On the level of structure sheaves, we set $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} B}$ which at $\mathfrak{p}$ should look like $\phi_\mathfrak{p} : A_{\phi^{-1}\mathfrak{p}} \to B_\mathfrak{p}, a/s \mapsto \phi(a)/\phi(s)$. This makes sense, since $s \notin \phi^{-1}\mathfrak{p} \implies \phi(s) \notin \mathfrak{p}$, and is automatically local.
Fix $U \subset \operatorname{Spec} A$. We need to build $f^\sharp(U) : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}U)$. We can identify the elements s of $\mathcal{O}_{\operatorname{Spec} A}(U)$ as compatible identifications $p \mapsto s(p)$ for $p \in U$. This is sent to $q \mapsto \phi_q(s(f(q)))$. They clearly glue, so we do have a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. (One can also do this on distinguished open sets, since they form a basis.)
Conversely, if we are given a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, then $\phi = f^\sharp(\operatorname{Spec} A) : A \to B$ is a map of rings (and is really our only possible choice). Let's show that this induces f back. We have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[f_\mathfrak{p}]{} & B_\mathfrak{p} \end{array}$$

Since $f_\mathfrak{p}$ is local, we are forced to conclude (with a gun to our heads) that $f(\mathfrak{p}) = \phi^{-1}\mathfrak{p}$, i.e. ϕ induces exactly the topological map underlies f. $f^\sharp$ also agrees with the induced map of ϕ on stalks. It's not hard to show from there that they are in fact the same, so we are done. □

Definition 4.4. A morphism $f : X \to Y$ is an open immersion if it induced an isomorphism of X onto an open subscheme $U \subset Y$.
f is a closed immersion (or closed embedding) if the induced topological map is a homeomorphism onto a closed subset of Y and $f^\sharp$ is surjective.

Example 4.1. 1. If we take the inclusion map $k[X, Y] \to k[X, X^{-1}, Y, Y^{-1}]$, then the induced map $\operatorname{Spec} k[X, X^{-1}, Y, Y^{-1}] \to \operatorname{Spec} k[X, Y]$ is an open immersion which can be thought of as the inclusion of the plane without the axes into the plane.
2. If A is any ring and $I \subset A$ is an ideal, then the map $\operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the quotient map is a closed immersion. This can be viewed as the inclusion of $\mathbb{V}(I)$ into $\operatorname{Spec} A$, only there can be many different closed immersions onto the set $\mathbb{V}(I)$.
For a more concrete example, $\operatorname{Spec} k[t]/(t^n) \to \operatorname{Spec} k[t], n \geq 1$ are closed immersions with the same image (namely the origin of the axis), but they are very clearly different.

Definition 4.5. A closed subscheme of Y is an equivalence class of closed immersions $X \to Y$, where $X \to Y, X' \to Y$ are equivalent if there is an isomorphism $X \to X'$ making

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\cong}\downarrow & \nearrow & \\ X' & & \end{array}$$

commute.

4.2 Fibre Products

We are looking to define the correct product of schemes. Although we're gonna see that we end up getting a lot more than that. For example, if $X_1, X_2 \hookrightarrow Y$ are closed subschemes, then the fibre products of them over Y will turn out to be the "correct" definition of $X_1 \cap X_2$. Taking the fibre product over a point, on the other hand, will give the right notion of fibres, giving them a scheme structure. There's also this thing about base changess, e.g. we want to have $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$.

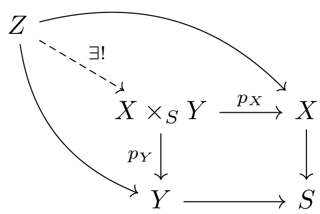
Definition 4.6. Let $X \to S, Y \to S$ be morphisms in some category $\mathcal{C}$. The fibre product of them is an object $X \times_S Y$ with morphisms (called projections) $p_X : X \times_S Y \to X, p_Y : X \times_S Y \to Y$ of $\mathcal{C}$ fitting into a commutative diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

such that whenever we have a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

There is a unique morphism $Z \to X \times_S Y$ such that the diagram



commutes.

If $\mathcal{C}$ also has a final object F, we write $X \times Y$ to denote $X \times_F Y$.

Remark. 1. If a fibre product exists, then it is necessarily unique.
2. The first diagram in the definition is also known as a pullback diagram, Cartesian diagram, base change diagram, and so on.

Example 4.2. If we are in the category of sets, then the fibre products of $f : X \to S, g : Y \to S$ is $X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$.

It's also worth noting that $X \times_S Y \cong (X \times Y) \times_{S \times S} S$ (where $S \to S \times S$ is the diagonal, i.e. the unique map induced by the commuting square of id_S).

Theorem 4.2. *Fibre products always exist in the category of schemes.*

Proof. Step 1: Suppose X, Y, S are all affine. Say $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, S = \operatorname{Spec} R$. Given $X \to S, Y \to S$, they must be induced by ring homomorphisms $R \to A, R \to B$. We can then form their tensor product $A \otimes_R B$. By the universal property of tensor products, we can take $X \times_S Y = \operatorname{Spec} A \otimes_R B$. This is certainly their fibre product in the category of affine schemes. To show that it is also their fibre product in the category of schemes, one extend the proof of Theorem 4.1 to see that morphisms $Z \to \operatorname{Spec} Q$ is in one-to-one correspondence with ring homomorphisms $Q \to \Gamma(Z, \mathcal{O}_Z)$.
Step 2: If $X \times_S Y$ exists and $U \subset X$ is any open subscheme, then $U \times_S Y$ also exists. Indeed, it is just the open subscheme $p_X^{-1} U \subset X \times_S Y$.
Step 3: Suppose X is covered by open sets $\{X_i\}_i$ and each $X_i \times_S Y$ exists, then $X \times_S Y$ also exists. This is done by gluing, which works by Step 2.
Step 4: For Y, S affine and any X, $X \times_S Y$ exists by Steps 1 and 3.
Step 5: For S affine and any X, Y, $X \times_S Y$ exists by Steps 3 and 4.
Step 6: For general X, Y, S, we cover S by affines S_i. Let X_i, Y_i be the preimages of S_i under $X \to S, Y \to S$, respectively. Then $X_i \times_{S_i} Y_i$ exists by Step 5. Moreover, $X_i \times_S Y_i$ exists because it is just $X_i \times_{S_i} Y_i$. Glue 'em with Step 3.
Step 7: Profit. □

Probably should mention that $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes.

Example 4.3. 1. We have

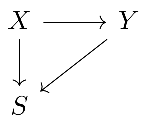
$$\mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \mathbb{Z}[X_1, \ldots, X_n] \otimes_{\mathbb{Z}} \mathbb{C} = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n] = \mathbb{A}^n_{\mathbb{C}}$$

Consequently we also have $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}$ by gluing.
2. In general, if $A_\bullet$ is an $\mathbb{N}$-graded ring finitely generated in degree 1 over $A_0 = \mathbb{Z}$, then $A_\bullet = \mathbb{Z}[X_1, \ldots, X_n]/I$ for some homogeneous ideal I. Then

$$\operatorname{Proj} A_\bullet \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C} = \operatorname{Proj} \mathbb{C}[X_1, \ldots, X_n]/I\mathbb{C}[X_1, \ldots, X_n]$$

3. Consider $C = \operatorname{Spec} \mathbb{C}[X, Y]/(Y - X^2) \to \operatorname{Spec} \mathbb{C}[X, Y] = \mathbb{A}^2_{\mathbb{C}}$ and $L = \operatorname{Spec} \mathbb{C}[X, Y]/(Y) \to \operatorname{Spec} \mathbb{C}[X, Y] = \mathbb{A}^2_{\mathbb{C}}$. Then $C \times_{\mathbb{A}^2_{\mathbb{C}}} L = \operatorname{Spec} \mathbb{C}[X]/(X^2)$ by computing the tensor product. This is a double point in $\mathbb{A}^1$, which does seem like the morally correct intersection of C and L screaming their tangency at the origin. If we instead look at $L' = \mathbb{C}[X, Y]/(Y - 1)$, then $L' \times_{\mathbb{A}^2_{\mathbb{C}}} C = \operatorname{Spec} \mathbb{C} \times \mathbb{C}$.

From now on, if $S = \operatorname{Spec} A$ is affine, we might write $X \times_A Y$ in place of $X \times_{\operatorname{Spec} A} Y$. A similar convention is used if any of X, Y is affine.
Fix a base scheme S (the "base scheme"), which could be any scheme but you're welcome to take stuff like $\operatorname{Spec} \mathbb{C}, \mathbb{A}^1_k, \operatorname{Spec} \mathbb{Z}$ as primary examples.
An S-scheme X is a morphism $X \to S$ (sometimes called its structure morphism), and a morphism of S-schemes $X \to Y$ (an "S-morphism") is a morphism $X \to Y$ such that the diagram



commutes. When $S = \operatorname{Spec} A$ is affine, we write A-scheme (resp. A-morphism) in place of S-scheme (resp. S-morphism) for brevity. Our previous discussions

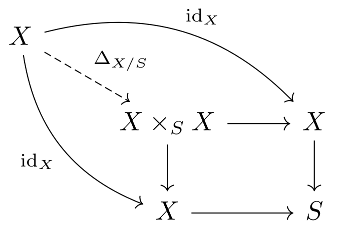
imply that fibre products exist in the category of S-schemes. Probably worth pointing out that the identity $S \to S$ is the final object in this category.
Note also that since $\operatorname{Spec} \mathbb{Z}$ is the final object in the category of schemes, the category of schemes is exactly the same as (i.e. equivalent to) the category of $\mathbb{Z}$-schemes.
Why do we do this? It happens that one should morally introduce most properties of schemes in a relative setting.

4.3 Separatedness

Recall that a topological space X is Hausdorff if and only if the diagonal $\Delta_X = \{(a, a) \in X \times X : a \in X\}$ is closed.

Definition 4.7. A morphism $X \to S$ is separated if the diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion. We say an S-scheme X is separated if its structure morphism is separated.

Just in case you're confused, $\Delta_{X/S}$ is the unique morphism that makes



commute.
If $X \to S$ is a morphism of affine schemes, say induced by $R \to A$ ($S = \operatorname{Spec} R, X = \operatorname{Spec} A$), then the diagonal morphism $X = \operatorname{Spec} A \to \operatorname{Spec} A \otimes_R A = X \times_S X$ is induced by $A \otimes_R A \to A, a \otimes b \mapsto ab$.

Proposition 4.3. *Morphisms of affine schemes are separated.*

Proof. The ring homomorphism $A \otimes_R A \to A, a \otimes b \mapsto ab$ is surjective. $\square$

Proposition 4.4. *Let $f : X \to S$ be a morphism of schemes. The diagonal $\Delta_{X/S} : X \to X \times_S X$ factors as $X \to U \to X \times_S X$ where $\mu : X \to U$ is a closed immersion and $\nu : U \to X \times_S X$ is an open immersion.*

Proof. Omitted, but you're welcome to either do it as an exercise or consult the usual textbooks. For a sketch: Cover S by affines $\{V_i\}_i$ and cover each $f^{-1}(V_i)$ by affine opens $\{U_{ij}\}_j$. The preceding proposition shows that each Δ_{U_{ij}/V_i} is a closed immersion, and we can build U out of $U_{ij} \times_{V_i} U_{ij}$. $\square$

Definition 4.8. A morphism $X \to Y$ is a locally closed immersion if it factorises as $X \to U \to Y$ where $X \to U$ is a closed immersion and $U \to Y$ is an open immersion.

Example 4.4. The inclusion of the axis with origin removed into $\mathbb{A}^2_{\mathbb{C}}$ is not an open nor closed immersion, but of course it is a locally closed immersion.

Definition chasing reveals that if $f : X \to Y$ is a locally closed immersion such that $f(X)$ is a closed subset of Y, then it is in fact a closed immersion. Consequently, $X \to S$ is separated if and only if $\Delta_{X/S}$ has closed image.

Example 4.5. 1. (Non-example) Consider the line X with two origins built by gluing two copies of $\mathbb{A}^1_k$. We get a structure morphism $X \to \operatorname{Spec} k$, which we claim is not separated. Indeed, $X \times_k X$ is the plane with doubled axes and quadrupled origins. And yet the image of $\Delta_{X/k}$ is only the diagonal of the plane with two of the four origins, which is not closed.
2. For any ring A, we build the A-scheme $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \ldots, X_n] \to \operatorname{Spec} A$, which we'll soon show is separated.
3. Open and closed immersions are separated. So are compositions of separated morphisms.
4. Base changes of separated morphisms are separated. To wit, suppose

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ W & \longrightarrow & S \end{array}$$

is a fibre product. Then f_2 is separated if f_1 is.

Proposition 4.5. *Let A be any ring, then $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \ldots, X_n] \to \operatorname{Spec} A$ is separated.*

Proof. We want to show that the image of $\Delta : \mathbb{P}^n_A \to \mathbb{P}^n_A \times_A \mathbb{P}^n_A$ is closed. Closedness can be checked on an open cover of the target. $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ is covered by affine opens $U_i \times_A U_j$ where $U_i = \mathbb{P}^n_A \setminus \mathbb{V}(X_i) = \operatorname{Spec}(A[X_0, \ldots, X_n][1/X_i])_0$. We have

$$\Delta^{-1}(U_i \times_A U_j) = U_i \times_{\mathbb{P}^n_A} U_j = U_i \cap U_j = \operatorname{Spec} A\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right]\left[\frac{X_i}{X_j}\right]$$

So Δ restricts to a morphism $U_i \cap U_j \to U_i \times_A U_j$, which is a morphism of affine schemes induced by

$$A\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}, \frac{Y_0}{Y_j}, \ldots, \frac{Y_n}{Y_j}\right] \to A\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right]\left[\frac{X_i}{X_j}\right]$$

given by simply replacing Y_i's with X_i's. This is surjective, therefore gives rise to a closed embedding. □

4.4 Properness and Valuative Criteria

Properness in scheme theory plays the role of compactness in differential geometry. $\mathbb{P}^n_A \to \operatorname{Spec} A$ will be proper but $\mathbb{A}^n_A \to \operatorname{Spec} A$ will not.

Definition 4.9. A morphism $f : X \to Y$ is locally of finite type if for any $V = \operatorname{Spec} A \subset Y$ affine, $f^{-1}V$ can be covered by affine schemes $U_i = \operatorname{Spec} B_i \subset X$ such that the induced ring homomorphisms $A \to B_i$ is of finite type, i.e. B_i is a finitely generated A-algebra. It is of finite type if we can take the open cover to be finite.

Definition 4.10. A morphism $f : X \to Y$ is closed if it is closed on the level of topological spaces. It is universally closed if for any morphism $Z \to Y$, the projection (base change) $X \times_Y Z \to Z$ is closed.
f is proper if it is separated, of finite type, and universally closed.

Remark. $\mathbb{A}^n_k \to \operatorname{Spec} k$ is not universally closed (example sheet), despite being closed. As we will see later, closed subschemes of $\mathbb{P}^n_k \to \operatorname{Spec} k$ are proper.

Let's recall

Definition 4.11. A discrete valuation ring (DVR) is a local PID.

Example 4.6. 1. $\mathbb{C}[[t]]$ is a DVR with unique maximal ideal (t) (usually interpreted as the "germ of a curve"). $\mathbb{Z}_{(p)}, \mathbb{Z}_p$ are also DVRs.
2. (Non-example) $\mathbb{C}[[t_1, t_2]]$ is not a DVR, despite being local.

If R is a DVR which is not a field, then $\operatorname{Spec} R$ consists of two points $\{\eta, \mathfrak{m}\}$, where $\eta = (0)$ is the generic point and $\mathfrak{m} \leq R$ is the unique maximal ideal. The closed sets are exactly $\{\mathfrak{m}\}, \operatorname{Spec} R$.
By the way, if $(\pi) = \mathfrak{m}$ then we call π a uniformiser of R. Also, there's a discrete valuation on $K = \operatorname{FF}(R)$, i.e. a map $v : K^\times \to \mathbb{Z}$ such that $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$, given by $v(u\pi^n) = n$ for any $u \in R^\times, n \in \mathbb{Z}$. Then $R = \{a \in K : v(a) \geq 0\} \cup \{0\}$ and $\mathfrak{m} = \{a \in K : v(a) > 0\} \cup \{0\}$.

Example 4.7. On $\mathbb{C}[[t]]$, we can take $\pi = t$ and $v(f)$ is the "order of zero/pole" of $f \in \mathbb{C}((t)) = \operatorname{FF}(\mathbb{C}[[t]])$ at 0.

From now on, all schemes are assumed to be locally Noetherian (i.e. every affine open is the spectrum of a Noetherian ring).

Theorem 4.6 (Valuative Criteria)**.** *Let $f : X \to Y$ be a morphism of finite type. Then f is separated if and only if for any DVR R with fraction field K and a commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

of solid arrows, there exists at most one lift $\operatorname{Spec} R \to X$ *filling in the dashed arrow.*
f is universally closed if and only if for any DVR R with fraction field K and a commutative diagram as above, there exists at least one lift $\operatorname{Spec} R \to X$ *filling in the dashed arrow.*

Example 4.8. 1. The line with doubled origins over k is not separated by taking $R = k[[t]]$ (example sheet).
2. $\mathbb{A}^n_k$ is not proper despite being separated and of finite type, again using $R = k[[t]]$ (again in example sheet).

Corollary 4.7. *(i)* $\mathbb{P}^n_A \to \operatorname{Spec} A$ *is proper.*
(ii) Compositions of proper morphisms are proper.
(iii) Closed immersions are proper. In particular, closed subschemes of $\mathbb{P}^n_A$ are proper (over $\operatorname{Spec} A$ *as well), hence everything from* Proj *constructure is proper.*
(iv) Base changes of proper morphisms are proper.

You'll do most of the proof in example sheet. The part that needs a small amount of intellectual input is:

Theorem 4.8. $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ *satisfies the existence part of valuative criterion.*

Proof. Let R be a DVR and K its fraction field. Write $T = \operatorname{Spec} R, U = \operatorname{Spec} K$. Cue diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \big\downarrow & \nearrow & \big\downarrow \\ T & \longrightarrow & \operatorname{Spec} \mathbb{Z} \end{array}$$

Observe that if $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[X_0, \ldots, X_n]$, then $\mathbb{P}^{n-1}_{\mathbb{Z}}$ is a closed subscheme structure on $\mathbb{V}(\mathbb{X}_i)$. If $U \to \mathbb{P}^n_{\mathbb{Z}}$ lands in any of $\mathbb{V}(X_i)$, then we are done by induction. Otherwise, the image u of U is contained in $\bigcap_i (\mathbb{P}^n_{\mathbb{Z}} \setminus \mathbb{V}(X_i))$.
In this case, X_i/X_j lives in $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}},u}$ for all i, j. So we can pull them back to $f_{ij} \in K$ satisfying $f_{ii} = 1$ and $f_{ij} f_{jk} = f_{ik}$.
Let $\alpha_i = v(f_{i0})$. Suppose α_k is the smallest of these, then $v(f_{ik}) \geq 0$ for all i. Define $\mathbb{Z}[X_0/X_k, \ldots, X_n/X_k] \to R$ via $X_i/X_k \mapsto f_{ik}$. This gives $T \to \mathbb{P}^n_{\mathbb{Z}} \setminus \mathbb{V}(X_k) \to \mathbb{P}^n_{\mathbb{Z}}$ which extends $U \to \mathbb{P}^n_{\mathbb{Z}}$. □

Corollary 4.9. *$\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper.*

Proof. It is the base change of $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ along $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. □

5 Sheaves of Modules

When we take the spectrum of a ring R, we get a scheme $\operatorname{Spec} R$. But the ring wants to bring a plus-one, namely the category of R-modules.

5.1 Modules on a Ringed Space

Fix a ringed space $(X, \mathcal{O}_X)$.

Definition 5.1. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ of abelian groups such that for every $U \subset X$ open, we have a multiplication map $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ compatible with restriction which makes $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module.

Sheaves of $\mathcal{O}_X$-algebras are defined analogously. By the way, basic module theoretic operations such as tensor products extend to sheaves of modules.

Example 5.1. 1. Let $X = \mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ be the projective n-space thought of as a classical variety, with structure sheaf given by regular functions. For an integer $d \in \mathbb{Z}$, we can define the sheaf $\mathcal{O}_X(d)$ of $\mathcal{O}_X$-modules whose value on $U \subset X$ is given by

$$\{P/Q \text{ homogenous rational function} : \deg P - \deg Q = d, Q \text{ nonvanishing on } U\}$$

with the obvious restriction maps.
2. Given a ring A and an A-module M, we can take a sheaf $\tilde{M}$ on $\operatorname{Spec} A$ such that if $f \in A$ we have $\tilde{M}(U_f) = M_f$ with restriction maps given by localisation. By the theory of sheaf on a base we had before, $\tilde{M}$ does exist, is unique, and is an $\mathcal{O}_{\operatorname{Spec} A}$-module. It's sometimes also denoted as M^{sh}.

Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}_X$-modules on Y, hence a sheaf of $\mathcal{O}_Y$-modules via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. This is called the pushforward of $\mathcal{F}$ along f.

Conversely, for $\mathcal{G}$ a sheaf of $\mathcal{O}_Y$-modules, $f^{-1}\mathcal{G}$ is a sheaf of $f^{-1}\mathcal{O}_Y$-modules. $f^\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ allows us to form the sheaf $f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ (the sheafification of the tensor product presheaf) of $\mathcal{O}_X$-modules, known as the pullback of $\mathcal{G}$.

Example 5.2. Let $i : Y = \operatorname{Spec} A/I \to X = \operatorname{Spec} A$ be a closed immersion of affine varieties. Then $i_*\mathcal{O}_Y$ is the sheaf associated to the A-module A/I. On the other hand, we can sheafify I to get a sheaf $\tilde{I}$ of $\mathcal{O}_{\operatorname{Spec} A}$-modules.

5.2 Quasicoherent Sheaves of Modules

Let $(X, \mathcal{O}_X)$ be a (locally Noetherian) scheme. If you don't care about coherent sheaves of modules then the stuff here are also true for general schemes.

Definition 5.2. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasicoherent if there exists an affine open cover $\{U_i\}_i$ of X such that $\mathcal{F}|_{U_i}$ is the sheaf of modules associated to an $\Gamma(U_i, \mathcal{O}_X)$-module. It is coherent if we can choose the $\Gamma(U_i, \mathcal{O}_X)$-module to be finite.

Example 5.3. 1. $\mathcal{O}_X^{\oplus m}$ is always a quasicoherent $\mathcal{O}_X$-module on any scheme. If m were finite then is it also coherent. These are called free $\mathcal{O}_X$-modules, and m is known as its rank.
2. If $i : Y \to X$ is a closed immersion, then $i_*\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-modules. If $X = \operatorname{Spec} A$ is affine, then $Y = \operatorname{Spec} A/I$ for some ideal $I \leq A$ and $i_*\mathcal{O}_Y = \widetilde{A/I}$.

Proposition 5.1. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X, then $\mathcal{F}$ is quasicoherent (resp. coherent) if and only if for any open affine $U \subset X$, $\mathcal{F}|_U$ is the sheaf of modules associated to an $\Gamma(U, \mathcal{O}_X)$-module (resp. finite $\Gamma(U, \mathcal{O}_X)$-module).*

In particular, quasicoherent $\mathcal{O}_{\operatorname{Spec} A}$-modules are the same as A-modules. Let's prove the proposition.

Lemma 5.2. *Suppose $X = \operatorname{Spec} A$. Fix $f \in A$. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_{\operatorname{Spec} A}$-module. Let $s \in \Gamma(X, \mathcal{F})$. Then:*
(i) If $s|_{U_f} = 0$, then $f^n s = 0$ for some n.
(ii) If $t \in \Gamma(U_f, \mathcal{F})$, then there is some $m \geq 0$ such that $f^m t$ is the restriction of a global section $\tilde{t} \in \Gamma(X, \mathcal{F})$.

Proof. By quasicoherence, there is an open cover on which the restrictions of $\mathcal{F}$ comes from a module. Since distinguished open sets form a basis, we can assume WLOG that the cover consists of distinguished opens. Let $(U_{g_i})_i$ be such a cover, so $\mathcal{F}|_{U_{g_i}}$ is the sheaf associated to an A_{g_i}-module.
The rest is just algebra, trust me. □

Proof of Proposition 5.1. Given $U = \operatorname{Spec} A \subset X$, then $\mathcal{F}|_U$ is also quasicoherent. So it suffices to show that every quasicoherent module $\mathcal{M}$ on $\operatorname{Spec} A$ comes from an A-module M. We are gonna guess $M = \Gamma(\operatorname{Spec} A, \mathcal{M})$.
Indeed, restriction gives a natural map $\tilde{M} \to \mathcal{M}$. The lemma tells us that this map is an isomorphism on a distinguished open cover, which means that it is an isomorphism. □

Remark. 1. So quasi-coherence is "affine-local".
2. Kernels, cokernels and images of morphisms of quasicoherent $\mathcal{O}_X$-modules

(i.e. $\mathcal{O}_X$-linear sheaf morphisms) are quasicoherent.
3. We write $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to be the sheaf whose sections on U are $\mathcal{O}_X$-linear morphisms $\mathcal{F}|_U \to \mathcal{G}|_U$, and $\mathcal{F}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Example 5.4. Let $A_\bullet$ be a graded ring subject to the usual assumptions. Let $M_\bullet$ be a graded $A_\bullet$-module. Now if U_f is a distinguished affine associated to $f \in A_+$, then we set $\tilde{M}_\bullet(U_f) = (M_\bullet[1/f])_0$. It's easy to check that these glue to give a quasicoherent $\mathcal{O}_{\mathrm{Proj}(A_\bullet)}$-module $\tilde{M}_\bullet$, sometimes also denoted as $\mathrm{Proj}\, M_\bullet$ or $M_\bullet^{\mathrm{sh}}$.

Definition 5.3. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is locally free if there is an open cover $(U_i)_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus m_i}$ for some m_i. They are also known as (algebraic) vector bundles.
A locally free $\mathcal{O}_X$-module is called a line bundle, or an invertible sheaf, if we can take $m_i \equiv 1$, i.e. if it is locally isomorphic to $\mathcal{O}_X$.

Example 5.5. Let $\mathbb{P}^n_A = \mathrm{Proj}\, A[X_0, \ldots, X_n]$. For $d \in \mathbb{Z}$, we consider the graded $A[X_0, \ldots, X_n]$-module whose degree k piece is $(A[X_0, \ldots, X_n])_{k+d}$. The sheaf associated to it is denoted $\mathcal{O}_{\mathbb{P}^n_A}(d)$.
Of course, there is an analogous construction for anything of the form $X = \mathrm{Proj}(A_\bullet)$, and the resulting sheaf is denoted $\mathcal{O}_X(d)$.

Proposition 5.3. *$\mathcal{O}_X(d)$ is always a line bundle.*

Example 5.6. Earlier we've shown that $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$. Let's now calculate $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1))$. Cover $\mathbb{P}^1_k$ (with coordinates $[X_0 : X_1]$) by $U = \mathrm{Spec}\, k[X_1/X_0] = \mathbb{P}^1_k \setminus \mathbb{V}(X_0)$ and $U' = \mathrm{Spec}\, k[X_0/X_1] = \mathbb{P}^1_k \setminus \mathbb{V}(X_1)$. Then $\Gamma(U, \mathcal{O}_{\mathbb{P}^1_k}(1))$ consists of rational functions on X_0, X_1 with degree 1 and denominator nonvanishing at $X_1 = 0$. Similarly for $\Gamma(U', \mathcal{O}_{\mathbb{P}^1_k}(1))$. So by gluing $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1))$ consists of linear homogeneous polynomials in X_0, X_1.
More generally, $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d))$ consists of degree d homogenous polynomials in X_0, X_1.

Just some terminology.
If $X \to Y$ is a morphism and $\mathcal{F}$ is a quasicoherent sheaf on Y, then $f^*\mathcal{F}$ is a quasicoherent sheaf on X. Let $\mathcal{L}$ be a line bundle on X, we say $\mathcal{L}$ is basepoint-free if there is a morphism $f : X \to \mathbb{P}^n$ (over some fixed base – we'll start to leave out that part) such that $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$. If f can additionally be taken to be a closed immersion, then we say $\mathcal{L}$ is very ample. $\mathcal{L}$ is called ample if $\mathcal{L}^{\otimes r}$ is very ample for some $r > 0$.

6 Divisors

In the world of rings, prime ideals of height 1 play special roles. For example, if A is a UFD (or something weaker, as we'll see), then height 1 primes are principal. The divisor theory on schemes wants to generalise this idea. It comes in two flavours: Weil divisors and Cartier divisors.

6.1 Weil Divisors

Ehh let's just take a detour to talk about some topology.

Definition 6.1. For a topological space X, its dimension $\dim X$ is the length n of the longest chain of nonempty closed irreducible subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ in X. We say $\dim X = \infty$ if the length of such a chain is not bounded above.

Example 6.1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

Definition 6.2. Given a closed subset $Z \subset X$, the codimension of Z in X is the length n of the longest chain of closed irreducible subsets $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ in X.

If X is any Noetherian topological space, for example the topological space of a Noetherian scheme, then any closed subset $Z \subset X$ has a finite decomposition into irreducible components.
Back to geometry. Let X be a Noetherian integral separated scheme which is regular in codimension 1, in the sense that if $\mathfrak{p}$ is a height 1 prime ideal in A and $\operatorname{Spec} A \subset X$ is an affine open, then $A_{\mathfrak{p}}$ is a DVR.

Example 6.2. $\mathbb{A}^n, \mathbb{P}^n$ satisfy this condition. More generally, if $\operatorname{Spec} A$ is normal, then it is regular in codimension 1.

Definition 6.3. A prime divisor on X is an integral closed subscheme of X of codimension 1. A Weil divisor on X is an element of the free abelian group $\operatorname{Div} X$ on the set of prime divisors.
A Weil divisor $\sum_D n_D[D]$, where the sum ranges in some finite set of prime divisors D, is effective if $n_D \geq 0$ for all D.

Let X be integral and suppose $\operatorname{Spec} A \subset X$ is any affine open, then A is an integral domain, therefore has a unique generic point $(0) \in \operatorname{Spec} A \subset X$. In fact, this point is independent of the choice of A, and is the unique generic point of X. It is usually denoted by $\eta = \eta_X \in X$.

Definition 6.4. The function field $\mathcal{k}(X)$ of X is $\mathcal{O}_{X,\eta}$.

So $\mathcal{k}(X)$ is just the fraction field of A for any affine open $\operatorname{Spec} A \subset X$.

Definition 6.5. For $f \in \mathcal{k}(X)^\times$, its divisor is

$$\operatorname{div} f = \sum_{Y \subset X \text{ prime divisor}} n_Y(f)[Y]$$

where $n_Y(f)$ is the valuation of f, viewed as an element of the fraction field of the DVR $\mathcal{O}_{X,\eta_Y}$.

Proposition 6.1. $\operatorname{div} f$ *is a bona fide Weil divisor.*

In other words, $n_Y = 0$ for all but finitely many Y.

Proof. Take $U = \operatorname{Spec} A \subset X$ affine such that $f \in A$. $Z = X \setminus U$ is closed of codimension at least 1. Since Z has a finite irreducible decomposition, only finitely many closed subset of X of codimension 1 is contained in $X \setminus U$. So all but finitely many n_Y is negative. Applying this conclusion to $1/f$ shows that all but finitely many n_Y is positive, so all but finitely many n_Y is nonzero. □

Alternatively way to finish the proof. Other Y with $n_Y > 0$ correspond to the irreducible components of $\mathbb{V}(f)$, but there are only finitely many of them. □

Definition 6.6. Divisors of the form div $f \in \operatorname{Div} X$ are called principal divisors. They form a subgroup $\operatorname{Prin}(X) \leq \operatorname{Div} X$. Its quotient is called the (Weil) divisor class group $\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X)$.

Example 6.3. 1. If A is a UFD, then $\operatorname{Cl}(\operatorname{Spec} A) = 0$. In particular, $\operatorname{Cl}(\mathbb{A}^n_A) = 0$ for any UFD A.
2. $\operatorname{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$, and is generated by the class of a hyperplane.

Proposition 6.2 (Excision)**.** *Let $Z \subset X$ be closed and $U = X \setminus Z$.*
(a) There is a surjection $\operatorname{res} : \operatorname{Cl}(X) \to \operatorname{Cl}(U)$ *given by restriction.*
(b) If Z has codimension at least 2, then res *is an isomorphism.*
(c) If Z is irreducible of codimension 1, then we have a right exact sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl}(X) \xrightarrow{\operatorname{res}} \operatorname{Cl}(U) \longrightarrow 0$$

Proof. (a) The restriction map takes a prime divisor D in X to $D \cap U$, which is either empty or has codimension 1 in U, so it gives a well-defined map $\operatorname{Div} X \to \operatorname{Div} U$. To push it into the class group, observe that $\mathcal{k}(X)^\times = \mathcal{k}(U)^\times$ which means that $\operatorname{div}(f) \cap U = \operatorname{div}(f|_U)$. This gives $\operatorname{res} : \operatorname{Cl}(X) \to \operatorname{Cl}(U)$. This is surjective since the closure of $D \cap U$ is D for any prime divisor D not contained in Z.
(b) There isn't even a prime divisor contained in Z.
(c) The only prime divisor contained in Z is Z. □

6.2 Cartier Divisors

Sometimes we want a divisor theory in a more general situation.
Recall that if A is a UFD then all height 1 prime ideals are principal.

Definition 6.7. If X is a scheme such that $\mathcal{O}_{X,x}$ is a UFD for every $x \in X$, we say X is locally factorial.

In this case, you would expect codimension 1 phenomena to be captured by closed subschemes locally cut out by a single function.
In general, "locally principal subschemes" (which will give us Cartier divisors) and "integral subschemes of codimension 1" (which gave us Weil divisors) can be quite different. But they can also be quite similar if your vision is blurry enough.

Definition 6.8. Let X be a scheme. Consider the presheaf on a base sending an affine open $U = \operatorname{Spec} A \subset X$ to $S^{-1}A$ where S is the multiplicative set of non-zerodivisors in A. Sheafify this to get a sheaf $\mathscr{K}_X$ of rings. If we pick out the invertible elements, we get a sheaf $\mathscr{K}_X^*$ of multiplicative groups.
Similarly, we have the sheaf on a base $U \mapsto A^\times$ which sheafifies to give a sheaf $\mathcal{O}_X^*$ of abelian groups.

Definition 6.9. A Cartier divisor on X is a global section of $\mathscr{K}_X^*/\mathcal{O}_X^*$.

Example 6.4. If $X = \mathbb{A}^1_k$, then a Cartier divisor is just a single rational function up to scalar.

In other words, a Cartier divisor is specified by the data $\{(U_i, f_i) : U_i \subset X, f_i \in \mathscr{K}_X^*(U_i)\}$ where U_i form an open cover of X and on $U_i \cap U_j$ we have $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$.
If a Cartier divisor can be represented by $\{(X, f)\}$, then we call it a principal Cartier divisor. The resulting class group is denoted $\mathrm{CaCl}(X)$ (obligatory chemistry joke goes here).
Let X now be a Noetherian integral separated scheme, regular in codimension 1. Every Cartier divisor $\mathscr{D}$ gives rise to a Weil divisor in the following way:
Specify $\mathscr{D}$ by $\{(U_i, f_i) : U_i \subset X, f_i \in \mathscr{K}_X^*(U_i)\}$ where the data are compatible with each other in the way described above. For a prime divisor $Y \subset X$, take any i such that $\eta_Y \in U_i$. Now declare $n_Y(\mathscr{D})$ to be the valuation of f_i at η_Y.
This is independent of the choice of i. If (U_j, f_j) is another choice, then $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$, but then f_i/f_j has to have valuation 0, so the valuation of f_j at η_Y is the same as the valuation of f_i.
All but finitely many Y has $n_Y(\mathscr{D}) = 0$ (exercise), so this process gives us a Weil divisor $D = \sum_Y n_Y(\mathscr{D})[Y]$.

Proposition 6.3. *Let X be a Noetherian integral separated scheme, regular in codimension 1. Suppose in addition that X is locally factorial, then the above construction gives a bijection between Cartier divisors and Weil divisors on X. Furthermore, this process respects principal divisors, hence descends to an isomorphism* $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$.

Brief sketch of proof. All height 1 primes are principal in a UFD. Given a Weil divisor D and a point $x \in X$, we can restrict D to $\operatorname{Spec} \mathcal{O}_{X,x}$ (technically by taking a fibre product), which gives us a principal divisor $\operatorname{div} f_x$ on $\operatorname{Spec} \mathcal{O}_{X,x}$. By a topological argument, it extends to an open U_x around x, which gives the local data of a Cartier divisor. □

Let $\mathscr{D}$ be a Cartier divisor, we consider the subsheaf $\mathcal{L}(\mathscr{D}) \hookrightarrow \mathscr{K}_X$ defined by the following process:
Describe $\mathscr{D}$ by the data $\{(U_i, f_i) : U_i \subset X, f_i \in \mathscr{K}_X^*(U_i)\}$, and set $\mathcal{L}(\mathscr{D})(U_i)$ to be the $\mathcal{O}_X$-submodule generated by f_i^{-1}. It's straightforward to check that this is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1, i.e. a line bundle.

Definition 6.10. The Picard group $\mathrm{Pic}(X)$ of X is the set of isomorphism classes of line bundles on X made into a group via tensor product over $\mathcal{O}_X$.

It goes without saying but the identity is $\mathcal{O}_X$ and the inverse of $\mathcal{L} \in \mathrm{Pic}(X)$ is its dual $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.
Our construction gives a map from the set of Cartier divisors on X to $\mathrm{Pic}(X)$. If X is integral, this is surjective and only depends on the Cartier divisor class. So we get an isomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$.

7 Sheaf Cohomology

7.1 Daydreaming about a Cohomology Theory

Let $\mathcal{F}$ be a sheaf of abelian groups on any topological space X. We can extract a group from $\mathcal{F}$ by taking its group of global sections $\Gamma(X, \mathcal{F})$, which is easy to deal with but loses a lot of information.

Example 7.1. 1. Consider $X = \mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{F}_1 = \mathcal{O}_X, \mathcal{F}_2 = \underline{\mathbb{C}}$, then $\Gamma(X, \mathcal{F}_1) = \Gamma(X, \mathcal{F}_2) = \mathbb{C}$ but these two sheaves encode very different data.
2. Consider $X = \mathbb{A}^2_{\mathbb{C}}, Y = \mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}$, then $\Gamma(X, \mathcal{O}_X) = \Gamma(Y, \mathcal{O}_Y)$ but X, Y too are very different.

So we want to measure the loss of information from taking only global sections. We'll do this by building abelian groups $H^i(X, \mathcal{F})$ for $i \geq 0$, known as the sheaf cohomology of $\mathcal{F}$, which will have nice properties such as:
(i) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
(ii) Suppose $\mathcal{F}$ is a sheaf on Y, then any continuous $f : X \to Y$ gives rise to a map $f^* : H^i(Y, \mathcal{F}) \to H^i(X, f^{-1}\mathcal{F})$.
(iii) If X is a CW-complex, then there is a canonical isomorphism $H^i(X, \underline{\mathbb{Z}}) \to H^i(X; \mathbb{Z})$ where the latter denotes singular cohomology.
(iv) A short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H^i(X, \mathcal{F}') \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}'') \longrightarrow$$
$$\longrightarrow H^{i+1}(X, \mathcal{F}') \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{F}'') \longrightarrow \cdots$$

starting with $0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots$.

7.2 Injective Resolutions

Definition 7.1. An R-module I is injective if for any diagram of solid arrows

$$\begin{array}{ccccc} & & I & & \\ & & \uparrow & \nwarrow^{\beta} & \\ 0 & \longrightarrow & A & \xrightarrow[f]{} & B \end{array}$$

with bottom row exact (i.e. f injective), there is some $\beta : B \to I$ filling in the dashed arrow.

Remark. Taking $f : \mathbb{Z} \to \mathbb{Z}, 1 \mapsto n$ shows that for every x in an injective $\mathbb{Z}$-module I there is some $y \in I$ such that $ny = x$. In fact, injective $\mathbb{Z}$-modules are precisely the divisible groups.

Example 7.2. $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{C}^\times$ are divisible groups, hence injective abelian groups.

Definition 7.2. A sheaf $\mathcal{I}$ of abelian groups is injective if for any diagram of solid arrows

$$\begin{array}{ccccc} & & \mathcal{I} & & \\ & & \uparrow & \nwarrow^{\beta} & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow[f]{} & \mathcal{B} \end{array}$$

with bottom row exact, there is some $\beta : \mathcal{B} \to \mathcal{I}$ filling in the dashed arrow.

Definition 7.3. Let A be an R-module. An injective resolution of A is a long exact sequence of the form

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

with each I_i injective R-modules.
Similarly, an injective resolution of a sheaf $\mathcal{A}$ is a long exact sequence of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \cdots$$

with each $\mathcal{I}_i$ injective sheaves of abelian groups.

We sometimes write $\mathcal{A} \to \mathcal{I}_\bullet$ to denote an injective resolution like this.

Proposition 7.1. *Every abelian group has an injective resolution.*

Proof. Every abelian group A admits an injective homomorphism f into an injective abelian group I_0. This can be done by writing it as a quotient of a free abelian group and shoving everything into $\mathbb{Q}$. And then we inject $\operatorname{coker} f$ into another injective I_1 and keep going. $\square$

Corollary 7.2. *Every sheaf of abelian groups has an injective resolution.*

Proof. Using the same "keep going" argument as before, it suffice to inject any sheaf $\mathcal{F}$ into an injective sheaf. At every $x \in X$, there is some injective abelian group I_x such that $\mathcal{F}_x$ injects into I_x. Consider $i_x : \{x\} \hookrightarrow X$, then $\mathcal{F}$ injects into $\mathcal{I} = \prod_{x \in X} (i_x)_* \underline{I_x}$, which is injective since

$$\mathrm{Hom}_{(\mathsf{Sh}/X)}(\mathcal{G}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{(\mathsf{Sh}/X)}(\mathcal{G}, (i_x)_* \underline{I_x}) = \prod_{x \in X} \mathrm{Hom}_{(\mathsf{Ab})}(\mathcal{G}_x, I_x)$$

and that infinite direct products of injectives are injective. $\square$

7.3 Cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups on X and $\mathcal{F} \to \mathcal{I}_\bullet$ an injective resolution. Since $\Gamma(X, -)$ sends zero to zero, we have a cochain complex

$$0 \longrightarrow \Gamma(X, \mathcal{I}_0) \longrightarrow \Gamma(X, \mathcal{I}_1) \longrightarrow \cdots$$

Definition 7.4. For $i \geq 0$, the i-th sheaf cohomology of $\mathcal{F}$ is

$$H^i(X, \mathcal{F}) = \frac{\ker(\Gamma(X, \mathcal{I}_i) \to \Gamma(X, \mathcal{I}_{i+1}))}{\mathrm{Im}(\Gamma(X, \mathcal{I}_{i-1}) \to \Gamma(X, \mathcal{I}_i))}$$

Proposition 7.3. *(i)* $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
(ii) $H^i(X, \mathcal{F})$ *is independent of the choice of injective resolution for* $\mathcal{F}$.
(iii) Given a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

there are connecting homomorphisms $H^i(X,\mathcal{F}'') \to H^{i+1}(X,\mathcal{F}')$ *such that we have a long exact sequence*

$$\cdots \longrightarrow H^i(X,\mathcal{F}') \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{F}'') \longrightarrow H^{i+1}(X,\mathcal{F}') \longrightarrow H^{i+1}(X,\mathcal{F}) \longrightarrow H^{i+1}(X,\mathcal{F}'') \longrightarrow \cdots$$

starting with $0 \to H^0(X,\mathcal{F}') \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{F}'') \to H^1(X,\mathcal{F}') \to \cdots$.

Proof. (i) Follows from the left-exactness of $\Gamma(X,-)$.
(ii) (iii) Diagram chasing. □

Theorem 7.4 (Grothendieck)**.** *If X is a Noetherian topological space of dimension n and $\mathcal{F}$ is any sheaf of abelian groups on X, then $H^i(X,\mathcal{F}) = 0$ for all $i > n$.*

7.4 Čech Cohomology

Nobody has ever seen an injective resolution for a nonstupid sheaf and lived. There's a different way to compute sheaf cohomology, namely via the Čech complex.
Let $\mathcal{F}$ be a sheaf of abelian groups on X and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X with I a well-ordered set. Write $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$.

Definition 7.5. The group of Čech p-cochains is

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p})$$

We can define a coboundary map $d = d^p : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ as follows: For $\alpha \in C^p(\mathcal{U},\mathcal{F})$, we set

$$(d\alpha)_{i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\cdots \hat{i}_k \cdots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}$$

Proposition 7.5. $d \circ d = 0$.

We then have a cochain complex $C^\bullet(\mathcal{U},\mathcal{F})$ known as the Čech complex.

Definition 7.6. The i-th Čech cohomology of the pair $(\mathcal{U},\mathcal{F})$ is $\check{H}^i(\mathcal{U},\mathcal{F}) = \ker d_i / \operatorname{Im} d_{i-1}$.

Example 7.3. Take $X = S^1$ and $U = S^1 \setminus \{1\}, V = S^1 \setminus \{-1\}, \mathcal{U} = \{U, V\}$. For $\mathcal{F} = \underline{\mathbb{Z}}$, $C^0(\mathcal{U},\mathcal{F}) \cong \mathbb{Z}^2$ and $C^1(\mathcal{U},\mathbb{F}) \cong \mathbb{Z}^2$. $d^0 : \mathbb{Z}^2 \to \mathbb{Z}^2$ takes (a,b) to $(b-a, b-a)$, so $\check{H}^0(\mathcal{U},\underline{\mathbb{Z}}) \cong \mathbb{Z}$, $\check{H}^1(\mathcal{U},\underline{\mathbb{Z}}) \cong \mathbb{Z}$.

Remark. $\check{H}^i(\mathcal{U},-)$ sucks if $\mathcal{U}$ sucks (think about what happens when $\mathcal{U} = \{X\}$).

Theorem 7.6. *Suppose X is a Noetherian scheme and $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module. Suppose also that $\mathcal{U}$ is a cover of X such that each $U_{i_0} \cap \cdots \cap U_{i_p}$ is affine (e.g. if X is separated and $\mathcal{U}$ consists of affine opens). Then the natural map $\check{H}^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$ is an isomorphism.*

Remark. In particular, $H^i(X,\mathcal{F}) = 0$ for $i > 0$ if X is the spectrum of an Noetherian ring. This is actually the main ingredient of the theorem's proof. In fact, (under the Noetherian hypothesis) X is affine iff $H^i(X,\mathcal{F}) = 0$ for all $i > 0$ and quasicoherent $\mathcal{F}$.

7.5 Cohomology of Projective Space

Let k be a ring and write $\mathbb{P}^n = \mathbb{P}^n_k$. Recall that on $X = \mathbb{P}^n$ we have quasicoherent sheaves $\mathcal{O}_X(d) = \mathcal{O}(d)$. Let $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(d)$ which is still a quasicoherent sheaf. Observe that $H^i(X, \mathcal{G} \oplus \mathcal{G}') = H^i(X, \mathcal{G}) \oplus H^i(X, \mathcal{G}')$.

Theorem 7.7. *We have the following isomorphism of graded k-modules:*
(i) $H^0(\mathbb{P}^n, \mathcal{F}) \cong k[X_0, \dots, X_n]$.
(ii) $H^n(\mathbb{P}^n, \mathcal{F}) \cong (X_0 \cdots X_n)^{-1} k[X_0^{-1}, \dots, X_n^{-1}]$.
(iii) $H^i(\mathbb{P}^n, \mathcal{F}) = 0$ for $i \neq 0, n$.

In particular, if k is a field, then $h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}$ for $d \geq 0$ and $h^n(\mathbb{P}^n, \mathcal{O}(d)) = \binom{-d-1}{n}$, where $h^i = \dim_k H^i$.

Proof. (i) $H^0 = \Gamma$.
(ii) We'll compute it with Čech cohomology. Take $\mathcal{U} = \{U_i\}_{i=0}^n$ where $U_i = \mathbb{P}^n \setminus \mathbb{V}(X_i)$ (where we identify $\mathbb{P}^n = \operatorname{Proj} k[X_0, \dots, X_n]$). Observe that $\mathcal{F}(U_{i_1 \cdots i_p}) = k[X_0, \dots, X_n]_{X_{i_0} \cdots X_{i_p}}$. So we have

$$C^{n-1}(\mathcal{U}, \mathcal{F}) = \bigoplus_{i=0}^n k[X_0, \dots, X_n]_{X_0 \cdots \hat{X}_i \cdots X_n}, C^n(\mathcal{U}, \mathcal{F}) = k[X_0, \dots, X_n]_{X_1 \cdots X_n}$$

and of course that $C^{n+1}(\mathcal{U}, \mathcal{F}) = 0$. Therefore

$$\begin{aligned} \check{H}^n(\mathcal{U}, \mathcal{F}) &\cong \frac{\operatorname{Span}_k\{X_0^{k_0} \cdots X_n^{k_n} : k_i \in \mathbb{Z}\}}{\operatorname{Span}_k\{X_0^{k_0} \cdots X_n^{k_n} : k_i \in \mathbb{Z}, \exists i, k_i > 0\}} \\ &\cong \operatorname{Span}_k\{X_0^{k_0} \cdots X_n^{k_n} : k_i \in \mathbb{Z}_{<0}\} \end{aligned}$$

(iii) If $n = 1$ then there is nothing to prove. We proceed by induction. Consider the closed immersion $\iota : \mathbb{P}^{n-1} \to \mathbb{P}^n$ onto $\mathbb{V}(X_0)$. We then have a short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{\times X_0} \mathcal{O} \longrightarrow \iota_* \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0$$

Tensoring this exact sequence with $\mathcal{O}(d)$ (check injectivity locally) gives

$$0 \longrightarrow \mathcal{O}(d-1) \xrightarrow{\times X_0} \mathcal{O}(d) \longrightarrow \iota_* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0$$

Note that pushforwards commute with cohomology, so we get exact sequences

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\times X_0} H^0(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}, \mathcal{F}) \longrightarrow H^1(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\times X_0} H^1(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^p(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\times X_0} H^p(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^{n-1}(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\times X_0} H^{n-1}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}, \mathcal{F}) \longrightarrow H^n(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\times X_0} H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0$$

for $1 < p < n-1$. Not much else to say, we are just done. □

Now let k be a field.

Example 7.4. $X = \mathbb{A}^2_k \setminus \{(0,0)\}$, then $H^1(X, \mathcal{O}_X)$ is infinite-dimensional. In particular, X is not affine.

Proposition 7.8. *If X is proper over k and $\mathcal{F}$ is coherent, then $H^p(X, \mathcal{F})$ is finite-dimensional for all p.*

Example 7.5. Suppose $X = \mathbb{V}(f_d) \subset \mathbb{P}^2_k$ where f_d is a degree d homogeneous polynomial in X_0, X_1, X_2. WLOG $(1 : 0 : 0) \notin X$, then $U = X \cap (\mathbb{P}^2_k \setminus \mathbb{V}(X_1)), V = X \cap (\mathbb{P}^2_k \setminus \mathbb{V}(X_2))$ allows one to calculate $h^0(X, \mathcal{O}_X) = 1, h^1(X, \mathcal{O}_X) = (d-1)(d-2)/2$.

Definition 7.7. Suppose X is proper over k. The Euler characteristic of a coherent sheaf $\mathcal{F}$ on X is

$$\chi(X, \mathcal{F}) = \sum_{p=0}^{\infty} (-1)^p h^p(X, \mathcal{F})$$

It's easy to show that a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

gives $\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'')$.
One other thing, if for $a \in \mathbb{Z}$ we define $\binom{a}{n} = a(a-1)\cdots(a-n+1)/n!$ and $X_d = \mathbb{V}(f_d) \subset \mathbb{P}^3$ for a degree d homogeneous polynomial f_d in $X_0, \ldots, X_3$, then $\chi(X_d, \mathcal{O}_{X_d}) = 1 - \binom{3-d}{3}$. But the left hand side depends only on the isomorphism class of X_d! So for distinct $d, d' \geq 3$, X_d is not isomorphic to $X_{d'}$.

8 Dhruv's Rant, Blåhaj Edition

A morphism $X \to Y$ is flat if for all $x \in X, x \mapsto y$, the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. Suppose X is integral and $X \to \operatorname{Spec} k$ is a separated morphism of finite type with k algebraically closed of characteristic 0. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is supposed to be the cotangent space to X at x. We say X is smooth if the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ is constant. We say $X \to Y$ is smooth if it is flat and every fibre (after a base change to an algebraic closure) is smooth. Fix $k = \mathbb{C}$ for now. By a curve we'll mean a connected scheme of dimension 1 which is smooth and projective over $\mathbb{C}$. Let C be a curve. Its genus is $g = h^1(C, \mathcal{O}_C)$. We want to understand maps $C \to \mathbb{P}^r$.
Of course, "understand" is too loose as a term. What we seek are invariants of these maps, and try to understand the relationship between them.
The dimension r of the target is of course an invariant. We can also talk about the degree d of such a map, namely the number of intersections between its image and a general hyperplane.
Let $W^r_d(C)$ be the set of (isomorphism classes of) maps $C \to \mathbb{P}^r$ of degree d which do not factor through a hyperplane.

Theorem 8.1 (Brill-Noether, but actually Griffith-Harris)**.** *If C is chosen generically, then $W^r_d(C) \neq \varnothing$ if and only if $g - (r+1)(g-d+r) \geq 0$.*

How would a proof look like? The idea is the following: First we study something slightly more versatile than smooth curves. Using sheaf cohomology techniques, we can instead study singular curves. We can consider a family $\mathcal{C} \to \operatorname{Spec} R$ of curves (where R is a DVR) whose general fibre is smooth but whose special fibre is a bunch of genus 1 curves (intersecting each other in a non-terrible way). This allows one to do induction on genus; and the genus 1 calculation can be done explicitly.
Let's move on to surfaces, i.e. connected scheme of dimension 2, smooth and projective over $\mathbb{C}$. Two types of easy yet interesting examples are smooth hypersurfaces in $\mathbb{P}^3$ and products of curves. Note that these two classes of surfaces are essentially disjoint except for $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 8.2 (Kodaira-Enriques Classification of Surfaces)**.** *Yeah, we're not gonna state the theorem. It's a classification of surfaces, trust me.*

We are, however, going to introduce some invariants of a surfaces used in the classification. The easiest way to come up with them is to look at $h^i(X, \mathcal{F})$ for "interesting" quasicoherent sheaves. For example, we can look at $\mathcal{F} = \mathcal{O}_X$. We can also look at the cotangent sheaf: Since X is separated, $\Delta : X \to X \times X$ is a closed immersion. So we get a quasicoherent sheaf of ideals $\mathcal{I}$ on $X \times X$. The cotangent sheaf of X is then defined a $\Delta^*(\mathcal{I}/\mathcal{I}^2)$. It turns out that Ω_X is locally free, and its stalks are cotangent spaces.
The classification of surfaces is build using numbers extracted from $h^i(X, \mathcal{F})$ for sheaves $\mathcal{F}$ obtained from Ω_X using some linear algebra constructions.
Let's now talk about moduli spaces 'coz why not. Fix a base scheme S, we get a category (Sch/S) of schemes over S.
Let X be a scheme over S. Then it gives rise to a functor $\hat{h}_X : (\mathsf{Sch}/S)^{\mathrm{op}} \to (\mathsf{Sets})$ taking $T \to S$ to $X(T) = \operatorname{Hom}_S(T, X)$ and $t : T' \to T$ to $\operatorname{Hom}_S(T, X) \to \operatorname{Hom}_S(T', X), f \mapsto f \circ t$.
How do we interpret this functor? If $X = \mathbb{A}^1_k \to \operatorname{Spec} k$, then $X(k)$ naturally corresponds to k, so $\hat{h}_X$ can be interpreted as if it's telling us what the "actual points" of X are. This is why $\hat{h}_X$ is known as the functor of points of X.

Lemma 8.3 (Yoneda Lemma)**.** *X can be "reconstructed" from $\hat{h}_X$. In other words, $X \mapsto \hat{h}_X$ is a fully faithful functor $(\mathsf{Sch}/S) \to [(\mathsf{Sch}/S)^{\mathrm{op}}, (\mathsf{Sets})]$.*

However, we immediately face the following problem: If I write down a functor which I feel is sufficiently geometric, would it be $\hat{h}_X$ for some scheme X? If it is, we say the functor is representable.
Fix $X = \mathbb{P}^n_k$. We can write down the functor $H_X : (\mathsf{Sch}/k)^{\mathrm{op}} \to (\mathsf{Sets})$ which sends T to closed subschemes Z of $X \times_k T$ such that $Z \to X \times_k T \to T$ is flat. If $T = \operatorname{Spec} k$, then everything is flat over T, so $H_X(T)$ is the set of closed subschemes of $X = \mathbb{P}^n_k$.

Theorem 8.4 (Grothendieck)**.** *$H_X = \hat{h}_Y$ for some scheme Y.*

Y is known as the Hilbert scheme of X/S. It's easy to show that Y satisfies the valuative criterion given its existence (we don't actually need to know its construction).
Let's consider another functor $\mathcal{M}_g : (\mathsf{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathsf{Sets})$ sending T to flat morphisms $\mathcal{C} \to T$ with every fibre a smooth curve of genus g. Now, $\mathcal{M}_g$ is not representable, but it turns out to be pretty close to being representable.

One has to invent something called Deligne-Mumford stacks to make sense of something geometric that represents this functor.
In any case, let's pretend $\mathcal{M}_g \to \operatorname{Spec} \mathbb{C}$ gives a space. We can put a $\mathbb{C}$-analytic structure on it, and it turns out that its topological Euler characteristic if $(2-2g)^{-1}\zeta(1-2g)$. See ya.