# Complex Methods *

Zhiyuan Bai

Compiled on May 31, 2021

This document serves as a set of revision materials for the Cambridge Mathematical Tripos Part IB course *Complex Methods* in Lent 2021. However, despite its primary focus, readers should note that it is NOT a verbatim recall of the lectures, since the author might have made further amendments in the content. Therefore, there should always be provisions for errors and typos while this material is being used.

## Contents



*Based on the lectures under the same name taught by Dr. U. Sperhake in Lent 2021.

# 0 Background Material

Something you've already known.

## 0.1 Complex Numbers

A complex number $z \in \mathbb{C}$ can be described by its real and imaginary parts $z = x + iy, x, y \in \mathbb{R}$ or (in the case where its polar form $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. As we have been doing all along, we use the notations $\operatorname{Re} z = x = r\cos\theta, y = \operatorname{Im} z = r\sin\theta, |z| = r = \sqrt{x^2+y^2}, \bar{z} = x - iy, \theta = \arg z$. As trivial as these stuff might be, there is a subtle, yet hugely significant point about the argument $\arg z$. Of course it is funky enough at $z = 0$, but even for general $z$, as one easily spot, it is only defined modulo $2\pi$. Of course, we can have

**Definition 0.1.** The principle argument is the (unique) value of $\arg z$ that lies within $(-\pi, \pi]$.

which is a proper definition, but disregarded the law on adding angles that one might assume. Another tempting thing to do is to write $\arg z = \arctan(y/x)$, which however does not work in many cases since its range is only $(-\pi/2, \pi/2)$ (so whenever $\operatorname{Re} z < 0$ there'll be mess). We will come back to this problematic issue with the argument and what not in a bit because, customarily, we want to write down some properties about complex numbers first.

**Proposition 0.1.**

$$|z|^2 = r^2 = z\bar{z}, \operatorname{Re} z = \frac{z+\bar{z}}{2}, \operatorname{Im} z = \frac{z-\bar{z}}{2i}$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Side note: a less regarded but easily derived version of the triangle inequality is $||z_1| - |z_2|| \le |z_1 + z_2|$.

*Proof.* LOL. □

For $z \in \mathbb{C} \setminus \{1\}$, one can verify that

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

So if $|z| < 1$, then we can set $n \to \infty$ to get

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

which turns out to be the Taylor series of $(1-z)^{-1}$ around 0.

**Definition 0.2.** A set $D \subset \mathbb{C}$ is an open set if

$$\forall z_0 \in D, \exists \epsilon > 0, |z - z_0| < \epsilon \implies z \in D$$

A neighbourhood of $z \in \mathbb{C}$ is an open set $D \subset \mathbb{C}$ that contains $z$.

## 0.2 Trigonometric and Hyperbolic Functions

Note that this is an applied course so we took a lot of things for granted without specifying details.
We have the relation $e^{i\theta} = \cos\theta + i\sin\theta$ which is known as Euler's formula, so

**Definition 0.3.**

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Correspondingly, we have the hyperbolic functions

**Definition 0.4.**

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

It is then clear from the definitions that $\cos(ix) = \cosh x, \cosh(ix) = \cos x$ and $\sin(ix) = i\sinh(x), \sinh(ix) = i\sin x$. Also from these definitions we can readily obtain

$$\cos(\alpha+\beta) = (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta)$$

$$\sin(\alpha+\beta) = (\sin\alpha)(\cos\beta) + (\sin\beta)(\cos\alpha)$$

$$\cosh(\alpha+\beta) = (\cosh\alpha)(\cosh\beta) + (\sinh\alpha)(\sinh\beta)$$

$$\sinh(\alpha+\beta) = (\sinh\alpha)(\cosh\beta) + (\sinh\beta)(\cosh\alpha)$$

## 0.3 Calculus of Real Functions in High Dimensions

As one expect, the calculus in $\mathbb{C}$ has some analogy to the calculus in $\mathbb{R}^2$ in the sense that a complex function $f(z)$ can be regarded as $f(x+iy) = u(x,y) + iv(x,y), u, v : \mathbb{R}^2 \to \mathbb{R}$.

**Definition 0.5.** For $\Omega \subset \mathbb{R}^n$, we denote by $C^m(\Omega)$ the set of functions $f : \Omega \to \mathbb{R}$ whose partial derivatives exist and are continuous up to order $m$.

The existence of partial derivatives in $\mathbb{R}^2$ actually does not mean much.

**Example 0.1.** Take

$$f(x) = \begin{cases} x, \text{ if } y = 0 \\ y, \text{ if } x = 0 \\ \text{Any rubbish, otherwise} \end{cases}$$

Then $\partial f/\partial x = \partial f/\partial y = 1$ at $(0,0)$ but $f$ might even not be continuous at $(0,0)$.

So to say a multivariate function is differentiable, we need some stronger notion.

**Definition 0.6.** Let $\Omega \subset \mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ if there exists a linear form $A : \mathbb{R}^n \to \mathbb{R}$ with

$$f(x + \Delta x) - f(x) = A\Delta x + r(\Delta x)$$

for some $r$ such that $r(\Delta x)/|\Delta x| \to 0$ as $\Delta x \to 0$. If this is the case, then $A$ (which is obviously unique if exists) is called the derivative of $f$ at $x$.
$f$ is continuously differentiable if, in addition, all partial derivatives are continuous.

This definition generalises the usual notion of one-dimensional differentiation.

**Proposition 0.2.** *$f$ is continuously differentiable if and only if all partial derivatives of its partial derivatives are continuous.*

How very nice is this.

*Proof.* GOTO Analysis. □

**Definition 0.7.** A sequence of functions $(f_k) : \Omega \to \mathbb{R}$ is uniformly convergent with limit $f$ if

$$\forall \epsilon > 0, \exists n \in \mathbb{N}, \forall k \geq n, \forall x \in \Omega, |f_k(x) - f(x)| < \epsilon$$

**Proposition 0.3.** *If $f_n \to f$ uniformly, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

*Proof.* Analysis. □

**Example 0.2.** The geometric series converges uniformly for $|z| < 1$.

# 1 Analytic Functions

## 1.1 The Riemann Sphere

We can identify the space of complex numbers $\mathbb{C}$ as $\mathbb{R}^2$, algebraically and topologically, by the identification $\mathbb{C} \ni z = x + iy \leftrightarrow (x, y) \in \mathbb{R}^2$. Algebraically, the addition on $\mathbb{C}$ descends natually from the one on $\mathbb{R}^2$, and the complex multiplication is done by $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. Nothing new here, until we realise that there is a natural geometrical extension of $\mathbb{C}$.

**Definition 1.1.** The extended complex domain is $\mathbb{C}_\infty = \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

The right way to picture this geometrically is to imagine it as the unit 2-sphere whose north pole is $\infty$ and the rest are identified with points on $\mathbb{C}$ via stereographic projection. [1] In this way, it is actually quite intuitive that $\infty$ is not at all anything strange from a geometrical point of view. In fact, it does behave exactly like every other points. In practice, we say $f$ has a property at $z = \infty$ if $z \mapsto f(1/z)$ has that property at $z = 0$.

[1] I'd put a photo here, but I'm lazy. Google it.

## 1.2 Complex Differentiation and Analytic Functions

**Definition 1.2.** Let $U$ be open in $\mathbb{C}$. A function $f : \mathbb{U} \to \mathbb{C}$ is differentiable at $z \in U$ if the limit

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists.

*Remark.* When we say this limit exists, we mean that it should exist whichever direction we choose to approach the limit $\delta z \to 0$.

**Definition 1.3.** A complex function $f : U \to \mathbb{C}$ is analytic at $z \in U$ if $z$ has a neighbourhood $D \subset U$ such that $f$ is differentiable at every $w \in D$.

We will see later that analytic functions have many nice properties than you'd ever imagine. For example, analytic functions are automatically infinitely differentiable and, even better, has a Taylor series locally which behaves as you want it to be. These are very obviously false in the case of $\mathbb{R}$. So there are indeed a lot of differences in the calculus of $\mathbb{R}$ and $\mathbb{C}$.
Fortunately, we can still get some tools in $\mathbb{R}^2$ to work in $\mathbb{C}$. Write $f(x+iy) = u(x,y) + iv(x,y)$, then if we approach the limit from the real line ("$\delta z = \delta x$"), then

$$\begin{aligned}\lim_{\delta x \to 0} \frac{f(z+\delta x) - f(z)}{\delta x} &= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned}$$

Similarly, if we approach along the imaginary axis,

$$\lim_{\delta y \to 0} \frac{f(z+\delta y) - f(z)}{\delta x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

But the two limits must equal! So we obtain the Cauchy-Riemann equations:

**Proposition 1.1.** *A complex function $f(z) = u(x,y) + iv(x,y)$ is (complex) differentiable if and only if $u, v$ are differentiable (in the multivariate sense) and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Note that the "if" parts is not necessarily true if we merely require the partial derivatives of $u, v$ exist.

*Proof.* The "only if" parts are illistrated in above discussion. For the "if" parts see Complex Analysis. ☐

Adding in what we already know about multivariate differentiability,

**Corollary 1.2.** *If $f(z) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations and the partial derivatives of $u, v$ are continuous in a neighbourhood of $z$, then $f$ is (complex) differentiable.*

There is an alternative viewpoint to this: We know $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$, so a complex function $f(z) = f(x + iy) = f(x, y)$ can be written in the form $f = f(z, \bar{z})$. Then, as one can check, $f$ is differentiable if and only if $\partial f/\partial \bar{z} = 0$.

Usual rules we have seen in real differentiation still holds.

**Proposition 1.3.** *1.* $(f + g)' = f' + g'$.
*2.* $(fg)' = fg' + f'g$.
*3.* $(f \circ g)' = (f' \circ g)g'$.

*Proof.* 1 is obvious. For 2, let

$$\varpi = \frac{f(z+h) - f(z)}{h} - f'(z), w = \frac{g(z+h) - g(z)}{h} - g'(z)$$

Then

$$\begin{aligned}(gf)'(z) &= \lim_{h\to 0}((g'(z) + w)f(z) + (f'(z) + \varpi)g(z) + (g'(z) + w)(f'(z) + \varpi)h) \\ &= (fg' + f'g)(z)\end{aligned}$$

3 is exercise. □

**Example 1.1.** 1. $f(z) = z$ is differentiable over all of $\mathbb{C}$ and has $f'(z) \equiv 1$. We say functions differentiable in all of $\mathbb{C}$ entire.
2. $f(z) = \exp z$ is also entire with derivative itself.
3. $f(z) = z^n$ (hence any polynomial) is entire as well with $f'(z) = nz^{n-1}$.
4. $f(z) = 1/z$ is analytic in $\mathbb{C} \setminus \{0\}$ where it has derivative $-z^{-2}$. Combining this with product and chain rules gives

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

So rational functions are also analytic where it is well-defined.
5. As linear combinations of analytic functions $\cos, \sin, \sinh, \cosh$ are all entire with

$$\sin' = \cos, \cos' = -\sin, \cosh' = \sinh, \sinh' = \cosh$$

6. $(\tan') = 1/\cos^2$ where it is well-defined.
7. One can show that log is analytic and $\log'(z) = 1/z$ for the principal branch except at the nonnegative real line.

We can check the Cauchy-Riemann equations on these examples as well. When calculating derivative in practice, if we are sure about the function being differentiable, then we can choose any direction that we find comfortable with to apply the limit. Or, alternatively, we can simply used the formulas of the limit when approaching from real/imaginary axis that we obtained earlier when dealing with Cauchy-Riemann equations.

Of course, we also want some examples of non-analytic functions.

**Example 1.2** (Non-examples)**.** 1. $f(z) = \operatorname{Re} z$ is nowhere analytic since it does not satisfy Cauchy-Riemann.
2. $f(z) = |z|$ is nowhere analytic because of the same reason.
3. $f(z) = \bar{z}$ is not analytic either.
4. $f(z) = |z|^2$ satisfies Cauchy-Riemann at $z = 0$ but nowhere else, so $f$ is nowhere analytic.

## 1.3 Harmonic Functions

**Definition 1.4.** Let $U \subset \mathbb{R}^2$ be open. A function $f(x, y) : U \to \mathbb{R}$ is harmonic if $f_{xx} + f_{yy} = 0$.

**Definition 1.5.** If $u, v : \mathbb{U} \to \mathbb{R}$ satisfy Cauchy-Riemann, then they are called harmonic conjugates.

So the real and imaginary parts of an analytic function are harmonic conjugates. Note that if we know one of the harmonic conjugates, then we can use Cauchy-Riemann to determine the other (given nice enough smoothness criteria) up to a constant.

**Example 1.3.** If $u, v$ are harmonic conjugates and $u = x^2+y^2$, then $v = 2xy$ up to a constant. In this case, one can easily observe that $f = u+iv = (x+iy)^2 = z^2$ (up to a constant) is actually an analytic function.

**Proposition 1.4.** *If $u, v$ are harmonic conjugates, then they are both harmonic.*

Hence the real and imaginary parts of an analytic function are harmonic.

*Proof.* Just apply Cauchy-Riemann. □

## 1.4 Multi-valued Functions and Branch Cuts

Recall that $\log z = \log|z| + \arg z$ is only defined modulo $2\pi i$. The argument always does not behave as bad as such. If we walk along, say, a circle that does not wrap along 0, then we can always find a branch such that arg behaves well. But if our loop wraps around 0 and assume continuity, then something funny will happen: When we go back to where we started, arg changed by a nonzero multiple of $2\pi$. As you can expect, this gets something to do with the point 0.

**Definition 1.6.** A branch point of a function $f$ is a point $z_0$ that "cannot be encircled by a curve $C$ such that $f$ is single-valued and continuous along $C$".

**Example 1.4.** 1. $f(z) = \log(z - a)$ has a branch point at $z = a$.
2. $f(z) = \log(z - 1) - \log(z + 1)$ has two points $z = \pm 1$.
3. $f(z) = z^\alpha = r^\alpha e^{i\alpha\theta}$ has a branch point at 0 iff $\alpha \notin \mathbb{Z}$.
4. $f(z) = \log(z)$ has a branch point at $z = \infty$ as well. The same can be said about $f(z) = z^\alpha$.
5. (Non-example) $f(z) = \log(z - 1) - \log(z + 1)$ actually does not branch at $\infty$.

Then, how do we handle this mess? Via branch cuts.

**Example 1.5.** For $\log z = \log|z| + i \arg z$, we draw a red line (branch cut) at $i\mathbb{R}_{\leq 0}$. Then we can define log as the principal branch of it, which is continuous along any curve not crossing the branch cut. Then, log is analytic in the region of $\mathbb{C}$ removing the branch cut with derivative $(\log(z))' = 1/z$.
Since what we want is just that the curve does not encircle the origin, there can be other possible branch cuts, as long as it starts from 0 and extends to $\infty$. This even includes non-straight lines (don't let it intersect itself though).

So we get three interrelated branch-thingies for multi-valued functions: The branch point: a point we cannot enclose; The branch cut: a red line we cannot cross; The branch: the choice of values $f(z)$ on the complex plane with the branch removed such that it behaves well (e.g. analytic) there.
We always have the freedom to choose branch cuts and branches, just like the log case we just considered, but not branch points. If we specify the function, branch cut and its value at one point, then (usually) the function is determined.

## 1.5 Riemann Surfaces (non-examinable)

The branch thingies still feel like a mess, right? There is a much much better way to do this, namely Riemann surfaces. What we we do (in this particular occasion) is to regard the branches of $f(z)$ as copies of (subsets) of $\mathbb{C}$ stacked on each other. For the case of log, imagine twisting the complex plane with one (fixed) branch cut removed, duplicate infinitely many copies indexed by $\mathbb{Z}$, and glue them together along the respective cuts. Then moving around 0 will bring you to another "level". This allows us to regard log as an analytic function from this surface to $\mathbb{C}$.

## 1.6 Multiple Branch Cuts

**Example 1.6.** 1. Take $g(z) = (z(z-1))^{1/3}$ has branch points, as you expect, at $z = 0$ and $z = 1$. Then we can take a 2-segnent branch cut $(-\infty, 0] \cup [1, \infty)$.
2. $f(z) = \log(z-1) - \log(z+1)$ has branch points $z = \pm 1$, and we can choose a branch cut $(-\infty, -1] \cup [1, \infty)$. Actually, we can also use a simpler branch cut $[-1, 1]$.

The second example above prompts another question: Which branch is better? Well, there is an obvious problem with using $(-\infty, -1] \cup [1, \infty)$, namely you are prohibited from curves that encloses both 1 and $-1$, which can be perfectly legitimate (check this!). On the other hand, using $[-1, 1]$ does not handle the curves between $-1$ and 1. So choosing which branch cut depends on what you want to study. On the Riemann sphere, you can see they are basically topologically the same.
Incidentally, could we have used $[0, 1]$ as a branch cut for $g(z) = (z(z-1))^{1/3}$? No, because $g$ has a branch point at $\infty$ while $f$ does not.

**Proposition 1.5.** *Suppose $f(z)$ has branch points $z_1, z_2, \ldots \in \mathbb{C}_\infty$. Suppose we have a set of (simple) curves satisfying:*
*1. Every branch point must have a curve ending on it.*
*2. Both ends of each curve must end on branch points.*
*Then this set of curves can be a complete branch cut of $f$.*

*Proof.* Well. □

## 1.7 Möbius Maps

**Definition 1.7.** A Möbius map is a map $m : \mathbb{C}_\infty \to \mathbb{C}_\infty$ that sends $z$ to $(az+b)/(cz+d)$ ($m(\infty) = a/c, m(-d/c) = \infty$) where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

It is easy to see that $m$ is bijective with its inverse given by the Möbius map $z \mapsto (-dz+b)/(cz-a)$.

**Definition 1.8.** A circline is either a circle or a line on the plane, which is basically circle on the Riemann sphere.

**Proposition 1.6.** *Any circline in* $\mathbb{C}$ *is given by* $\{z \in \mathbb{C} : |z - z_1| = \lambda |z - z_2|\}$ *where* $z_1 \neq z_2$ *and* $\lambda > 0$.

*Proof.* Elementary geometry. □

**Proposition 1.7.** *A Möbius map maps a circline to a circline.*

*Proof.* Calculations. □

A circline is determined by any 3 distinct points on it (which can include $\infty$).

**Proposition 1.8.** *Let* $\alpha, \beta, \gamma \in \mathbb{C}_\infty$ *be distinct and* $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in \mathbb{C}_\infty$ *distinct. Then there exists a (unique) Möbius map sending* $\alpha \mapsto \tilde{\alpha}, \beta \mapsto \tilde{\beta}, \gamma \mapsto \tilde{\gamma}$.

*Proof.* Let $M_{a,b,c}$ be the Möbius map sending $z \mapsto (\beta-\gamma)(z-\alpha)/((\beta-\alpha)(z-\gamma))$. Then $M_{a,b,c}$ sends $a$ to 0, $b$ to 1 and $\gamma$ to $\infty$. $M^{-1}_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}} \circ M_{\alpha,\beta,\gamma}$ then does the trick. Uniqueness is obvious. □

## 1.8 Conformal Mappings

There are some simple maps on the complex plane. We want to study their geometrical meanings.

**Example 1.7.** 1. Take $z \mapsto z^2$, which squares the modulus and double the argument. So in general, it pushes the points away from the unit circle, either to 0 or to $\infty$. Likewise for $z \mapsto z^\alpha$ for $\alpha > 1$ (choose branch if appropriate).
2. Take (a branch of) $z \mapsto z^{1/2}$, then it halves the argument and pulls everything towards the unit circle. Similar for $z^\alpha$ for (branches of) $\alpha \in (0,1)$.
3. The exponential function exp relate the real part of the input argument to the modulus of the output and the imaginary part of the input argument to the argument of the output. So it maps rectangles (parallel to the axes) on the complex plane to sections of annuli.
4. So a branch of log maps sections of annuli (that avoids the branch cut) to rectangles.

**Definition 1.9.** $f : U \to W$ (where $U, W \subset \mathbb{C}$ are open) is conformal if $f$ is analytic and has nonzero derivative on $U$. If $f$ is also bijective, it is called a conformal equivalence.

**Proposition 1.9.** *A conformal map preserves the angle (magnitude and direction) between two intersecting curves (that are differentiable at the intersection).*

*Proof.* Let $z_1(t)$ be a curve and $z_0 = z_1(t_0)$, then $\arg z_1'(t_0)$ is the angle $z_1$ made with the real direction at $t = t_0$. So if $f$ is conformal, then $\zeta_1(t) = f(z_1(t))$ has tangent

$$\zeta_1'(t_0) = \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{t=t_0} \left.\frac{\mathrm{d}z_1}{\mathrm{d}t}\right|_{t=t_0} = f'(z_0)z_1'(t_0)$$

So the angle $\zeta_1(t)$ made with the real direction at $t = t_0$ is

$$\vartheta = \arg(\zeta_1'(t_0)) = \arg(z_1'(t_0) f'(z_0)) = \theta + \arg f'(z_0)$$

So if $z_2(t)$ is a second curve that also goes through $z_0$, then the angle it made with the real direction at $z_0$ also just gets rotated by $\arg f'(z_0)$, so as a result we have the proposition. □

*Remark.* The converse of the proposition is also true, but we are not gonna prove it here.

If we are given an analytic map $f : U \to \mathbb{C}_\infty$, we usually can understand its image by looking at $f(\partial U) = \partial f(U)$ and taking an example point in the interior to know the orientation of its interior.

**Example 1.8.** 1. If $f(z) = az + b, a, b \in \mathbb{C}$ with $a \neq 0$, then $f$ is conformal in all of $\mathbb{C}$ and its action is to (anti-clockwisely) rotate $z$ by an angle $\arg a$, rescale by $|a|$ and translate by $b$.
2. $f(z) = z^2$ is conformal except at $z = 0$. It maps the quarter-circle $U = \{z \in \mathbb{C} : 0 < |z| < 1, 0 < \arg z < \pi/2\}$ to the half-circle $V = \{z \in \mathbb{C} : 0 < |z| < 1, 0 < \arg z < \pi\}$. Observe that the right angles at $z = 1, i$ are preserved at $w = 1, -1$ when we map $\bar{U}$ to $\bar{V}$ via $f$. However, then angle at 0 is not preserved. It is, however, doubled (this is, of course, part of a general phenomenon – see example sheet).
3. Often, we know $U, V$ and wish to find an analytic (or conformal) map $f$ taking $U$ to $W$. For example, we might take $U = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ and $W = \{w \in \mathbb{C} : -\pi/4 < \arg w < \pi/4\}$. Our first step is to halve the angle $z \mapsto z^{1/2}$ (with branch chosen as, say, $\arg z \in (-\pi/2, 3\pi/2)$) which brings us to the sector $\{z \in \mathbb{C} : \pi/4 < \arg z < 3\pi/4\}$. A clockwise rotation by $\pi/2$ via $z \mapsto -iz$ brings us to $W$. Composing them together gives $f(z) = -iz^{1/2}$ (with the branch choice discussed earlier).
4. $f(z) = e^z$ is conformal everywhere and, as we have seen, maps rectangles to sectors of annuli. And with a suitable choice of branch, an logarithm does the reverse.
5. Recall that Möbius maps take circlines to circlines. Consider $f(z) = (z - 1)/(z + 1)$ and the domain $U = \{z \in \mathbb{C} : |z| < 1\}$. Then $f(\partial U) = i\mathbb{R} \cup \{\infty\}$ by looking at the image of $-1, i, 1$. Also $f(0) = -1$, so $f(U) = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. In fact, if we let

$$U_1 = \{z \in \mathbb{C} : |z| > 1, \arg z \in (0, \pi/2)\}$$

$$U_2 = \{z \in \mathbb{C} : 0 < |z| < 1, \arg z \in (0, \pi/2)\}$$

$$U_3 = \{z \in \mathbb{C} : 0 < |z| < 1, \arg z \in (\pi/2, \pi)\}$$

$$U_4 = \{z \in \mathbb{C} : |z| > 1, \arg z \in (\pi/2, \pi)\}$$

then $f(U_i) = U_{i+1}$ (index modulo 4). The same is true if we reflect everything along the real axis.
6. The same sort of analysis can be done for $f(z) = 1/z$.
7. We want to map the upper half disk $\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ to the disk $\{z \in \mathbb{C} : |z| < 1\}$. Notably, $z \mapsto z^2$ does not work. We start with $z \mapsto (z - 1)/(z + 1)$ which maps the upper half-disk to the second quadrant $\{x + iy : x < 0, y > 0\}$. Then $z \mapsto z^2$ actually is helpful since it brings us to

$\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. Rotating $z \mapsto iz$ leads to the region $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ which can be easily mapped to the unit disk by, again, $z \mapsto (z-1)/(z+1)$.

Let $U \subset \mathbb{R}^2$ be a "tricky" domain, and $V \subset \mathbb{R}$ be a "nice" domain. Write $z = x + iy \in U$ and $\zeta = u + iv \in V$. Suppose we already know a conformal equivalence $f : U \to V$ with $f = u+iv$. Let $G : V \to \mathbb{C}, G(\zeta) = \Phi(u,v)+i\Psi(u,v)$ be analytic. Then $\Phi$ and $\Psi$ are harmonic. Clearly, $g = G \circ f$ is also analytic. In particular, if $g(z) = \phi(x,y) + i\psi(x,y)$ then $\phi, \psi$ are both harmonic.
This gives a way to construct harmonic functions on a "tricky" domain from known harmonic functions on a "nice" domain that receives an analytic mapping from it: First, find simple domain $V$ and conformal equivalence $f : U \to V$ with $f = u + iv$. Then, translate the boundary condition, say $\phi|_{\partial U} = \phi_0$ into conditions $\Phi|_{\partial V} = \Phi_0$ via $f$. If $V$ is nice enough, then we can solve $\nabla^2\Phi = 0$ subject to this boundary condition. Then $\phi(x,y) = \Phi(u(x,y), v(x,y))$ solves $\nabla^2\phi = 0$ on $U$ subject to the boundary condition.

**Example 1.9.** Suppose we want to solve $\nabla^2\phi = 0$ on the first quadrant $U = \{(x,y) : x, y > 0\}$ subject to $\phi(x,0) = 0, \phi(0,y) = 1$. So $f(z) = \log z$ (principal branch) maps the quadrant to the strip $V = \{x + iy : y \in (0, \pi/2)\}$. We then want to solve $\nabla^2\Phi = 0$ subject to $\Phi(u,0) = 0, \Phi(u,\pi/2) = 1$. This is easy: Taking $\Phi(u,v) = 2v/\pi$ works. Push it back to $U$ gives $\phi(x,y) = (2/\pi)\arg z = (2/\pi)\arctan(y/x)$.

# 2 Contour Integration and Cauchy's Theorem

## 2.1 Contours and Integrals

In complex differentiation, we had infinitely many directions to consider the limit. Correspondingly ,in complex integration, we have infinitely many paths from two points on which we might choose to integrate our function. However, in contrast to differentiation, we do not demand path independence.

**Definition 2.1.** A curve in $\mathbb{C}$ is a continuous map $\gamma : [0,1] \to \mathbb{C}$.

Of course, we can change $[0,1]$ to other intervals. Also, we sometimes denote by $\gamma$ the image of it.

**Definition 2.2.** A closed curve is a curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = \gamma(1)$.
A simple curve is a curve that is injective except maybe at endpoints.

**Definition 2.3.** A contour is a piecewise differentiable curve.

We write $-\gamma$ as the "reverse" of $\gamma$ i.e. $(-\gamma)(t) = \gamma(1-t)$. We can also join two curves

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(2t), \text{ for } t \leq 1/2 \\ \gamma_2(2t-1), \text{ for } t \geq 1/2 \end{cases}$$

Given that $\gamma_1(1) = \gamma_2(0)$. One can verify that these two operations are always well-defined when dealing with curves or contours.

**Definition 2.4.** The contour integral of a function $f$ along the contour $\gamma$ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^1 f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

This is very similar to integrating a vector field. By the chain rule, the value of the integral is independent of the specific parameterisation we use.
An alternative way to characterise it is the following: We dissect the interval $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_n = 1$ and write $z_i = \gamma(t_i)$. Let $\delta t_k = t_{k+1} - t_k, \delta z_k = z_{k+1} - z_k$, then what we want is

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\max_k \delta t_k \to 0} \sum_{k=0}^{n-1} f(z_n)\delta z_n$$

which, as one can easily see, is equivalent to the definition above (assuming everything is nice and integrable).

**Example 2.1.** Let $f(z) = 1/z$ and $\gamma_1, \gamma_2$ be unit half-circles $\gamma_i(\theta) = e^{i\theta}$ over $[\pi, 0]$ for $i = 1$ and $[-\pi, 0]$ for $i = 2$. Then

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = -i\pi, \int_{\gamma_2} f(z)\,\mathrm{d}z = i\pi$$

**Proposition 2.1.** *We have the following rules of integration:*
*1.*

$$\int_{\gamma_1+\gamma_2} = \int_{\gamma_1} + \int_{\gamma_2}$$

*2.*

$$\int_{-\gamma} = -\int_\gamma$$

*3. If $f$ is differentiable along a contour $\gamma$ from $a$ to $b$, then*

$$\int_\gamma f'(z)\,\mathrm{d}z = f(b) - f(a)$$

*4. Integration by parts and substitution works exactly the same rule as usual.*
*5. The length of a curve $\gamma$ is*

$$L = \int_\gamma |\mathrm{d}z| = \int_0^1 |\gamma'(t)|\,\mathrm{d}t$$

*6. If we are integrating over a closed contour, then it might depend on the direction but does not depend on the specific starting point.*

By convention, a closed curve is taken counterclockwise unless otherwise specified.

*Proof.* Obvious. □

*Remark.* The third rule above implies that if a function has an antiderivative, then its integral is independent of the path. This, however, does not contradict out example above. In fact, this shows exactly that $z \mapsto 1/z$ does not have an antiderivative on the unit circle.

**Definition 2.5.** An open set $D \subset \mathbb{C}$ is a (connected) domain if it is path-connected, i.e. for any $z_1, z_2 \in D$, there exists a curve connecting them. $D$ is a simply connected domain if it is a connected domain and every curve in $D$ encloses points in $D$.

## 2.2 Cauchy's Theorem

**Theorem 2.2.** *If $f$ is analytic on a simply connected domain $D$ and $\gamma$ is a closed contour in $D$, then*

$$\oint_\gamma f(z)\,\mathrm{d}z$$

*Proof.* Naturally we don't prove the full version of it as it is an applied course. Nonetheless, we are able to give the proof to a weaker statement assuming Green's theorem.
Let $M$ be the region bounded by $\gamma$. Write $f(z) = u(x,y)+iv(x,y)$ and assuming $u, v$ are both continuously differentiable,

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\oint_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y)\\ &= \iint_M \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y + i\iint_M \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y\\ &= 0\end{aligned}$$

by Cauchy-Riemann equations. □

## 2.3 Deforming Contours

**Proposition 2.3.** *Let $\gamma_1, \gamma_2$ be two contours from $a$ to $b$ and suppose $f(z)$ is analytic on a simply connected domain containing both contours, then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

*Proof.* $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$ is a closed curve. □

All these hints that any analytic function has analogous characteristic of an exact differential when the domain =is nice. Indeed, we can verify that it is a closed differential by Cauchy-Riemann equations. Its antiderivative in a simply-connected domain can be given by

$$F(z) = \int_{\gamma_z} f(w)\,\mathrm{d}w$$

where $\gamma_z$ is a path from a fixed $z_0$ to $z$. This is well-defined by the preceding proposition.
Cauchy's theorem also allows us to deform contours. If we can continuously deform a closed contour $\gamma_1$ to another closed contour $\gamma_2$, then by cutting a gap between them (to form a closed contour out of them) and sending the width of that gap to 0 gives

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z$$

## 2.4 Cauchy's Integral Formula

**Theorem 2.4** (Cauchy's Integral Formula)**.** *Let $f$ be an analytic function on a domain $D$ and $z_0 \in D$. Suppose $\gamma$ is a simplex closed contour in $D$ that* *encircles $z_0$, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z, f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

Note that the formula does not hold in general for $z_0$ outside the region bounded by $\gamma$.
In the proof, we assumed that we can switch orders of certain limiting operations. These can be justified using tools from analysis, which is not in the scope of this course.

*Proof.* Let $\gamma_\epsilon$ be a counterclockwise circle of radius $\epsilon > 0$ around $z_0$, then $f(z)/(z - z_0)$ is analytic between $\gamma$ and $\gamma_\epsilon$. Then

$$\oint_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = \oint_{\gamma_\epsilon} \frac{f(z)}{z-z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{f(z_0+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,\mathrm{d}\theta \to i2\pi f(z_0)$$

Differentiate the formula $n$ times (we can do this this the integral has to be differentiable) gives the second formula. □

Consequently, knowing how $f$ behaves on $\gamma$ gives $f(z)$ for everypoint $z$ encircled by $\gamma$. In particular, this is an alternative proof of the uniqueness of the solution to a Dirichlet problem. Also, the theorem implies that any analytic $f$ is, in fact, infinitely differentiable. We can actually get much more stuff out of this formula.

**Theorem 2.5** (Liouville's Theorem)**.** *A bounded entire function is constant.*

*Proof.* Suppose there exists $c_0 \in \mathbb{R}$ such that $|f| \leq c_0$ over $\mathbb{C}$. For any $z_0 \in \mathbb{C}, r > 0$, let $\gamma_r$ be a counterclockwise circular contuor of radius $r$ around $z_0$, then

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\right| \leq \frac{1}{2\pi}\oint_{\gamma_r} \frac{c_0}{r^2}\,\mathrm{d}z = \frac{c_0}{r} \to 0$$

as $r \to \infty$. Consequently $f' \equiv 0$, hence $f$ is constant. □

# 3 Taylor Series, Laurent Series and Singularities

## 3.1 Taylor Series and Laurent Series

Recall that in the real numbers, for some $f$ infinitely differentiable at $x_0$, we can form the formal Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

which might or might not converge nicely. In the complex numbers, however, the situation is much nicer.

**Proposition 3.1.** *Let $f(z)$ be analytic on an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. Then $f$ admits a Laurent series*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n$$

*that converges to $f(z)$ on $A$.*
*If $f$ is analytic at some $z_0$, it has a Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*that converges to $f(z)$ in a sufficiently small open disc centered at $z_0$.*

*Proof.* WLOG $z_0 = 0$. Let $z \in \mathbb{C}$ with $R_1 < r_1 < |z| < r_2 < R_2$. Let $\gamma_1, \gamma_2$ be counterclockwise circular contours with radii $r_1, r_2$ around $z_0 = 0$. Join them together via a small gap (with suitably chosen position) to form a closed contour $\gamma$ whose interior contains $z$. Cauchy integral formula (sending the width of the gap to 0) then gives

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

On $\gamma_1, |\zeta/z| < 1$, so

$$\begin{aligned} -\frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta &= \frac{1}{2\pi i z} \oint_{\gamma_1} \frac{f(\zeta)}{1 - \zeta/z} \, \mathrm{d}\zeta \\ &= \frac{1}{2\pi i z} \oint_{\gamma_1} f(\zeta) \sum_{m=0}^{\infty} \left(\frac{\zeta}{z}\right)^m \mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \sum_{m=0}^{\infty} z^{-m-1} \oint_{\gamma_1} f(\zeta) \zeta^m \, \mathrm{d}\zeta \\ &= \sum_{n=-\infty}^{-1} a_n z^n \\ \text{where } a_n &= \frac{1}{2\pi i} \oint_{\gamma_1} f(\zeta) \zeta^{-n-1} \, \mathrm{d}\zeta \end{aligned}$$

(For full rigour, refer to the theory of uniform convergence.) Similarly, we can evaluate the other part of the expression (but this time using $|z/\zeta| < 1$ on $\gamma_2$) to get the terms of nonnegative indices. In this case, we have

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_2} f(\zeta) \zeta^{-n-1} \, \mathrm{d}\zeta$$

Combining both gives the Laurent series. Exactly the same idea gives the Taylor series whose coefficient has the same form as above (with the path encircling $z_0$ chosen such that $f$ is analytic on a disk around $z_0$ containing it). It also has the form we are familiar with, i.e. $a_n = f^{(n)}(z_0)/n!$, by either Cauchy integral formula or general theory about power series. □

*Remark.* One can show that the Laurent series (to which Taylor series is a special case) is unique.

**Example 3.1.** Take $z_0 = 0$.
1. $f(z) = e^z/z^3$ has Laurent series

$$e^z/z^3 = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!}$$

2. $f(z) = e^{1/z}$ has Laurent series

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

3. $f(z) = (z-a)^{-1}$ has Taylor series

$$\frac{1}{z-a} = \sum_{n=0}^{\infty} \left( -\frac{1}{a^{n+1}} \right) z^n$$

for $|z| < |a|$ and

$$\frac{1}{z-a} = \sum_{n=-\infty}^{-1} a^{-n-1} z^n$$

for $|z| > |a|$.
4. Now take $z_0 = 1$ and consider $f(z) = e^z/(z^2-1)$. Write $\zeta = z - z_0 = z - 1$, then

$$\begin{aligned} f &= e\frac{e^\zeta}{\zeta(\zeta+2)} = e\frac{e^\zeta}{2\zeta}\frac{1}{1+\zeta/2} = \frac{e}{2\zeta}\left(\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(-\frac{\zeta}{2}\right)^n\right) \\ &= \frac{e}{2}\left(\frac{1}{z-1} + \frac{1}{2} + \cdots\right) \end{aligned}$$

5. $f(z) = z^{-1/2}$ cannot be expanded as a Laurent series around 0 since 0 is its branch point.

## 3.2 Zeros and Singularities

**Theorem 3.2.** *Any polynomial $p(z)$ of degree $n \geq 1$ can be factorised into exactly $n$ linear factors in $\mathbb{C}$.*

**Definition 3.1.** The zeros of a function are the points where it vanishes (duh). A zero $z_0$ of a function $f$ analytic at $z_0$ is of order $n$ if its Taylor expansion $f(z) = \sum_k a_k (z-z_0)^k$ has its first nonzero coefficient at $a_n$. Equivalently, $f(z) = (z-z_0)^n \tilde{f}(z)$ for some $\tilde{f}$ analytic at $z_0$ and $\tilde{f}(z_0) \neq 0$.
A simple zero is a zero of order 1.

**Example 3.2.** 1. $f(z) = z^3 + iz^2 + z + i = (z-i)(z+i)^2$ has a simple zero at $i$ and a zero of order 2 ("double zero") at $-i$.
2. $f(z) = \sinh z$ has simple zeros at $z = in\pi, n \in \mathbb{Z}$.
3. $f(z) = (\sinh z)^3 = \sinh^3 z$ has zeros of order 3 ("triple zero") at $z = in\pi, n \in \mathbb{Z}$.

**Definition 3.2.** A singularity of $f$ is a point $z_0$ where it is undefined or not analytic.
$z_0$ is an isolated singularity if $f$ is analytic on a punctured disk centered at it; Otherwise it is a non-isolated singularity.

**Example 3.3.** 1. $f(z) = 1/\sinh z$ has isolated singularities at $in\pi, n \in \mathbb{Z}$.
2. $f(z) = 1/(\sinh(1/z))$ has isolated singularities at $z = 1/(in\pi), n \in \mathbb{Z} \setminus \{0\}$ and a non-isolated singularity at 0.

3. $f(z) = 1/\sinh z$ has a non-isolated singularity at $\infty$.
4. $f(z) = \log z$ (any branch) has a non-isolated singularity at 0. Similarly, branch points of branched functions are non-isolated singularities.

For isolated singularities, $f$ can be expanded as a Laurent series near it.

**Definition 3.3.** Suppose $z_0$ is an isolated singularity of $f$ and $f$ has a Laurent series $\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ near it.
If $\forall n < 0, a_n = 0$, then $z_0$ is a removable singularity. Otherwise, if $\exists N \in \mathbb{N}$ such that $a_{-N} \neq 0$ and $\forall n < -N, a_n = 0$, then we say $f$ has a pole of order $N$ at $z_0$. If no such $N$ exists, then we say $z_0$ is an essential singularity.

If $z_0$ is a removable singularity, then we can naturally extend $f$ to $z_0$ analytically as the Laurent series which is just a Taylor series.

**Example 3.4.** 1. $f(z) = 1/z$ has a pole of order 1 ("simple pole") at 0.
2. $f(z) = (\cos z)/z$ has a simple pole at 0.
3. $f(z) = z^2/((z-1)^3(z-i)^2)$ has a double pole at $z = i$.
4. If $f$ has a zero of order $n$ at $z_0$, then $1/f$ has a pole of order $n$ at $z_0$.
5. $f(z) = z^2$ has a double pole at $\infty$.
6. $f(z) = e^{1/z}$ has an essential singularity at 0.
7. $f(z) = \sin(1/z)$ has an essential singularity at 0.
8. $f(z) = (e^z - 1)/z$ has a removable singularity at 0.
9. $f(z) = (\sin z)/z$ has a removable singularity at 0.
10. Let $P, Q$ be polynomials with zeros at $z_0$ of order $m, n$ respectively. Let $f = P/Q$, then $z_0$ is a removable singularity of $f$ if $m \geq n$ and is a pole of order $n - m$ if $n > m$.

**Proposition 3.3.** *Suppose $f(z)$ have an essential singularity at $z_0$, then any neighbourhood $D$ of $z_0$ has $|\mathbb{C} \setminus f(D)| \leq 1$.*

Obviously we are not gonna prove this.

## 3.3 Residues

**Definition 3.4.** Suppose $f$ has an isolated singularity at $z_0$. The residue $\operatorname{Res}_{z_0} f$ of a function $f$ at $z_0$ is the coefficient of $(z-z_0)^{-1}$ in the Laurent series of $f$ near $z_0$.

**Proposition 3.4.** *Suppose $f$ has a pole of order $n$ at $z = z_0$, then*

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}((z-z_0)^n f(z))$$

*In particular, if $z_0$ is a simple pole, then* $\operatorname{Res}_{z_0} f = \lim_{z\to z_0}((z-z_0)f(z))$.

*Proof.* Obvious. □

**Example 3.5.** 1. $f(z) = e^z/z^3 = z^{-3} + z^{-2} + (1/2)z^{-1} + 1/6 + \cdots$, so $\operatorname{Res}_0 f = 1/2$.
2. $f(z) = e^z/(z^2-1)$ has a simple pole at $z = 1$, so $\operatorname{Res}_1 f = e/2$ by the proposition.
3. $f(z) = 1/(z^8 - w^8)$ has a simple pole at $w$, so by L'Hopital's rule $\operatorname{Res}_w f = 1/(8w^7)$.

4. $f(z) = \sin(\pi z)$ has simple zeros at $z = ni, n \in \mathbb{Z}$ and we have $\operatorname{Res}_{ni} 1/f = 1/(\pi \cosh(in\pi)) = (-1)^n/\pi$.
5. $f(z) = \sinh^3 z$ has simple zero at, say, $z = i\pi$. We have $\operatorname{Res}_{i\pi} 1/f = 1/2$ by series expansion.

Why do we care about residues?

**Theorem 3.5.** *Let $\gamma$ be a simple closed contour in counterclockwise direction and $f$ analytic inside $\gamma$ except for an isolated singularity $z_0$, then*

$$\oint_\gamma f(z)\,\mathrm{d}z = i2\pi \operatorname*{Res}_{z=z_0} f(z)$$

*Proof.* Expand $f$ into Laurent series and switch summation and integral. □

# 4 Calculus of Residues

## 4.1 The Residue Theorem in Full Form

**Theorem 4.1.** *Suppose $f$ is analytic in a simply connected domain $D$ except a finite number of isolated singularities $z_1, \ldots, z_k$. Let $\gamma$ be a simple closed counterclockwise contour in $D$ that encircles all $z_1, \ldots, z_k$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = i2\pi \sum_{m=1}^{k} \operatorname*{Res}_{z=z_m} f(z)$$

*Proof.* Followed fairly easy from Theorem 3.5. □

## 4.2 Integrals along the Real Axis

Often, we want to use residue theorem on integrals that do not involve a closed contour. For example, we might be only interested in the integration over part of real axis.

**Example 4.1.** 1. Suppose we want to integrate

$$2I = \int_0^\infty \frac{\mathrm{d}x}{1+x^2}$$

Consider the contour $\gamma$ that is the boundary of the upper half-disk of radius $R$. It can be parameterised by $\gamma = \gamma_0 + \gamma_R$ where $\gamma_0(t) = t, t \in [-R, R]$ and $\gamma_R(t) =$. Now

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{1+t^2} = \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_0} \frac{\mathrm{d}z}{1+z^2} \\ &= \frac{1}{2}\lim_{R\to\infty}\left(2\pi i \operatorname*{Res}_{z=i} \frac{1}{1+z^2} - \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2}\right) \\ &= \frac{\pi}{2} - \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} \end{aligned}$$

Now,

$$\lim_{R\to\infty}\left|\int_{\gamma_R}\frac{\mathrm{d}z}{1+z^2}\right| \le \lim_{R\to\infty}\int_{\gamma_R}\frac{|\mathrm{d}z|}{|1+z^2|} \le \lim_{R\to\infty}\int_{\gamma_R}\frac{|\mathrm{d}z|}{|1-|z|^2|}$$
$$= \lim_{R\to\infty}\frac{1}{R^2-1}\int_{\gamma_R}|\mathrm{d}z| = \lim_{R\to\infty}\frac{\pi R}{R^2-1} = 0$$

Combining both gives $I = \pi/2$. As one can verify, if we choose another contour which is the boundary of the lower half-disk, then the corresponding minus signs cancel out and ultimately yield the same result.

2. We want to evaluate

$$I = \int_0^\infty \frac{\mathrm{d}x}{1+x^3}$$

Let $f(z) = 1/(1+z^3)$. Note that $f$ is invariant under rotation by $e^{2\pi i/3}$. So naturally we consider the contour $\gamma_0+\gamma_R+\gamma_1$ where $\gamma_0(t) = t, t\in[0,R], \gamma_1(t) = e^{2\pi i/3}t, t\in[R,0]$ and $\gamma_R(\theta) = e^{2\pi it/3}, t\in[0,1]$. Then

$$\lim_{R\to\infty}\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_0^\infty \frac{\mathrm{d}t}{1+t^3} = I$$

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_\infty^0 \frac{e^{2\pi i/3}\,\mathrm{d}t}{1+(e^{2\pi i/3}t)^3} = -e^{2\pi i/3}I$$

$$\lim_{R\to\infty}\left|\int_{\gamma_R}\frac{\mathrm{d}z}{z^3}\right| \le \lim_{R\to\infty}\frac{2}{3}\pi R\sup_{z\in\gamma_R}\left|\frac{1}{1+z^3}\right| = \lim_{R\to\infty}\frac{2\pi}{3}\frac{R}{1+R^3} = 0$$

So

$$I = \frac{2\pi i}{1-e^{2\pi i/3}}\left(\operatorname*{Res}_{z=e^{i\pi/3}} f(z) + \operatorname*{Res}_{z=e^{i\pi}} f(z) + \operatorname*{Res}_{z=e^{i5\pi/3}} f(z)\right)$$
$$= \frac{1}{1-e^{2\pi i/3}}i\frac{2\pi}{3}e^{-2\pi i/3} = \frac{2\pi}{3\sqrt{3}}$$

3. Consider

$$I = \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}, a>0$$

Let $f(z) = 1/((z^2+a^2)^2)$ where we have $\mathrm{Res}_{ia} f = 1/(i4a^3)$. Let $\gamma = \gamma_0+\gamma_R$ be the same contour as in our first example (boundary of the half-circle centered at 0 with radius $R$). Then

$$\lim_{R\to\infty}\left|\int_{\gamma_R}\frac{\mathrm{d}z}{(z^2+a^2)^2}\right| = \lim_{R\to\infty}\pi R O(R^{-4}) = 0$$

So

$$I = \frac{1}{2}2\pi i\frac{1}{i4a^3} = \frac{\pi}{4a^3}$$

## 4.3 Integrals of Trigonometric Functions

For integrals of the form

$$\int_0^{2\pi} f(\cos\theta,\sin\theta)\,\mathrm{d}\theta$$

We'd obviously like to use the substitution $z = e^{i\theta}$, i.e. integrating along the unit circle.

**Example 4.2.** Suppose we have $a > 1$ and we want to evaluate

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

Let $\gamma$ be the unit circle (oriented counterclockwise), then

$$\begin{aligned} I &= \oint_\gamma \frac{-i\,\mathrm{d}z}{z(a + (z + z^{-1})/2)} = -2i \oint_\gamma \frac{\mathrm{d}z}{z^2 + 2az + 1} \\ &= (-2i)(2\pi i) \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{1}{z^2 + 2az + 1} = (-2i)(2\pi i)\frac{1}{2\sqrt{a^2 - 1}} \\ &= \frac{2\pi}{\sqrt{a^2 - 1}} \end{aligned}$$

## 4.4 Branch Cuts and Key Hole Contours

For branched functions, we use a key hole contour to avoid the branch cut.

**Example 4.3.** 1. Consider

$$I = \int_0^\infty \frac{x^\alpha}{1 + \sqrt{2}x + x^2}, 0 \neq \alpha \in (-1, 1)$$

The function $f(z) = z^\alpha/(1 + \sqrt{2}z + z^2)$ has a branch point at 0. We choose the branch cut to be $\mathbb{R}_{\geq 0}$ (so arg $\in (0, 2\pi)$). The contour we shall consider is built from the anticlockwise boundary $\gamma_R$ of the circle centred at 0 with radius $R$ and the clockwise boundary $\gamma_\epsilon$ of one that has radius $\epsilon << R$. To avoid the branch cut, we add a slit consisting of $\gamma_1$ (pointing outwards) and $\gamma_2$ (pointing inwards) near the branch cut that joins $\gamma_R$ and $\gamma_\epsilon$. We will take the limit $R \to \infty, \epsilon \to 0$ and $\gamma_1, \gamma_2$ tends to the branch cut. In this limit, we have

$$\int_{\gamma_R} \frac{z^\alpha}{1 + \sqrt{2}z + z^2}\,\mathrm{d}z = 2\pi R O(R^{\alpha-2}) \to 0$$

$$\int_{\gamma_\epsilon} \frac{z^\alpha}{1 + \sqrt{2}z + z^2}\,\mathrm{d}z \to \int_{2\pi}^0 \frac{\epsilon^\alpha e^{i\alpha\theta}}{1 + \sqrt{2}\epsilon e^{i\theta} + \epsilon^2 e^{i2\theta}} i\epsilon e^{i\theta}\,\mathrm{d}\theta = O(\epsilon^{\alpha+1}) \to 0$$

We parameterise $\gamma_1(t) = te^{i\delta\theta}$ where $t \in [\epsilon, R]$ and we take the limit $\delta\theta \to 0^+$. So

$$\int_{\gamma_1} \frac{z^\alpha}{1 + \sqrt{2}z + z^2}\,\mathrm{d}z \to \int_0^\infty \frac{t^\alpha e^{i\alpha\delta\theta}}{1 + \sqrt{2}te^{i\delta\theta} + t^2 e^{i2\delta\theta}} \to \int_0^\infty \frac{t^\alpha}{1 + \sqrt{2}t + t^2} = I$$

Similarly $\gamma_2$ is parameterised as $\gamma_2(t) = te^{i\delta\theta}$ with $t \in [\epsilon, R]$ and $\delta\theta \to 2\pi^-$, which gives

$$\int_{\gamma_2} \frac{z^\alpha}{1 + \sqrt{2}z + z^2}\,\mathrm{d}z \to -e^{i2\alpha\pi} I$$

Hence

$$\begin{aligned} I &= \frac{1}{1 - e^{i2\alpha\pi}} \lim_{\epsilon\to 0, R\to\infty, \gamma_1, \gamma_1 \to \mathbb{R}_{\geq 0}} \oint_{\gamma_1+\gamma_R+\gamma_2+\gamma_\epsilon} \frac{z^\alpha}{1 + \sqrt{2}z + z^2}\,\mathrm{d}z \\ &= \frac{2\pi i}{1 - e^{i2\alpha\pi}} \left( \operatorname*{Res}_{z=e^{i3\pi/4}} \frac{z^\alpha}{1 + \sqrt{2}z + z^2} + \operatorname*{Res}_{z=e^{i5\pi/4}} \frac{z^\alpha}{1 + \sqrt{2}z + z^2} \right) \\ &= \frac{2\pi i}{1 - e^{i2\alpha\pi}} \left( \frac{e^{i3\pi\alpha/4}}{\sqrt{2}i} - \frac{e^{i5\pi\alpha/4}}{\sqrt{2}i} \right) = \sqrt{2}\pi \frac{\sin(\alpha\pi/4)}{\sin(\alpha\pi)} \end{aligned}$$

## 4.5 Rectangular Contours

Sometimes streching a rectangular contour to $\infty$ can also help.

**Example 4.4.** 1. Say we want to integrate

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh x} \, \mathrm{d}x, \alpha \in (-1, 1)$$

Consider $\gamma_0(t) = t, t \in [-R, R], \gamma_R(t) = R + it, t \in [0, \pi], \gamma_1(t) = t + i\pi, t \in [R, -R], \gamma_{-R}(t) = -R + it, t \in [\pi, 0]$. The contour we have in mind is of course $\gamma = \gamma_0 + \gamma_R + \gamma_1 + \gamma_{-R}$. We have

$$\lim_{R\to\infty} \int_{\gamma_0} \frac{e^{\alpha z}}{\cosh z} \, \mathrm{d}z = I, \lim_{R\to\infty} \int_{\gamma_1} \frac{e^{\alpha z}}{\cosh z} \, \mathrm{d}z = e^{i\alpha\pi} I$$

For $\gamma_{\pm R}$, note that $|\cosh(R + it)| = \sqrt{\cos^2 t + \sinh^2 R} \geq \sinh R$, so

$$\left| \int_{\gamma_R} \frac{e^{\alpha z}}{\cosh z} \, \mathrm{d}z \right| \leq \int_0^{\pi} \frac{|e^{\alpha R} e^{\alpha i t}|}{|\sinh R|} \, \mathrm{d}t = \frac{e^{\alpha R}}{\sinh R} \pi = O(e^{(\alpha - 1)R}) \to 0$$

as $R \to \infty$. Likewise for $\gamma_{-R}$. Therefore

$$\begin{aligned} I &= \frac{1}{1 + e^{i\alpha\pi}} \int_{\gamma} \frac{e^{\alpha z}}{\cosh z} \, \mathrm{d}z = \frac{2\pi i}{1 + e^{i\alpha\pi}} \operatorname*{Res}_{z = i\pi/2} \frac{e^{\alpha z}}{\cosh z} \\ &= \frac{2\pi i}{1 + e^{i\alpha\pi}} (-i e^{i\alpha\pi/2}) = \frac{\pi}{\cos(\alpha\pi/2)} \end{aligned}$$

2. Consider

$$I = \oint_{\gamma} f(z) \, \mathrm{d}z, f(z) = \frac{1}{z^2 \tan(\pi z)}$$

with contour $\gamma$ the counterclockwise boundary of the square whose vertices are located at $\pm(N + 1/2) \pm i(N + 1/2)$. $f$ has a pole of order 3 at 0 and simple poles at $z = n \in \mathbb{Z}$. We have $\operatorname{Res}_0 f = -\pi/3$ and $\operatorname{Res}_n f = 1/(n^2\pi)$ for $n \in \mathbb{Z} \setminus \{0\}$. The right edge of $\gamma$ can be parameterised as $z(t) = N + 1/2 + it, t \in [-(N + 1/2), (N + 1/2)]$. One can show that $|\tan(\pi z(t))| \geq 1$, therefore

$$\left| \int_{-N-1/2}^{N+1/2} \frac{i}{z(t)^2 \tan(\pi z(t))} \, \mathrm{d}t \right| \leq \int_{-N-1/2}^{N+1/2} \frac{1}{z(t)^2} \, \mathrm{d}t = O(N^{-1}) \to 0$$

as $N \to \infty$. Playing a similar game with the other edges then shows the contour integral over $\gamma$ vanishes, therefore

$$0 = \lim_{N\to\infty} \oint_{\gamma} f(z) \, \mathrm{d}z = 2\pi i \left( -\frac{\pi}{3} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\pi n^2} \right) \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

## 4.6 Jordan's Lemma

In some previous examples, we observe that when integrating certain functions along an arc, the integral vanishes as the radius goes to infinity. This prompts the following lemma:

**Lemma 4.2** (Jordan's Lemma). *Suppose $f$ is analytic in $\mathbb{C}$ except for a finite number of singularities. If $f(z) \to 0$ as $|z| \to \infty$, then for any $\lambda > 0, \mu < 0$,*

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)e^{i\lambda z}\,\mathrm{d}z = \lim_{R\to\infty} \int_{\bar{\gamma}_R} f(z)e^{i\mu z}\,\mathrm{d}z = 0$$

*where $\gamma_R$ and $\bar{\gamma}_R$ are the upper and lower half-circles of radius $R$ respectively.*

*Proof.* One can show that $\sin x \geq 2/\pi$ for $x \in [0, \pi/2]$. Parameterise $\gamma_R(\theta) = Re^{i\theta}$ for $\theta \in [0, \pi]$. Then

$$\begin{aligned}
\left|\int_{\gamma_R} f(z)e^{i\lambda z}\,\mathrm{d}z\right| &= \left|\int_0^\pi f(Re^{i\theta})e^{i\lambda Re^{i\theta}} iRe^{i\theta}\,\mathrm{d}\theta\right| \\
&\leq R\int_0^\pi |f(Re^{i\theta})|e^{-\lambda R\sin\theta} R\,\mathrm{d}\theta \\
&\leq 2R \sup_{z\in\gamma_R} |f(z)| \int_0^{\pi/2} e^{-\lambda R\sin\theta}\,\mathrm{d}\theta \\
&\leq 2R \sup_{z\in\gamma_R} |f(z)| \int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\,\mathrm{d}\theta \\
&= \frac{\pi}{\lambda}(1 - e^{-\lambda R}) \sup_{z\in\gamma_R} |f(z)| \to 0
\end{aligned}$$

as $R \to \infty$. Likewise for $\mu < 0$ and $\bar{\gamma}_R$. ☐

**Example 4.5.** 1. Consider

$$I = \int_0^\infty \frac{\cos(\alpha x)}{1 + x^2}\,\mathrm{d}x, \alpha > 0$$

Naturally we want to use Jordan's lemma, so we consider the contour to be the counterclockwise boundary $\gamma$ of the upper half-disk of radius $R$ centered at origin and send $R \to \infty$. Then by Jordan's lemma,

$$\begin{aligned}
I &= \mathrm{Re}\left(\int_{-\infty}^\infty \frac{e^{i\alpha x}}{1+x^2}\,\mathrm{d}x\right) = \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty}\int_\gamma \frac{e^{i\alpha z}}{1+z^2}\,\mathrm{d}z\right) \\
&= \frac{1}{2}\mathrm{Re}\left(i2\pi \operatorname*{Res}_{z=i} \frac{e^{i\alpha z}}{1+z^2}\right) = \frac{1}{2}\mathrm{Re}\left(i2\pi\frac{e^{-\alpha}}{2i}\right) = \frac{\pi}{2}e^{-\alpha}
\end{aligned}$$

2. Consider

$$I = \int_{-\infty}^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

We obviously want to consider $e^{iz}/z$ to use Jordan's lemma. But this function does not behave well enough at 0. So what we consider instead is the boundary $\gamma$ of the upper half-disk (with large enough radius $R$) centered at origin with a (half-)disk of radius $\epsilon$ around the origin removed. As usual, let $\gamma_R$ be the outer half-circle, $\gamma_\epsilon$ be the inner half-circle and $\gamma_\pm$ be the respective parts of the contour in $\pm\mathbb{R}_{>0}$. This allows us to write

$$\begin{aligned}
I &= \lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{-\epsilon} \frac{\sin x}{x}\,\mathrm{d}x + \int_\epsilon^R \frac{\sin x}{x}\,\mathrm{d}x\right) \\
&= \mathrm{Im}\left(\lim_{\epsilon\to 0, R\to\infty}\left(\int_{\gamma_-} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\gamma_+} \frac{e^{iz}}{z}\,\mathrm{d}z\right)\right)
\end{aligned}$$

Parameterise $\gamma_\epsilon(t) = \epsilon e^{i\theta}, \theta \in [\pi, 0]$. Then

$$\int_{\gamma_\epsilon} \frac{e^{iz}}{z} \, \mathrm{d}z = \int_\pi^0 \frac{1 + O(\epsilon)}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} \, \mathrm{d}\theta \to -i\pi$$

as $\epsilon \to 0$. Using Jordan's lemma on the $\gamma_R$ part gives $I = -\operatorname{Im}(-i\pi) = \pi$.

# 5 Transform Theory

## 5.1 Fourier Transforms

**Definition 5.1.** Suppsoe $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable with bounded variation and a finite number of discontinuities. The Fourier transform (FT) of it is defined as

$$\tilde{f}(k) = \mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x$$

Correspondingly, the inverse Fourier transform (IFT) is

$$f(x) = \mathcal{F}^{-1}[\tilde{f}](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, \mathrm{d}k$$

*Remark.* 1. There are other conventions for these stuff.
2. For functions that does not behave well, we can still sort of define a Fourier transform of it using distribution theory, but that is not our concern.
3. At discontinuities, the Fourier transform returns the average value of the limits from the sides.

Putting $\pm\infty$ as limits of the integral operator we are considering is sometimes too strong of a condition. What we actually want is

**Definition 5.2.** The Cauchy principal value of the integral of $g$ over $\mathbb{R}$ is

$$\mathrm{P} \int_{-\infty}^{\infty} g(x) \, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} g(x) \, \mathrm{d}x$$

There are many divergent integrals with convergent Cauchy principal values. The lecturer used the example $g(x) = x/(1 + x^2)$, but I enjoy the more brutal $g(x) = x$. We shall assume that all integrals we write over $(-\infty, \infty)$ are taken to be the Cauchy princial values.

**Example 5.1.** 1. Take $f(x) = e^{-x^2/2}$, then

$$\int_{-\infty}^{\infty} e^{-x^2/2} e^{-ikx} \, \mathrm{d}x = \int_{-\infty}^{\infty} e^{-(x+ik)^2/2} e^{-k^2/2} \, \mathrm{d}x = e^{-k^2/2} \int_{-\infty+ik}^{\infty+ik} e^{-z^2/2} \, \mathrm{d}z$$

Let $\gamma$ be the boundary of the rectangle whose vertices are located at $\pm R, \pm R+ik$. Break down the contour as $\gamma = \gamma_0 + \gamma_R + \gamma_1 + \gamma_{-R}$ where $\gamma_0(t) = t + ik, t \in [-R, R], \gamma_1(t) = t, t \in [R, -R], \gamma_{\pm R} = \pm R + it, t \in [0, k]$. Then the integral of $e^{-z^2/2}$ over $\gamma_0$ is the integral we are after; The integral over $\gamma_{\pm R}$ tends to 0 as $R \to \infty$; The integral over $\gamma_1$ is just the Gaussian integral. Putting these together gives

$$\int_{\gamma_0} e^{-z^2/2} \, \mathrm{d}z = -\int_{\gamma_1} e^{-z^2/2} \, \mathrm{d}z = \int_{-\infty}^{\infty} e^{-z^2/2} \, \mathrm{d}z = \sqrt{2\pi}$$

Consequently $\tilde{f}(k) = \sqrt{2\pi}e^{-k^2/2}$.
2. If a $f$ has FT $\tilde{f}(k) = (a+ik)^{-1}, a > 0$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a+ik}e^{ikx}\,\mathrm{d}k$$

Good place to use Jordan's lemma, don't you think? Let $\gamma_R, \bar{\gamma}_R$ be the upper and lower semicircles centered at origin with radii $R$ respectively. Suppose $\gamma_R$ is counterclockwise and $\bar{\gamma}_R$ is clockwise. Let $\gamma_0(t) = t, t \in [-R, R]$, then $\gamma_0 + \gamma_R$ and $\gamma_R + \bar{\gamma}_R$ are closed contours on which we can use Residue Theorem.
For $x > 0$, Jordan's lemma tells us

$$f(x) = \frac{1}{2\pi}\left(2\pi i \operatorname*{Res}_{k=ia}\frac{e^{ikx}}{a+ik}\right) = i(-ie^{-ax}) = e^{-ax}$$

For $x < 0$, observe that $\gamma_0 + \bar{\gamma}_R$ does not encircles a singularity, so we simply have $f(x) = 0$. Combining both cases, we have $f(x) = e^{-ax}1_{x>0}$.

## 5.2 Laplace Transforms

**Definition 5.3.** Let $f(t)$ be defined for all $t \geq 0$. Its Laplace transform is the linear operator

$$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t, s \in \mathbb{C}$$

assuming the integral exist.

For integrable functions with subexponential growth, this integral certainly converges. This will be the class of functions we are interested in. If $f(t)$ can be extended to $\mathbb{R}$ such that $\forall t < 0, f(t) = 0$, then $\mathcal{L}(f)(s) = \mathcal{F}(f)(-is)$.

**Example 5.2.** 1. $\mathcal{L}(1)(s) = 1/s$ for $\operatorname{Re} s > 0$. But observe that $s \mapsto 1/s$ can be defined over $\mathbb{C} \setminus \{0\}$ at some point of which the integral might not be defined! This is a process called analytic continuation.
2. $\mathcal{L}(\mathrm{id})(s) = 1/s^2$ for $\operatorname{Re} s > 0$, which again can be analytically continued to $\mathbb{C} \setminus \{0\}$.
3. $\mathcal{L}(t \mapsto e^{\lambda t})(s) = 1/(s-\lambda)$ for $\operatorname{Re}\lambda < \operatorname{Re} s$ but can be continued to $\mathbb{C} \setminus \{\lambda\}$.
4. $\mathcal{L}(t \mapsto \sin t)(s) = \mathcal{L}(t \mapsto (e^{it} - e^{-it})/(2i))(s) = 1/(s^2+1)$.

As everybody expect, Laplace transforms admit some nice properties with respect to usual operations. If we translate by $t_0 \in \mathbb{R}$, then

$$\mathcal{L}(t \mapsto f(t-t_0)H(t-t_0))(s) = e^{-st_0}F(s)$$

where $H$ is the Heaviside function. Scaling by $\lambda > 0$ gives $\mathcal{L}(t \mapsto f(\lambda t))(s) = F(s/\lambda)/\lambda$, and shifting by $s_0 \in \mathbb{C}$ gives $\mathcal{L}(t \mapsto e^{s_0 t}f(t))(s) = F(s-s_0)$.
Laplace transform, like Fourier transform, also behaves well on derivatives:

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0), \mathcal{L}(f'')(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$$

and so on. Conversely, $(\mathrm{d}/\mathrm{d}s)\mathcal{L}(f)(s) = \mathcal{L}(t \mapsto -tf(t))(s)$ or in general

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}(f)(s) = \mathcal{L}(t \mapsto (-t)^n f(t))(s)$$

Asymptotically, if $f(t) \to L$ as $t \to \infty$, then $s\mathcal{L}(f)(s) \to f(0)$ as $s \to \infty$ and $s\mathcal{L}(f)(s) \to L$ as $s \to \infty$. These properties can be easily verified.

**Example 5.3.** 1. We can evaluate $\mathcal{L}(t \mapsto t \sin t)(s) = -(\mathrm{d}/\mathrm{d}s)\mathcal{L}(t \mapsto \sin t)(s) = -(\mathrm{d}/\mathrm{d}s)(1/(s^2+1)) = 2s/(s^2+1)^2$.
2. $\mathcal{L}(t \mapsto t^n)(s) = (-1)^n(\mathrm{d}^n/\mathrm{d}s^n)\mathcal{L}(1)(s) = (-1)^n(\mathrm{d}^n/\mathrm{d}s^n)(1/s) = n!/s^{n+1}$. So for example we can evaluate

$$\Gamma(n) = \int_0^\infty e^{-t}t^{n-1}\,\mathrm{d}t = \mathcal{L}(t \mapsto t^{n-1})(1) = (n-1)!$$

3. For $a > 0$, we have $\mathcal{L}(t \mapsto \sin(at))(s) = a^{-1}\mathcal{L}(t \mapsto \sin t)(s/a) = a/(s^2+a^2)$.
4. We know $\mathcal{L}(t \mapsto e^{iat})(s) = 1/(s-ia) = (s+ia)/(s^2+a^2)$. Combining this with above gives $\mathcal{L}(t \mapsto \cos(at))(s) = s/(s^2+a^2)$.

Fourier transforms admit inverses. Since we can sort of relate one to the other, it is reasonable to guess Laplace transforms have inverses too.

**Proposition 5.1.** *We can compute the inverse Laplace transform $f(t)$ of $F(s)$ by the formula (known as the Bromwich inversion formula)*

$$f(t) = \mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}\,\mathrm{d}s$$

*For some $\alpha \in \mathbb{R}$ greater than the real parts of all singularities of $F$.*

*Proof.* By assumption $f(t)$ has a Laplace transform, so there exists $\alpha \in \mathbb{R}$ such that $g(t) = f(t)e^{-\alpha t}$ decays exponentially. Then $\tilde{g}(\omega) = \mathcal{F}(g)(\omega) = F(\alpha + i\omega)$. So

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha+i\omega)e^{i\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{(s-\alpha)t}$$

Substitution yields the result. ☐

**Proposition 5.2.** *Let $F(s)$ be the Laplace transform of $f(t)$. Suppose $F$ has a finite number of isolated singularities $s_k \in \mathbb{C}, 1 \le k \le n$. If $F \to 0$ as $s \to \infty$, then $f(t) = 0$ for $t < 0$ and*

$$f(t) = \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k}(F(s)e^{st})$$

*for $t > 0$.*

*Proof.* Let $\alpha$ be as in the preceding proposition, which is greater than the real part of all singularities of $F$.
For $t < 0$, consider the contour $\gamma = \gamma_0 + \bar{\gamma}_R$ where $\gamma_0(t) = \alpha + it, t \in [-R, R]$ and $\bar{\gamma}_R(t) = \alpha + Re^{i\theta}, \theta \in [\pi/2, -\pi/2]$.
If $F(s) = o(s^{-1})$ as $s \to \infty$, then the integral of $F(s)e^{st}$ over $\bar{\gamma}_R$ vanishes as $R \to \infty$ by simply bounding (making use of $t < 0$). If we are not that lucky, we can still cast (a variant of) Jordan's lemma to obtain the same result. The rest is just routine:

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}\,\mathrm{d}s = \frac{1}{2\pi i}\left(\lim_{R\to\infty}\oint_{\gamma_0+\bar{\gamma}_R} F(s)e^{st}\,\mathrm{d}s\right) = 0$$

For $t > 0$, we use a different contour $\gamma = \gamma_0 + \gamma_R$ where $\gamma_0$ is as before and $\gamma_R(t) = \alpha + Re^{i\theta}, \theta \in [\pi/2, 3\pi/2]$. Again by similar arguments the integral over $\gamma_R$ vanishes and

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(s)e^{st}\,\mathrm{d}s = \frac{1}{2\pi i}\left(\lim_{R\to\infty}\oint_{\gamma_0+\bar{\gamma}_R} F(s)e^{st}\,\mathrm{d}s\right)$$
$$= \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k}(F(s)e^{st})$$

as desired. ☐

**Example 5.4.** 1. Take $F(s) = 1/(s-1)$, then $f(t) = 0$ for $t < 0$ and $f(t) = e^t$ for $t > 0$.
2. Take $F(s) = s^{-n}$ for $N \in \mathbb{N}$, then $f(t) = 0$ for $t < 0$ and $f(t) = t^{n-1}/(n-1)!$ for $t > 0$.

The chief application of Laplace transform is, naturally, solving differential equations since it behaves nicely with respect to derivatives.

**Example 5.5.** 1. Suppose we want to solve $t\ddot{f} - t\dot{f} + f = 2$ subject to $f(0) = 2, \dot{f}(0) = -1$. Let $F(s) = \mathcal{L}(f)(s)$, then we have $\mathcal{L}(t\dot{f})(s) = -sF'(s) - F(s)$, $\mathcal{L}(t\ddot{f})(s) = -s^2F'(s) - 2sF(s) + f(0)$, $\mathcal{L}(2)(s) = 2/s$. So what's left over is (note that we already incorporated the boundary condition $f(0) = 2$ here)

$$s^2F'(s) + 2sF(s) = 2 \implies F(s) = \frac{2}{s} + \frac{A}{s^2}$$

where $A \in \mathbb{C}$ is a constant. Inverting this gives $f(t) = 2 + At = 2 - t$ by the boundary conditions.
2. PDEs are sure more fun. Consider the heat equation $f_t = f_{xx}$ on the region $x \in [0, 2], t \geq 0$ subject to boundary conditions $f(t, 0) = 0, f(t, 2) = 0, f(0, x) = 3\sin(2\pi x)$. Let $F(s, x)$ be the Laplace transform of $f$ wrt $t$, then by taking Laplace transform we get

$$\frac{\partial^2}{\partial x^2}F(s, x) - sF(s, x) = -f(0, x) = -3\sin(2\pi x)$$

which solves to (with the boundary conditions)

$$F(s, x) = \frac{3}{4\pi^2 + s}\sin(2\pi x) \implies f(t, x) = 3\sin(2\pi x)e^{-4\pi^2 t}$$

For Fourier transforms, we have a convolution theorem. We are nice to ourselves, so we also have a convolution theorem for Laplace transforms.

**Definition 5.4.** The convolution $f * g$ of $f, g : \mathbb{R} \to \mathbb{C}$ is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\,\mathrm{d}u$$

Easy to see that the convolution is commutative. If $f, g$ vanishes on $\mathbb{R}_{<0}$ (which frequently happens), then we simply have

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\,\mathrm{d}u = \int_0^t f(t-u)g(u)\,\mathrm{d}u$$

Recall that we have $\mathcal{F}(f * g) = \mathcal{F}[g]\mathcal{F}[f]$ for Fourier transforms. Correspondingly,

**Theorem 5.3.** $\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)$.

*Proof.* Just calculate.

$$\begin{aligned}\mathcal{L}(f*g)(s)&=\int_0^\infty\left(\int_0^t f(t-u)g(u)\,\mathrm{d}u\right)e^{-st}\,\mathrm{d}t\\&=\int_0^\infty\left(\int_u^\infty f(t-u)g(u)e^{-st}\,\mathrm{d}t\right)\,\mathrm{d}u\\&=\int_0^\infty\left(\int_0^\infty f(x)g(u)e^{-sx}e^{-su}\,\mathrm{d}x\right)\,\mathrm{d}u, x=t-u\\&=\left(\int_0^\infty f(x)e^{-sx}\,\mathrm{d}x\right)\left(\int_0^\infty g(u)e^{-su}\,\mathrm{d}u\right)=\mathcal{L}(f)\mathcal{L}(g)\end{aligned}$$

which is what we want. □

**Example 5.6.** 1. Consider $H(s)=1/(s(s^2+1))=F(s)G(s)$ where $F(s)=1/s, G(s)=1/(s^2+1)$. Suppose $H=\mathcal{L}(h)$, then by the convolution theorem,

$$h(t)=\int_0^t 1\cdot\sin u\,\mathrm{d}u=1-\cos t$$

2. Consider the ODE $4\ddot{f}+f=h$ subject to $f(0)=3,\dot{f}(0)=-7$. Laplace transform the ODE gives $(4s^2+1)F(s)-12s+28=H(s)$ which solves to

$$F(s)=3\frac{s}{s^2+1/4}-7\frac{1}{s^2+1/4}+\frac{H(s)}{4}\frac{1}{s^2+1/4}$$

which gives

$$f(t)=3\cos\frac{t}{2}-14\sin\frac{t}{2}+\frac{1}{2}\int_0^t h(t-u)\sin\left(\frac{u}{2}\right)\,\mathrm{d}u$$