TRIVIAL TENSOR PRODUCTS

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Abstract. In this short note, we classify all situations where a product of vector bundles on a smooth projective variety becomes trivial. The result itself is easy, but it might be of interest to others who are wondering the same question.

This came up and was resolved in a discussion with Junliang Shen.

1. Introduction

Fix a reasonable category of manifolds and a (compact) member X. One might ask the following question: Does every vector bundle E on X admit a "rational inverse" F, in the sense that $E \otimes F$ can be trivialised in the said category?

When X is a smooth compact manifold, techniques from K-theory reveals that the answer is yes[1]. One would, however, expect some kind of extra rigidity in the case of a smooth projective variety X. Indeed, the vector bundle $\mathcal{O} \oplus \mathcal{O}(1)$ already does not admit a rational inverse. One would then be interested in the classification of all vector bundles such that this is possible.

We show

Theorem 1.1. *Let X be a smooth projective variety. Suppose there are vector bundles E, F over X such that $E \otimes F \cong \mathcal{O}_X^N$ for some $N \in \mathbb{Z}$, then $E \cong L^{\oplus m}$ for some $m \in \mathbb{N}$ (and necessarily $F \cong (L^\vee)^{\oplus n}$ for some n).*

2. Vector bundles on smooth projective varieties

We recall some useful facts about vector bundles on smooth projective varieties. Chief of which is the following fundamental theorem of M. Atiyah:

Theorem 2.1 (Atiyah)**.** *The category of coherent sheaves on a smooth projective variety has the Krull-Schmidt property. That is, every object decomposes uniquely into a direct sum of indecomposables.*

Remark. When talking about decomposition and indecomposablility, it doesn't hurt if we restrict ourselves to the full subcategory of vector bundles. Indeed, every summand of a vector bundle is still a vector bundle (since projectives over local rings are free).

There certainly would be nice to have some way to check if a coherent sheaf is indecomposable. One useful criterion would be to examine its endomorphisms. It's easy to see that

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[1]Private communication with Junzhi Huang.

Lemma 2.2. *A coherent sheaf (or, in general, an object in a Krull-Schmidt category) is indecomposable if and only if the only idempotent in its endomorphism ring are* 0 *and* id.

In particular, any line bundle is indecomposable.

We also need a property of vector bundles that is rather special from a sheaf-theoretic point of view (perhaps less special if you actually think about them as vector bundles). For any coherent sheaves $\mathcal{M}, \mathcal{N}$, we certainly would have a map $\mathcal{M}^\vee \otimes \mathcal{N}^\vee \to (\mathcal{M} \otimes \mathcal{N})^\vee$, which is in general NOT an isomorphism.

Expectedly we can measure this discrepancy cohomologically. Indeed,

$$\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{O}_X) \otimes^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{N}, \mathcal{O}_X) \cong \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{M} \otimes^{\mathbf{L}} \mathcal{N}, \mathcal{O}_X)$$

Consequently,

Lemma 2.3. *If $\mathcal{M}, \mathcal{N}$ are vector bundles, then $\mathcal{M}^\vee \otimes \mathcal{N}^\vee \cong (\mathcal{M} \otimes \mathcal{N})^\vee$.*

3. The proof of the theorem

Suppose $E \otimes F$ is trivial. Then certainly its dual $(E \otimes F)^\vee \cong E^\vee \otimes F^\vee$ (Lemma 2.3) too would be trivial. So $E \otimes E^\vee \otimes F \otimes F^\vee \cong \underline{\mathrm{End}}(E) \otimes \underline{\mathrm{End}}(F)$ is trivial.

Now, we have a trace map $\underline{\mathrm{End}}(F) \to \mathcal{O}_{\mathrm{X}}$ which has a section given by $\mathrm{id} \in \mathrm{End}(F) = \Gamma(\underline{\mathrm{End}}(F))$. Therefore $\mathcal{O}_X$ is a summand of $\underline{\mathrm{End}}(F)$, hence $\underline{\mathrm{End}}(E)$ is a summand of a trivial vector bundle. This means that $\underline{\mathrm{End}}(E)$ is trivial by Theorem 2.1.

Then just choose any $x \in X$ and consider the restriction map $\mathrm{End}(E) \to \mathrm{End}(E_x)$, where E_x is the fibre of E at x. This is a homomorphism of rings, and is bijective as $\underline{\mathrm{End}}(E)$ is trivial. Therefore $\mathrm{End}(E) \cong \mathrm{GL}(\mathrm{rank}\, E, \mathbb{C})$.

This has a nontrivial idempotent as soon as $\mathrm{rank}\, E \geq 2$. By Lemma 2.2, E cannot be indecomposable unless it is a line bundle.

Because of Theorem 2.1, if $E \otimes F$ is trivial then $P \otimes F$ is trivial for any summand P of E. Applying the above argument to every indecomposable factor of E, we see that E itself must be a sum of line bundles. Using Theorem 2.1 again then shows that it must be a sum of copies of the same line bundle. This establishes Theorem 1.1.