HILBERT SCHEMES OF PLANAR CURVES ON SURFACES

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Abstract. This is an elementary calculation done in the midst of a summer project under Dr Dhruv Ranganathan.

We study the Hilbert schemes of planar curves in hypersurfaces of degree d in $\mathbb{P}^3$. We've shown that, for $d \geq 4$, this Hilbert scheme of a generic hypersurface of degree d is isomorphic to a countable disjoint union of copies of $\mathbb{P}^3$.

Throughout this article we work over $\mathbb{C}$.

For a degree $d \geq 4$ projective hypersurface S in $\mathbb{P}^3$, we are interested in the Hilbert scheme $H(S)$ parameterising planar curves contained in S. It has the decomposition

$$H(S) = \coprod_{k=1}^{\infty} H_k(S) \text{ where } H_k(S) = \mathrm{Hilb}^{P_k}_{S/\mathbb{C}}, P_k(T) = kT + 1 - \binom{k-1}{2}$$

where, essentially, $H_k(S)$ parameterises plane curves of degree k contained in S.

Another noteworthy fact about $H(S)$ is that it is smooth when S is smooth. Indeed, from [1, Corollary 2.7] we know that $\mathrm{Hilb}_{S/\mathbb{C}}$, of which $H(S)$ is a open subscheme, is smooth since S has irregularity 0.

Theorem 1. *Suppose S is very general, then*

$$H_k(S) \cong \begin{cases} \mathbb{P}^3 & \text{if } d \mid k; \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. As S is very general, we may as well assume that it's smooth.

We shall first deal with the case where $k = d$ by producing a morphism $\psi : (\mathbb{P}^3)^\vee \to H_d(S)$ which is a bijection on $\mathbb{C}$-points. Here, $(\mathbb{P}^3)^\vee$ denotes the dual projective space parameterising planes in $\mathbb{P}^3$. Since both $\mathbb{P}^3$ and $H_d(S)$ are smooth and projective, ψ would then have to be an isomorphism by Zariski's Main Theorem.

To give a morphism $(\mathbb{P}^3)^\vee \to H_d(S)$ is the same as to give a flat family $Z \subset S \times (\mathbb{P}^3)^\vee \to (\mathbb{P}^3)^\vee$ with Hilbert polynomial P_d. To construct ψ, we take Z to be the family whose fibre over a geometric point $\mathrm{Spec}\, \bar{k} \to (\mathbb{P}^3)^\vee$ (corresponding to a plane $V \subset \mathbb{P}^3_{\bar{k}}$) is the closed subscheme of $S_{\bar{k}}$ defined by the ideal of V restricted to $S_{\bar{k}}$ ("intersection of V with $S_{\bar{k}}$"). Alternatively, Z may be described as the relative Hilbert scheme parameterising pairs $(p, V) \in S \times (\mathbb{P}^3)^\vee$ with $p \in V$.

It follows that Z as such is a flat family, and on the level of $\mathbb{C}$-points it takes $V \in (\mathbb{P}^3)^\vee$ to the curve produced by the intersection of V with S. This is a injection, for if two different planes intersects S at the same curve C, then C is contained in the intersection

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of the two planes, which is a line. It is also a surjection: By the theorem of Noether-Lefchetz, for very general choice of S, any curve $C \subset S$ must be the complete intersection intersection of S with another surface S', say of degree d'. But then C must have degree dd' by Bézout's theorem, hence S' is a plane.

This Noether-Lefchetz argument also shows that $H_k(S) = \varnothing$ for $d \nmid k$, and that any plane curve on S must be irreducible (and its support has degree exactly d).

It remains to deal with the case where $k = dn$ for some $n > 1$.

Consider the map $\phi(\bar{k})$ that brings $C \in H_d(S)(\bar{k})$ to the curve with ideal $\mathcal{I}_C^n$, which is an element of $H_{dn}(S)(\bar{k})$. This comes from a morphism $\phi : H_d(S) \to H_{dn}(S)$ using the same procedure as before. Again, due to Zariski's main theorem, to conclude ϕ is an isomorphism it suffices to show that $\phi(\bar{k})$ is a bijection.

It is an injection since taking support recovers C. It is a surjection again due to Noether-Lefchetz: Any curve in $H_d(S)(\bar{k})$ is a complete intersection, hence an effective Cartier divisor. But any effective Cartier divisor on S is determined by its support and multiplicity by the well-known comparison theorem between Weil and Cartier divisors, so we have surjectivity. □

References

[1] Fogarty, J. Algebraic families on an algebraic surface. *American Journal of Mathematics 90*, 2 (1968), 511–521.