University of Cambridge

Part III Essay

Local Moduli of Abelian Varieties and the Serre-Tate Theorem

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Introduction

Suppose we have a moduli space $\mathcal{M}$ parameterising families of abelian varieties over a field k. Any abelian variety X_0 over k defines a closed point of $\mathcal{M}$, so it makes sense to talk about the tangent space at that point. Elements of this tangent space should correspond to elements of $\mathcal{M}(k[\epsilon]/(\epsilon^2))$ reducing to $X_0 \in \mathcal{M}(k)$.

In general, the study of infinitesimal neighbourhoods of X_0 in $\mathcal{M}$ can be viewed as the study of ways in which X_0 can be lifted across a closed immersion $\mathrm{Spec}(k) \to S$ for some Noetherian scheme S whose underlying set consists of one point. Of course, it is necessary and sufficient that $S = \mathrm{Spec}(R)$ for some Artinian local ring R with residue field k. One can also consider a more general situation of lifting an abelian *scheme* over a base ring R' across a surjection $R \to R'$ with nilpotent kernel.

This essay gives an account of classical results on problems of this sort, commonly known as the "deformation theory" of abelian schemes.

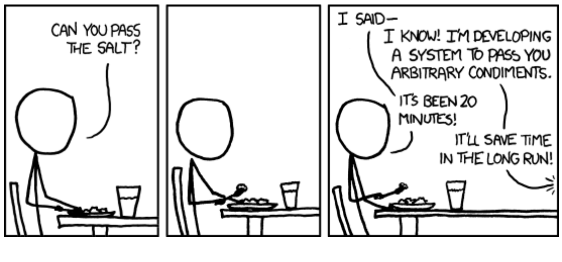
In Chapter 1, we develop obstruction theory, which gives a cohomological characterisation of infinitesimal deformations of smooth morphisms and smooth schemes. This would be a convenient tool when we go on to study the general deformation theory of abelian schemes in Chapter 2, where we show that abelian schemes can always be lifted across a surjection $R \to R'$ of Noetherian rings with nilpotent kernel.

We also construct the local moduli $\mathscr{M}_{X_0}$ associated to an abelian variety X_0 over k, which parameterises the deformations of X_0 over various Artinian local rings with residue field k. We prove, with the help of a criterion due to Schlessinger, that this local moduli is pro-representable by a power series ring in $(\dim X_0)^2$ variables.

We then specialise to the positive characteristic case. Fix a prime p which shall be nilpotent in all rings considered. We devote Chapter 3 to Drinfeld's proof of the classical theorem of Serre-Tate, which states that the deformation theory of abelian schemes in this case is controlled precisely by the deformation theory of their p-divisible groups.

Using the Serre-Tate theorem, we obtain in Chapter 4 a canonical way to equip $\mathscr{M}_{X_0}$ with a group structure for any ordinary abelian variety X_0 over an algebraically closed field k of characteristic p. We also sketch how one would use this to lift X_0 canonically to an abelian scheme over the ring $W(k)$ of Witt vectors.

We mainly work under the Noetherian hypothesis. Nonetheless, most of the main theorems hold in general (after minimal modifications) and can be deduced from the Noetherian case using standard techniques such as [2, IV_3, §8.9].



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1 Obstruction Theory

1.1 Smooth Morphisms and Infinitesimal Liftings

In differential geometry, a map between manifolds is a submersion at a point if its differential there is surjective. Mimicking this behaviour, one is inspired to take the following definition of a smooth morphism in the context of algebraic geometry, sometimes known as the Jacobian criterion.

Definition 1.1. A ring map $B \to A$ is standard smooth if there is an isomorphism of B-algebras

$$A \cong B[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$$

such that $n \geq m \geq 0$ and $\det((\partial f_i/\partial T_j)_{1\leq i,j\leq m}) \in A^\times$.

Definition 1.2. A morphism $f : X \to S$ is smooth at $x \in X$ if there exists affine opens $U = \operatorname{Spec}(A) \subset X$, $V = \operatorname{Spec}(B) \subset S$ such that $x \in U$, $f(U) \subset V$, and the induced ring map $B \to A$ is standard smooth. f is smooth if it is smooth at every $x \in X$.

One immediately sees that a smooth morphism is flat and locally of finite presentation. In particular, it is (universally) open.

Unsurprisingly, differentials over a smooth morphism are quite well-behaved: If $B \to A = B[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$ is standard smooth, then the module of differentials $\Omega_{A/B}$ is free on $\mathrm{d}T_{m+1}, \ldots, \mathrm{d}T_n$. Since smooth morphisms are covered by standard smooth ring maps, we have:

Proposition 1.1.1. *Suppose $f : X \to S$ is a smooth morphism, then $\Omega_{X/S}$ is a finite locally free $\mathcal{O}_X$-module. Furthermore, for any $x \in X$, $\operatorname{rank}_x \Omega_{X/S} = \dim_x X_{f(x)}$.*

Proof. [13, Lemma 02G1]. □

Definition 1.3. Suppose $f : X \to S$ is a smooth morphism with connected fibres. Then its relative dimension is the locally constant function $S \to \mathbb{Z}_{\geq 0}$ sending $s \in S$ to $\dim X_s$.

Example 1.1.1. If $B \to A = B[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$ is standard smooth, then $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ has relative dimension $n - m$.

Despite it being well-motivated, the Jacobian criterion is not usually convenient to work with, since one has to choose the coordinates $T_1, \ldots, T_n$. It turns out that there is a more canonical characterisation of smoothness.

Proposition 1.1.2 (Infinitesimal Lifting Criterion)**.** *A morphism $X \to S$ is smooth if and only if it is locally of finite presentation, and for every commutative diagram of solid arrows*

$$\begin{array}{ccc} X & \longleftarrow & \operatorname{Spec}(A') \\ \downarrow & \nwarrow & \downarrow \\ S & \longleftarrow & \operatorname{Spec}(A) \end{array}$$

with $A \to A'$ a surjection with nilpotent kernel, there is a morphism filling in the dashed arrow.

Proof. [13, Lemma 02H6]. □

In view of this, we make the following definition:

Definition 1.4. Fix a base scheme S and let $\mathcal{C}$ be a full subcategory of (Sch/S).

A functor $F : \mathcal{C}^{\mathrm{op}} \to (\mathsf{Sets})$ is formally smooth if, for any surjection $A \to A'$ with nilpotent kernel such that $\operatorname{Spec}(A), \operatorname{Spec}(A') \in \operatorname{ob}(\mathcal{C})$, the induced map $F(A) \to F(A')$ is surjective.

An S-scheme X is formally smooth if its functor of points is formally smooth on (Sch/S).

So Proposition 1.1.2 can be rephrased as saying that $X \to S$ is smooth if and only if it is locally of finite presentation and formally smooth.

Remark 1.1.1. Suppose that all closed subschemes of Y live in $\mathrm{ob}(\mathcal{C})$ whenever Y does. Then, to check the formal smoothness of F, it suffices to check the surjectivity of $F(A) \to F(A')$ when $A \to A'$ is a surjection with square-zero kernel. Indeed, any surjection $A \to A'$ with nilpotent kernel is a composite $A = A_N \to A_{N-1} \to \cdots \to A_0 = A'$ of surjections with square-zero kernels.

The criterion of Proposition 1.1.2 can be further simplified when S is locally Noetherian.

Definition 1.5. A surjection $\phi : A \to A'$ of Artinian local rings is small if $\mathfrak{m}_A \cdot \ker \phi = 0$.

Proposition 1.1.3. *When S is locally Noetherian, to check the lifting criterion in Proposition 1.1.2 it suffices to check the cases where $A \to A'$ is a small surjection of Artinian local rings.*

Proof. [13, Lemma 02HX]. □

1.2 Lifting Smooth Morphisms

Let $X \to S$ be smooth. Proposition 1.1.2 tells us that, whenever $Y' \to Y$ is a closed immersion of *affine* S-schemes with nilpotent ideal, we may lift a Y'-point of X to a Y-point of X.

However, there is no guarantee that this lifting will be unique. So if we want to study such a lifting across closed immersions $Y' \to Y$ of S-schemes in general, we cannot simply glue together local liftings. Our aim in this section is to study infinitesimal liftings in this general setting.

First, we analyse the space of such liftings assuming it's nonempty. Fix $g' : Y' \to X$. Let $\mathcal{I}$ be the sheaf of ideals of Y' in Y. We might as well assume that $\mathcal{I}^2 = 0$. Then $\mathcal{I}$ is also an $O_{Y'}$-module.

Lemma 1.2.1. *Let A, B be R-algebras and $f, g : B \to A$ maps of R-algebras. Suppose I is a square-zero ideal of A. Then A/I acts on I. Suppose in addition that the composites $\pi \circ f, \pi \circ g : B \to A/I$ agree (where $\pi : A \to A/I$ is the quotient map), then $f - g \in \mathrm{Der}_R(B, I)$ where B acts on I through A/I.*

Furthermore, for any $f : B \to A$ and $D \in \mathrm{Der}_R(B, I)$, $g = f + D : B \to A$ is a map of R-algebras with $\pi \circ f = \pi \circ g$.

Proof. Since $\pi \circ (f - g) = 0$, $f - g$ is an R-linear map into I.

For $a \in B, m \in I$, we have by definition of the B-action that $a \cdot m = f(a)m = g(a)m$. So $(f - g)(ab) = f(ab) - g(ab) = f(a)f(b) - g(a)g(b) = f(b)(f(a) - g(a)) + g(a)(f(b) - g(b)) = b \cdot (f - g)(a) + a \cdot (f - g)(b)$, i.e. $f - g$ is a derivation. Reversing this calculation shows the last statement. □

Globalising this argument, we conclude that the liftings of $g' : Y' \to X$ to $Y \to X$ differ by an element of $G = \mathrm{Der}_{\mathcal{O}_S}((g')^{-1}\mathcal{O}_X, \mathcal{I}) = \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, g'_*\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_{Y'}}((g')^*\Omega_{X/S}, \mathcal{I})$.

In other words, we have a simply transitive action of G on the set of liftings as given by the lemma. We capture this phenomenon with the following definition.

Definition 1.6. For a group G, a G-torsor is a nonempty G-set with simply transitive G-action.

Loosely speaking, a G-torsor is just G except we forget where the identity is.

Definition 1.7. Let $X \to S$ be a smooth morphism. Its relative tangent sheaf is the $\mathcal{O}_X$-module $T_{X/S} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) = \underline{\mathrm{Der}}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_X)$.

By Proposition 1.1.1, $T_{X/S}$ is finite locally free. We make the natural identification

$$G = H^0(Y', \mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X/S}).$$

Remark 1.2.1. There is a sheaf version of torsors:

Let $\mathcal{G}$ be a sheaf of abelian groups and $\mathcal{L}$ a sheaf of sets on the same topological space X. Suppose $\mathcal{G}$ acts on $\mathcal{L}$, in the sense that there is a map of sheaves $\mathcal{G} \times \mathcal{L} \to \mathcal{L}$ which becomes a group action on every open set of X. Then $\mathcal{L}$ is called a pseudo $\mathcal{G}$-torsor if, for each open $U \subset X$, either $\mathcal{L}(U)$ is empty or the $\mathcal{G}(U)$-action is simply transitive. It is called a $\mathcal{G}$-torsor if in addition that X can be covered by open sets on which $\mathcal{L}$ has sections.

Our argument essentially shows that the sheaf $\mathcal{L}$ of local liftings is a $(\mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X/S})$-torsor.

Theorem 1.2.2. *Fix a base scheme S. Let X be a smooth S-scheme. Suppose $j : Y' \to Y$ is a closed immersion with square-zero ideal $\mathcal{I}$ and let $g' : Y' \to X$ be a morphism.*

Suppose Y' is separated over $\mathrm{Spec}(\mathbb{Z})$*. Then there is a natural "obstruction element"*

$$\mathfrak{o} \in H^1(Y', \mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X/S})$$

such that $\mathfrak{o} = 0$ if and only if a lifting $g : Y \to X$ of g' exists.

Moreover, if $\mathfrak{o} = 0$, then the set of such liftings is a $H^0(Y', \mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X/S})$-torsor.

Remark 1.2.2. The theorem is in fact true without the separatedness assumption on Y' (see [4, Theorem 8.5.9(a)]). We introduce this extra hypothesis because we only need the separated case, and because we can explicitly write down the obstruction element in this case.

Proof. We first fix some notations: For an open set U of the common topological space for Y and Y', we write U for the associated open subscheme of Y and U' the open subscheme of Y'. We also write $\mathcal{G} = \mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X/S}$.

Fix an affine open cover $\{U_\alpha\}_\alpha$ of Y, which gives rise to an affine open cover $\mathcal{U} = \{U'_\alpha\}_\alpha$ of Y'. Since Y' is separated, we may identify $H^i(Y', \mathcal{G}) = \check{H}^i(\mathcal{U}, \mathcal{G})$ (cf. [13, Lemma 01XD]).

By Proposition 1.1.2, each $g'|_{U'_\alpha}$ can be lifted to some $g_\alpha : U_\alpha \to X$. On each overlap $U'_\alpha \cap U'_\beta$, both $g_\alpha|_{U_\alpha \cap U_\beta}$ and $g_\beta|_{U_\alpha \cap U_\beta}$ lift $g'|_{U'_\alpha \cap U'_\beta}$. So they differ by an element $\phi_{\beta\alpha} \in H^0(U'_\alpha \cap U'_\beta, \mathcal{G})$. This collection of data gives a Čech 1-cochain $\phi \in \check{C}^1(\mathcal{U}, \mathcal{G})$.

It is in fact a cocycle. Indeed, $(\partial\phi)_{\alpha\beta\gamma} = \phi_{\beta\gamma} - \phi_{\alpha\gamma} + \phi_{\alpha\beta} \in H^0(U'_\alpha \cap U'_\beta \cap U'_\gamma, \mathcal{G})$ acts trivially on the restriction of g_α to $U_\alpha \cap U_\beta \cap U_\gamma$ (which is a local lifting of g' there). But the action is simply transitive, so this section must vanish.

Furthermore, the class of this cocycle does not depend on the initial choice of $(g_\alpha)_\alpha$, for if $(\tilde{g}_\alpha)_\alpha$ is another set of local liftings, then $\phi - \tilde{\phi}$ is the coboundary of the 0-cochain given by the elements of $H^0(U'_\alpha, \mathcal{G})$ measuring the differences between the liftings. In addition, if we take a finer affine cover, then the resulting classes eventually agree in $H^1(Y', \mathcal{G})$. In particular, $\mathfrak{o} = [\phi] \in H^1(Y', \mathcal{G})$ also does not depend on the choice of covering.

The vanishing of $\mathfrak{o}$ is equivalent to ϕ being a coboundary. If a global lifting $g : Y \to X$ exists, then taking $g_\alpha = g|_{U_\alpha}$ shows that ϕ is a coboundary. Conversely, if $\phi = \partial\mu$ for some 0-cochain μ, then refining each g_α by μ_α gives rise to local data which glue to a global lifting $Y \to X$.

The last part of the theorem follows from what we have already discussed. □

Remark 1.2.3. The theory is usually applied to the following situation: Suppose X, Y are smooth S-schemes, $i : S' \to S$ is a closed immersion with square-zero ideal $\mathcal{I}$, and $g' : Y' \to X'$ is an S'-morphism (where $Y' = Y \times_S S'$, $X' = X \times_S S'$). Then an S-morphism $g : Y \to X$ has g' as its base-change to S' if and only if it lifts the composite $Y' \to X' \to X$.

But $Y' \to Y$ is a closed immersion with square-zero ideal since $S' \to S$ is. As Y is smooth (hence flat) over S, this ideal is in fact $f^*\mathcal{I}$ where $f : Y \to S$ is the structure map of Y.

By Theorem 1.2.2, such a lifting g exists if and only if an element

$$\mathfrak{o} = \mathfrak{o}(g', i) \in H^1(Y', f^*\mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X'/S'})$$

vanishes. And when $\mathfrak{o} = 0$, the set of liftings is a $H^0(Y', f^*\mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X'/S'})$-torsor.

Remark 1.2.4. We may understand the spaces $H^i(Y', f^*\mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X'/S'})$ after another base-change along a nilpotent closed immersion $S_0 \to S'$, since such an operation does not change the topological space.

For example, if $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ is given by a small surjection $R \to R'$ of Artinian local rings (with kernel I, say), we can take $Y_0 = \mathrm{Spec}(k)$ where $k = R'/\mathfrak{m}_{R'}$. We may then make a natural identification

$$H^i(Y', f^*\mathcal{I} \otimes_{\mathcal{O}_{Y'}} (g')^* T_{X'/S'}) = H^i(Y_0, I \otimes_k g_0^* T_{X_0/k}) = H^i(Y_0, g_0^* T_{X_0/k}) \otimes_k I$$

where $g_0 : Y_0 \to X_0$ is the base-change of $g' : Y' \to X'$ along $S_0 \to S'$. Here, we have regarded I as a k-vector space via the R'-action on it, which is allowed since $\mathfrak{m}_{R'} \cdot I = 0$.

Remark 1.2.5. From its construction, the obstruction element satisfies a "chain rule": Suppose $i : S' \to S$ is a closed immersion with square-zero ideal, X, Y, Z are smooth S-schemes, X', Y', Z' are their base-change to S', and $f' : X' \to Y'$, $g' : Y' \to Z'$ are S'-morphisms. Then $\mathfrak{o}(g' \circ f', i) = (f')^* \mathfrak{o}(g', i)$.

1.3 Infinitesimal Variations of Smooth Schemes

Fix a closed immersion $S' \to S$ with nilpotent ideal $\mathcal{I}$.

Let $X' \to S'$ a smooth morphism. We want to study the lifting of $X' \to S'$, i.e. Cartesian squares of the form

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array}$$

We will only be interested in smooth liftings.

Such a lifting, if exists, automatically inherits some good properties of the original scheme.

Lemma 1.3.1. *Suppose $S' \to S$ is a closed immersion of locally Noetherian schemes with nilpotent ideal and $X' \to S'$ is smooth and proper. Then any smooth lifting $X \to S$ is proper. Moreover, if the geometric fibres of $X' \to S'$ are connected, then the geometric fibres of $X \to S$ are also connected.*

Proof. Since $X \to S$ and $X' \to S'$ are the same on the level of topological spaces, $X \to S$ must too be quasicompact. It is also locally of finite type by the definition of smoothness.

Now $X' \to X$ is a closed embedding with nilpotent ideal, so for any reduced ring R the map $X'(R) \to X(R)$ is a bijection. Therefore the valuative criterion (cf. [5, Theorem II.4.7]) for $X' \to S' \to S$ implies the valuative criterion for $X \to S$. And all the geometric fibres of $X \to S$ are in fact geometric fibres of $X' \to S'$. □

So let's study the existence and (non-)uniqueness of smooth liftings. As in the previous section, our strategy is as follows: First of all, a smooth lifting always exists locally on X'.

Lemma 1.3.2. *For every $x \in X'$, there is some open $U' \subset X'$ containing x such that there exists a smooth morphism $U \to S$ with $U' = U \times_S S'$.*

We can of course make U affine by restriction. Then U' too has to be affine since $U' \to U$ is a closed immersion.

Sketch of proof. This is a local statement, so we can assume WLOG that X', S are affine and $X' \to S'$ is induced by a standard smooth ring map. But then we can simply lift it by lifting the polynomials $f_1, \ldots, f_m$ as in Definition 1.1. □

Knowing this, our task now is to measure the non-uniqueness of this lifting. The following general lemma is surprisingly convenient in this analysis.

Lemma 1.3.3. *Suppose $f : X \to Y$ is an S-morphism with X flat over S. Then f is an isomorphism if and only if its base-change $f \times \mathrm{id}_{S'} : X \times_S S' \to Y \times_S S'$ is.*

Proof. The "only if" part is immediate.

For the "if" part, first observe that $S' \to S$ is surjective and is a closed immersion. As its base-change, the projection $X \times_S S' \to X$ inherits these two properties. In particular, $X \times_S S' \to X$ is a homeomorphism. Similarly, $Y \times_S S' \to Y$ too is a homeomorphism. So f must be a homeomorphism since $f \times \mathrm{id}_{S'}$ is. Hence we may reduce to the affine case.

It suffices to show the following: Suppose $R \to R'$ is a surjective ring map with nilpotent kernel I (say $I^n = 0$), $u : M \to P$ is a morphism of R-modules with P flat over R, and $u \otimes_R R' : M/IM \to P/IP$ is an isomorphism, then u is an isomorphism.

Put $N = \ker u, Q = \operatorname{coker} u$. From the right-exactness of $- \otimes_R R'$, we get an exact sequence

$$M/IM \xrightarrow[\sim]{u \otimes_R R'} P/IP \longrightarrow Q/IQ \longrightarrow 0$$

which shows that $Q/IQ = 0$. So $Q = IQ = I^2 Q = \cdots = I^n Q = 0$. This also gives a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

whence the exact sequence

$$\mathrm{Tor}^1_R(P, R') \longrightarrow N/IN \longrightarrow M/IM \xrightarrow[\sim]{u \otimes_R R'} P/IP \longrightarrow 0.$$

But $\mathrm{Tor}^1_R(P, R') = 0$ as P is flat over R, hence $N/IN = 0$ which means that $N = IN = I^2 N = \cdots = I^n N = 0$. ☐

We are now ready to say something concrete. As usual, we consider only the case where $\mathcal{I}^2 = 0$. Take a smooth lifting $f : X \to S$ of $f' : X' \to S'$. Since f is flat, $X' \to X$ is automatically a closed immersion with square-zero ideal $f^*\mathcal{I}$. This becomes the $\mathcal{O}_{X'}$-module $(f')^*\mathcal{I}$ on X', where we interpret $\mathcal{I}$ as a $\mathcal{O}_{S'}$-module.

Suppose now that $\tilde{X} \to S$ is another lifting together with a map $X \to \tilde{X}$ making the diagram

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X' \\ \downarrow & \nwarrow & \downarrow \\ S & \longleftarrow & X \end{array}$$

commute, then this map must be an (S-)isomorphism by Lemma 1.3.3. In particular, $\tilde{X} \to S$ must also be smooth.

Note that such a map exists when X is affine by Proposition 1.1.2. So an affine U in Lemma 1.3.2 is unique up to (non-unique) isomorphism.

The non-canonicality of a smooth lifting can be easily measured by what we have done.

Proposition 1.3.4. *Suppose X' is separated over* $\mathrm{Spec}(\mathbb{Z})$*, $f' : X' \to S'$ is smooth, and $f : X \to S$ is a smooth lifting of it. The group $\mathrm{Aut}_S(X, S')$ of S-automorphisms of X which become the identity on X' is naturally isomorphic to $H^0(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'})$.*

Proof. Theorem 1.2.2 and Remark 1.2.3 shows that $\mathrm{Aut}_S(X, S')$ is a $H^0(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'})$-torsor. But the group action is clearly compatible with composition of automorphisms. Hence this becomes an isomorphism of groups. ☐

1.4 Lifting Smooth Schemes

Theorem 1.4.1. *Let $i : S' \to S$ be a closed immersion with square-zero ideal $\mathcal{I}$ and let $f' : X' \to S'$ be a smooth morphism. Suppose X' is separated over* Spec($\mathbb{Z}$). *Then there is a natural "obstruction element"*

$$\mathfrak{o} = \mathfrak{o}(X', i) \in H^2(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'})$$

such that $\mathfrak{o} = 0$ if and only if f' admits a smooth lifting $X \to S$.

Moreover, if $\mathfrak{o} = 0$, then the set

$$\mathscr{L}(X', i) = \{\text{isomorphism classes of pairs } (X, \phi) \text{ such that } X \to S \text{ is smooth and } \phi : X \times_S S' \to X' \text{ is an isomorphism}\}$$

is a $H^1(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'})$-torsor.

Remark 1.4.1. Again, the separatedness hypothesis can be removed. See [4, Theorem 8.5.9(b)].

Proof. Write $\mathcal{G} = (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'}$.

Pick an affine open cover $\mathcal{U} = \{U'_\alpha\}_\alpha$ of X' such that each $U'_\alpha \to S'$ can be lifted to a smooth affine scheme $U_\alpha \to S$. Write $j_\alpha : U'_\alpha \to U_\alpha$ which is a surjective closed immersion. For indices $\alpha_1, \dots, \alpha_l$, we denote by $U'_{\alpha_1 \cdots \alpha_l}$ the (affine) open $U'_{\alpha_1} \cap \cdots \cap U'_{\alpha_l}$.

As in the proof of Theorem 1.2.2, we note the isomorphism $\check{H}^i(\mathcal{U}, \mathcal{G}) = H^i(X, \mathcal{G})$.

Now $j_\alpha(U'_{\alpha\beta})$ and $j_\beta(U'_{\alpha\beta})$ are both affine smooth liftings of $U'_{\alpha\beta}$. By our discussion in the last section, we get an S-isomorphism $\xi_{\beta\alpha} : j_\alpha(U'_{\alpha\beta}) \to j_\beta(U'_{\alpha\beta})$. Write $\xi^\gamma_{\beta\alpha} : j_\alpha(U'_{\alpha\beta\gamma}) \to j_\beta(U'_{\alpha\beta\gamma})$ for its restriction.

$c_{\alpha\beta\gamma} = (\xi^\beta_{\gamma\alpha})^{-1} \circ \xi^\alpha_{\gamma\beta} \circ \xi^\gamma_{\beta\alpha}$ is an element of $\mathrm{Aut}_S(j_\alpha(U'_{\alpha\beta\gamma}), S') \cong H^0(U'_{\alpha\beta\gamma}, \mathcal{G})$ (Proposition 1.3.4). They give the data of a Čech 2-cochain $c \in \check{C}^2(\mathcal{U}, \mathcal{G})$. c is in fact a cocycle: Its boundary has components

$$(\partial c)_{\alpha\beta\gamma\delta} = c_{\beta\gamma\delta} - c_{\alpha\gamma\delta} + c_{\alpha\beta\delta} - c_{\alpha\beta\gamma} = -c_{\alpha\gamma\delta} + c_{\alpha\beta\delta} + c_{\beta\gamma\delta} - c_{\alpha\beta\gamma}.$$

On $j_\alpha(U'_{\alpha\beta\gamma\delta})$, this is represented by the automorphism

$$\left((\xi^\delta_{\gamma\alpha})^{-1} \circ (\xi^\alpha_{\delta\gamma})^{-1} \circ \xi^\gamma_{\delta\alpha}\right) \circ \left((\xi^\beta_{\delta\alpha})^{-1} \circ \xi^\alpha_{\delta\beta} \circ \xi^\delta_{\beta\alpha}\right) \circ \left(\xi^{-1}_{\beta\alpha} \circ ((\xi^\gamma_{\delta\beta})^{-1} \circ \xi^\beta_{\delta\gamma} \circ \xi^\delta_{\gamma\beta}) \circ \xi_{\beta\alpha}\right) \circ \left((\xi^\gamma_{\beta\alpha})^{-1} \circ (\xi^\alpha_{\gamma\beta})^{-1} \circ \xi^\beta_{\gamma\alpha}\right)$$

where everything cancels out to give the identity. Similar to the proof of Theorem 1.2.2, it's clear that the class $\mathfrak{o} = [c] \in H^2(X, \mathcal{G})$ does not depend on the choice of $\mathcal{U}$, the local liftings, or $\{\xi_{\beta\alpha}\}_{\beta\alpha}$. For example, if a different $\{\tilde{\xi}_{\beta\alpha}\}_{\beta\alpha}$ was chosen, then the resulting $\tilde{c}$ would differ from c by the coboundary of the 1-cochain $\tilde{\xi}^{-1}_{\beta\alpha} \circ \xi_{\beta\alpha} \in \mathrm{Aut}_S(j_\alpha(U'_{\alpha\beta}), S') \cong H^0(U'_{\alpha\beta}, \mathcal{G})$.

If a global lifting $X \to S$ exists, then we can take U_α to be the pullback of U'_α in X, and $\xi_{\beta\alpha}$ the identity map on $U_\alpha \cap U_\beta$. Then the local data of c are just the identity maps, so $c = 0$ which is a coboundary. Conversely, if $\mathfrak{o} = 0$, then $c = \partial\zeta$ for some 1-cycle ζ. We then modify each $\xi_{\beta\alpha}$ with $\zeta_{\beta\alpha}$. This new set of local data then gives $c = 0$, which means that the cocycle condition $\xi^\beta_{\gamma\alpha} = \xi^\alpha_{\gamma\beta} \circ \xi^\gamma_{\beta\alpha}$ holds. So we can glue these U_α together to obtain a global lifting $X \to S$.

Now suppose $\mathfrak{o} = 0$, then $\mathscr{L} \neq \varnothing$. Fix an "origin" $[(X, \phi)] \in \mathscr{L}$. For indices $\alpha_1, \dots, \alpha_l$, we write $X_{\alpha_1 \cdots \alpha_l}$ to denote the pullback of $U'_{\alpha_1 \cdots \alpha_l}$ in X.

Any Čech 1-cocycle $\zeta_{\alpha\beta} \in \mathrm{Aut}_S(X_{\alpha\beta}, S') \cong H^0(U'_{\alpha\beta}, \mathcal{G})$ provides a set of gluing data for $\{X_\alpha\}_\alpha$ since they satisfy the cocycle condition. So they glue to a scheme X^ζ and an isomorphism $X^\zeta \times_S S' \to X'$. This is isomorphic to (X, ϕ) if and only if there is a system of elements $\mu_\alpha \in \mathrm{Aut}_S(X_\alpha, S') \cong H^0(U'_\alpha, \mathcal{G})$ gluing to an isomorphism $X \to X^\zeta$, which is precisely saying that $\zeta = \partial\mu$ is a coboundary.

Therefore this defines an action of $H^1(X, \mathcal{G})$ on $\mathscr{L}$ with trivial stabilisers. It is also transitive. Indeed, given any $[(Y, \psi)] \in \mathscr{L}$, we always have a (noncanonical) isomorphism $\phi_\alpha : Y_\alpha \to X_\alpha$. Then $Y \cong X^\zeta$ where $\zeta_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} \in \mathrm{Aut}_S(X_{\alpha\beta}, S')$. □

Remark 1.4.2. From the construction of $\mathfrak{o}$, the following naturality properties are immediate:

(i) Suppose $f' : X' \to S'$, $g' : Y' \to S'$ are smooth morphisms and $h' : X' \to Y'$ is an S'-morphism. Then $\mathfrak{o}(X', i)$ and $\mathfrak{o}(Y', i)$ have the same image in $H^2(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} (h')^* T_{Y'/S'})$. In particular, $\mathfrak{o}$ is invariant under automorphisms.

(ii) Suppose $X' \to S'$, $g' : Y' \to S'$ are smooth. Then we have $\mathfrak{o}(X' \times_{S'} Y', i) = i_1(\mathfrak{o}(X', i)) + i_2(\mathfrak{o}(Y', i))$ where i_1, i_2 are the compositions of pr_1^*, pr_2^* with the split injections from the identification $T_{X' \times_{S'} Y'/S'} = \mathrm{pr}_1^* T_{X'/S'} \oplus \mathrm{pr}_2^* T_{Y'/S'}$. It's worth noting that i_1, i_2 are injective.

These facts are particularly useful when one attempts to show the vanishing of $\mathfrak{o}$, since the cohomology group itself rarely vanishes.

Remark 1.4.3. Same as Remark 1.2.4, if $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ comes from a small surjection $R \to R'$ with kernel I and residue field k, then an identification

$$H^i(X', (f')^*\mathcal{I} \otimes_{\mathcal{O}_{X'}} T_{X'/S'}) = H^i(X_0, T_{X_0/k}) \otimes_k I$$

can be made, where $X_0 \to \mathrm{Spec}(k) = S_0$ is the base-change of $X' \to S'$ along $S_0 \to S'$.

2 Deformation of Abelian Schemes

2.1 Abelian Schemes

We recall the definition and basic properties of abelian schemes.

Definition 2.1. Let $\mathcal{C}$ be a locally small category with products and a final object. A group (or group object) in $\mathcal{C}$ is an object $G \in \mathrm{ob}(\mathcal{C})$ together with a factorisation $\mathrm{Mor}_{\mathcal{C}}(-, G) : \mathcal{C}^{\mathrm{op}} \to (\mathsf{Grp}) \to (\mathsf{Sets})$ through the forgetful functor $(\mathsf{Grp}) \to (\mathsf{Sets})$. We say G is commutative if the factor $h_G : \mathcal{C}^{\mathrm{op}} \to (\mathsf{Grp})$ lands in the subcategory (Ab) of abelian groups.

A morphism $f : G \to G'$ between groups in $\mathcal{C}$ is called a group homomorphism if $h_G(T) \to h_{G'}(T)$ is a group homomorphism for all $T \in \mathrm{ob}(\mathcal{C})$.

Remark 2.1.1. For a group G in $\mathcal{C}$, the group operation, identity, and inverse maps on $h_G(T)$ define, by Yoneda Lemma, morphisms $m_G : G \times G \to G$, $e_G : S \to G$ (where S is the final object of $\mathcal{C}$), and $i_G : G \to G$. They satisfy several commutative diagrams corresponding to the usual group axioms. For example, associativity is given by the commutativity of the diagram

$$\begin{array}{ccccc} (G \times G) \times G & \xrightarrow{m_G \times \mathrm{id}_G} & G \times G & \xrightarrow{m_G} & G \\ \downarrow{\cong} & & & & \downarrow{\mathrm{id}_G} \\ G \times (G \times G) & \xrightarrow[\mathrm{id}_G \times m_G]{} & G \times G & \xrightarrow[m_G]{} & G \end{array}$$

Conversely, for any $G \in \mathrm{ob}(\mathcal{C})$ and any choice of m_G, e_G and i_G satisfying these axioms, we get a factorisation of $\mathrm{Mor}_{\mathcal{C}}(-, G)$ through (Grp) by giving each $\mathrm{Mor}_{\mathcal{C}}(T, G)$ the structure of a group via $(m_G)_T$, $(e_G)_T$ and $(i_G)_T$.

For $f, g : T \to G$, we write $f + g$ for $m_G \circ (f, g)$, $-f$ for $i_G \circ f$, 0 for $e_G \circ (T \to S)$, and so on. Of course, $+$ is only commutative when G is.

Definition 2.2. A (commutative) group in $\mathcal{C} = (\mathsf{Sch}/S)$ is called a (commutative) group scheme over S. An abelian scheme over S is a group scheme over S which is smooth, proper, and geometrically connected on every fibre. An abelian scheme over a field is called an abelian variety.

Remark 2.1.2. Suppose $G \to S$ is a group scheme and $S' \to S$ is a morphism, then $G \times_S S' \to S'$ has the natural structure of a group scheme via $h_{G'} = h_G \circ b^{\mathrm{op}}$ where $b : (\mathrm{Sch}/S') \to (\mathrm{Sch}/S)$ sends $X' \to S'$ to the composite $X' \to S' \to S$. Since the extra properties defining an abelian scheme are all stable under base-change, $G' \to S'$ is an abelian scheme whenever $G \to S$ is.

The properness of abelian schemes has strong consequences due to the following result:

Theorem 2.1.1 (Mumford's Rigidity Lemma)**.** *Suppose S is connected and locally Noetherian, $p : X \to S$ is proper, flat, and geometrically integral on every fibre, $q : Y \to S$ is separated, and $f : X \to Y$ is an S-morphism such that $f(X_s)$ is a single point for some $s \in S$. Then q has a section $\eta : S \to Y$ with $\eta \circ p = f$.*

Proof. This is a slightly weaker version of [10, Proposition 6.1]. □

Corollary 2.1.2. *Let p, q be as in Theorem 2.1.1, and suppose in addition that $q : Y \to S$ is in fact a group scheme over S. If $f, g : X \to Y$ are S-morphisms such that $f_s = g_s$ for some $s \in S$, then q has a section $\eta : S \to Y$ with $f = \eta \circ p + g$.*

Proof. $f - g = f + (-g)$ maps X_s to the image of the unit section of the group scheme $Y_s \to \mathrm{Spec}(\kappa(s))$, so we invoke Theorem 2.1.1 to conclude. □

Corollary 2.1.3. *Suppose $p : X \to S$ is proper, flat, and geometrically integral on every fibre, $q : Y \to S$ is a connected, locally Noetherian S-scheme admitting a section $\epsilon : S \to Y$. Suppose G is a separated group scheme over S and $F : X \times_S Y \to G$ is an S-morphism, then there are $F_X : X \to G$ and $F_Y : Y \to G$ such that $F = F_Y \circ \mathrm{pr}_2 + F_X \circ \mathrm{pr}_1$.*

Proof. Consider the Y-morphisms $f, g : X \times_S Y \to G \times_S Y$, where $f = (F, \mathrm{pr}_2)$ and g is the base-change of $F_X = F \circ (\mathrm{id}_X, \epsilon \circ p)$. Pick any $s \in S$, then $f_{\epsilon(s)} = g_{\epsilon(s)}$. Therefore Corollary 2.1.2 applies, and we take F_Y to be η composed with the first projection $G \times_S Y \to G$. ☐

For the rest of this section we fix a locally Noetherian base scheme S.

Theorem 2.1.4. *Suppose $X \to S$ is an abelian scheme, $G \to S$ is a separated group scheme, and $f : X \to G$ is an S-morphism such that $f \circ e_X = e_G$. Then f is a group homomorphism.*

Proof. When X is connected, this follows from Corollary 2.1.3 with $F = f \circ m_X$.

In general, we use the fact that connected components of locally Noetherian schemes are open (cf. [13, Lemma 0819]). So X is a disjoint union $X = \coprod_i X_i$ of connected open subschemes X_i.

Let $S_i = e_X^{-1}(X_i)$, then $S = \coprod_i S_i$ is a disjoint union of open subschemes. Note that S_i are nonempty since e_X is a section. Pulling each S_i back along the structure map of G gives open subschemes $G_i \subset G$ whose disjoint union is G.

We now base-change along the open immersions $S_i \to S$. In view of Remark 2.1.2, $X_i \to S_i$ is an abelian scheme, $G_i \to S_i$ is a separated group scheme, and the group structures on both come from restrictions.

f restricts to an S'-morphism $f_i : X_i \to G_i$ sending e_{X_i} to e_{G_i}. By the connected case, we know that each f_i is a group homomorphism. So f is also a group homomorphism. ☐

Corollary 2.1.5. *Any abelian scheme is commutative.*

Proof. Apply Theorem 2.1.4 to $i_X : X \to X$. ☐

Corollary 2.1.6. *For any section $e : S \to X$ of an S-scheme X, there is at most one group law $m : X \times X \to X$ making X an abelian scheme with identity e.*

Proof. For any two such group law, apply Theorem 2.1.4 to id_X. ☐

Lastly, we mention some facts about projectivity of abelian schemes.

Theorem 2.1.7. *(i) If $S = \mathrm{Spec}(A)$ for a normal domain A, then every abelian scheme over S is projective. In particular, every abelian variety is projective.*

(ii) For any affine S, any abelian scheme over S has the finite-affine property: Any finite set of points is contained in an affine open.

Proof. [3, pp. 5 – 7]. ☐

2.2 Lifting across Artinian Local Rings

Let $i : S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ be a closed immersion of Noetherian affine schemes such that the ring map $R \to R'$ has square-zero kernel $I \leq R$. We are interested in the following question: Given an abelian scheme $X' \to S'$, can we always lift it to an abelian scheme $X \to S$?

The first step is to lift X' to a smooth scheme.

Proposition 2.2.1. *Any abelian scheme $X' \to S'$ has a smooth lifting $X \to S$. Moreover, any smooth lifting of it is proper and geometrically connected on every fibre.*

Proof. In view of Theorem 1.4.1, we seek the vanishing of $\mathfrak{o}(X', i)$.

From Remark 1.4.2(ii), $\mathfrak{o}(X' \times_{S'} X', i) = i_1(\mathfrak{o}(X', i)) + i_2(\mathfrak{o}(X', i))$. On the other hand, if we apply the shearing automorphism $X' \times_{S'} X' \to X' \times_{S'} X'$ which on T-valued points is defined by $(x, y) \mapsto (x + y, y)$, we get $\mathfrak{o}(X' \times_{S'} X', i) = i_1(\mathfrak{o}(X', i)) + 2i_2(\mathfrak{o}(X', i))$ by Remark 1.4.2(i).

Therefore $i_2(\mathfrak{o}(X', i)) = 0$, hence $\mathfrak{o}(X', i) = 0$. This gives the existence of a smooth lifting. The next part follows from Lemma 1.3.1. □

This of course is not good enough: We still want to lift the group structure. The description in Remark 2.1.1 is convenient for this purpose, as we have already established an obstruction theory about lifting morphisms in Theorem 1.2.2.

The unit section $e_{X'} : S' \to X'$ can always be lifted to an S-morphism $e_X : S \to X$ (i.e. a section of $X \to S$) by Proposition 1.1.2 and Remark 1.2.3. By Corollary 2.1.6, e_X determines at most one group structure on X.

Finding one such group structure, however, requires intricate work. We shall first discuss the simpler case where $R \to R'$ is a small surjection of Artinian local rings. In this case, S and S' are both just one-point schemes.

Theorem 2.2.2. *Suppose $R \to R'$ is a small surjection of Artinian local rings. Then $X \to S$ has a group structure with unit section e_X.*

Proof. Note that $X' \times_{S'} X' \to X \times_S X$ is a closed immersion with square-zero ideal. This inspires us to try and lift $\mu' : X' \times_{S'} X' \to X'$ given by $(x, y) \mapsto x - y$.

Let $k = R/\mathfrak{m}_R = R'/\mathfrak{m}_{R'}$ and write $S_0 = \mathrm{Spec}(k)$, $X_0 = X' \times_{S'} S_0$. Note that X_0 is an abelian variety by Remark 2.1.2. Let $\mu_0 : X_0 \times_k X_0 \to X_0$ be the base-change of μ'.

By Theorem 1.2.2 and Remark 1.2.4, the obstruction $\mathfrak{o}$ to the existence of such a lifting is naturally an element of $H^1(X_0 \times_k X_0, \mu_0^* \Theta) \otimes_k I$, where $\Theta = T_{X_0/k}$.

Θ is in fact trivial, since any tangent vector v at the identity of X_0 gives rise to a global section of Θ by left-translation (cf. [9, p. 42, (iii)]). We write $\Theta = \mathcal{O}_{X_0} \otimes_k V$ where $V = H^0(X_0, \Theta)$. By the Künneth formula (cf. [13, Lemma 0BED]) and the fact that $H^0(X_0, \mathcal{O}_{X_0}) \cong k$,

$$\begin{aligned} H^1(X_0 \times_k X_0, \mu_0^* \Theta) \otimes_k I &\cong H^1(X_0 \times_k X_0, \mathcal{O}_{X_0 \times_k X_0}) \otimes_k V \otimes_k I \\ &\cong \big(\mathrm{pr}_1^* H^1(X_0, \mathcal{O}_{X_0}) \oplus \mathrm{pr}_2^* H^1(X_0, \mathcal{O}_{X_0})\big) \otimes_k V \otimes_k I. \end{aligned}$$

Let $g_1, g_2 : X' \to X' \times_{S'} X'$ be given by $x \mapsto (x, e_{X'})$ and $x \mapsto (x, x)$, respectively. Write $(g_1)_0$, $(g_2)_0$ for their base-change to S_0. Then $\mathrm{pr}_i \circ (g_i)_0 = \mathrm{id}_{X_0}$. So the vanishing of $\mathfrak{o}$ would follow from the vanishing of $(g_i)_0^* \mathfrak{o}$.

By Remark 1.2.5, $(g_i)_0^* \mathfrak{o}$ is the obstruction to the existence of a lifting of $\mu' \circ g_i : X' \to X'$. But such a lifting does exist for both: id_X lifts $\mu' \circ g_1$, and $e_X \circ (X \to S)$ lifts $\mu' \circ g_2$. So these obstructions must vanish, therefore $\mathfrak{o}$ vanishes, i.e. μ' lifts to some $\mu : X \times_S X \to X$.

Of course, only one such choice could possibly work. To make the choice, observe the following: The set of liftings μ of μ' is a $(V \otimes_k I)$-torsor by Theorem 1.2.2. On the other hand, if μ is any such lifting, then $\mu \circ \Delta_X$ lifts $\mu' \circ \Delta_{X'}$. But the set of liftings of $\mu' \circ \Delta_{X'}$ is also a $(V \otimes_k I)$-torsor again by Theorem 1.2.2, so the map $\mu \mapsto \mu \circ \Delta_X$ establishes a bijection between the set of liftings of μ' and the set of liftings of $\mu' \circ \Delta_{X'} = e_{X'} \circ (X' \to S')$. In particular, there is a unique lifting μ of μ' satisfying $\mu \circ \Delta_X = e_X \circ (X \to S)$.

We set $i_X(x) = \mu(e_X, x)$ and $m_X(x, y) = \mu(x, i_X(y))$ for T-valued points x, y. The desired group axioms all take the form $h_1 = h_2$, where $h_i : \tilde{X} = X \times_S \cdots \times_S X \to X$ are morphisms built from μ, e_X, diagonal, projections, and identity, and they satisfy $h_i \circ (e_X, \ldots, e_X) = e_X$.

Since $\mu_0(x, y) = x - y$, $h_1 = h_2$ always holds after base-change to S_0, therefore the image of $\mu \circ (h_1, h_2)$ is a single point. By Theorem 2.1.1, there is a section $\eta : S \to X$ such that

$\mu \circ (h_1, h_2) = \eta \circ (\tilde{X} \to S)$. Composing both sides with $(e_X, \ldots, e_X) : S \to \tilde{X}$ shows that $\eta = e_X$ since $h_i \circ (e_X, \ldots, e_X) = e_X$ and $\mu \circ (e_X, e_X) = \mu \circ \Delta_X \circ e_X = e_X \circ (X \to S) \circ e_X = e_X$.

Now consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_S X \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ S & \xrightarrow[e_X]{} & X \end{array}$$

The induced map $X \to (X \times_S X) \times_{\mu, e_X} S$ becomes an isomorphism after base-change to S' (since $\mu'(x, y) = x - y$), hence is itself an isomorphism by Lemma 1.3.3. Therefore this diagram is in fact Cartesian, which means that (h_1, h_2) factors through Δ_X, i.e. $h_1 = h_2$. □

Corollary 2.2.3. *Suppose $R \to R'$ is a small surjection of Artinian local rings, then any abelian scheme over R' lifts to an abelian scheme over R.*

Such a lifting, in fact, exists for any surjection of (Noetherian) rings $R \to R'$ with nilpotent kernel. Before discussing this more general case, let's first try to analyse the set of liftings in this special case while we're at it.

2.3 The Local Moduli

For a ring map $B \to A$ and a B-scheme X (or, in general, a functor $X : (\mathsf{Sch}/B)^{\mathrm{op}} \to (\mathsf{Sets})$), we write $X \otimes_B A$ for $X \times_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. For a B-morphism f, we write $f \otimes_B A$ for its base-change to $\mathrm{Spec}(A)$.

Fix a field k and a complete local Noetherian ring W with residue field $W/\mathfrak{m}_W = k$. We write (Art/W) for the category whose objects are Artinian local W-algebras R such that $W \to R$ is local and the induced map on residue fields $k = W/\mathfrak{m}_W \to R/\mathfrak{m}_R$ is an isomorphism, and whose morphisms are (necessarily local) W-homomorphisms.

Clearly, if W' is another complete local Noetherian ring and $W \to W'$ is a local homomorphism which induces an isomorphism on residue fields, then (Art/W') may be regarded as a full subcategory of (Art/W). In particular, (Art/W) contains (Art/k) as a full subcategory.

Example 2.3.1. $k[\epsilon]/(\epsilon^2)$ is an object of (Art/W).

Fix an abelian variety X_0 over k. Following [11, p. 273], we consider the local moduli functor $\mathscr{M} = \mathscr{M}_{X_0} : (\mathsf{Art}/W) \to (\mathsf{Sets})$ defined as follows: For any object R in (Art/W), we set

$$\begin{aligned} \mathscr{M}(R) = \{&\text{isomorphism classes of pairs } (X, \phi) \text{ such that} \\ &\qquad X \text{ is an abelian scheme over } R \text{ and } \phi : X \otimes_R k \to X_0 \text{ is an isomorphism}\}; \end{aligned}$$

and for any homomorphism $f : R \to R'$ in (Art/W), we set $\mathscr{M}(f)$ to be the function that takes $[(X, \phi)] \in \mathscr{M}(R)$ to $[(X \otimes_R R', \phi)] \in \mathscr{M}(R')$ where we make the natural identification $(X \otimes_R R') \otimes_{R'} k = X \otimes_R k$.

By what we have discussed in the previous section, lifting abelian schemes is the same as lifting the underlying smooth schemes. More precisely,

Lemma 2.3.1. *Let $\pi : R \to R'$ be a small surjection in (Art/W). For any $[(X', \phi)] \in \mathscr{M}(R')$, we have a natural bijection $\kappa : \mathscr{M}(\pi)^{-1}([(X', \phi)]) \to \mathscr{L}(X', \mathrm{Spec}(\pi))$ (cf. Theorem 1.4.1).*

Proof. Suppose $\mathscr{M}(\pi)([(Y, \psi)]) = [(X', \phi)]$. By definition, there exists an isomorphism $\psi^\flat : Y \otimes_R R' \to X'$ of abelian schemes such that $\phi \circ (\psi^\flat \otimes_{R'} k) = \psi$. We set $\kappa([(Y, \psi)]) = [(Y, \psi^\flat)] \in \mathscr{L}(X', \mathrm{Spec}(\pi))$.

This is well-defined, for if a different $\tilde{\psi}^\flat$ was chosen, then $\tilde{\psi}^\flat \circ (\psi^\flat)^{-1}$ is an automorphism of the abelian scheme X' whose base-change to k is the identity on X_0. So it must be the identity on X' by Corollary 2.1.2, i.e. $\tilde{\psi}^\flat = \psi^\flat$.

κ is surjective by Theorem 2.2.2. To see that it is also injective, suppose $\kappa([(Y,\psi)]) = \kappa([(Z,\mu)])$, then there is an isomorphism of schemes $b : Y \to Z$ such that $\mu^\flat \circ b = \psi^\flat$. So $\mu \circ (b \otimes_R k) = \psi$.

Consider $h = b - b \circ e_Y \circ f_Y : Y \to Z$ where f_Y, e_Y are the structure morphism and unit section of Y, respectively. Then $h \circ e_Y = e_Z$ is the unit section of Z, hence h is a homomorphism by Corollary 2.1.4.

Since μ and ψ are both homomorphisms, we must have $\mu \circ (h \otimes_R k) = \psi$. From this, we also know that $h \otimes_R k$ is an isomorphism, so h must be an isomorphism as well by Lemma 1.3.3. Therefore $[(Y,\psi)] = [(Z,\mu)]$. □

Another consequence of what we've done is that, if we view $(\mathsf{Art}/W)^{\mathrm{op}}$ as a full subcategory of (Sch/W), then:

Proposition 2.3.2. *$\mathscr{M}$ is formally smooth (cf. Definition 1.4).*

Proof. Due to Corollary 2.2.3, it suffices to show that any surjection of Artinian local rings (necessarily with nilpotent kernel) is a composite of small surjections.

Suppose R is any Artinian local ring and $I \lneq R$ is any ideal. Choose N such that $\mathfrak{m}_R^N = 0$, then $R \to R/I$ factors as a composite $R \to R/(\mathfrak{m}_R^{N-1} I) \to R/(\mathfrak{m}_R^{N-2} I) \to \cdots \to R/(\mathfrak{m}_R I) \to R/I$ of small surjections. □

For any functor defined on a category of algebras, one of the most important questions one can ask is whether the functor is representable. This is not quite the case for $\mathscr{M}$, but we have the next best thing.

Denote by (ProArt/W) the category whose objects are complete local Noetherian W-algebras $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^r$ is an object of (Art/W) for all $r \geq 1$, and whose morphisms are local W-homomorphisms. It is immediate that (Art/W) is a full subcategory of (ProArt/W).

Example 2.3.2. Any formal power series ring over W is an object of (ProArt/W).

Definition 2.3. A functor $F : (\mathsf{Art}/W) \to (\mathsf{Sets})$ is pro-representable by an object $\mathcal{O}$ of (ProArt/W) if it is naturally isomorphic to $\mathrm{Hom}_{(\mathsf{ProArt}/W)}(\mathcal{O}, -)|_{(\mathsf{Art}/W)}$.

Theorem 2.3.3. *$\mathscr{M}$ is pro-representable by $\mathcal{O} = W[[t_{1,1}, \ldots, t_{g,g}]]$ where $g = \dim X_0$.*

To prove this theorem, we need some general theory surrounding pro-representability.

Definition 2.4. Let $\mathcal{C}$ be a category with fibre products and a final object S. A functor $F : \mathcal{C} \to (\mathsf{Sets})$ is left-exact if $F(S) = \{*\}$ and the natural map $F(X \times_Y Z) \to F(X) \times_{F(Y)} F(Z)$ is a bijection.

Suppose F is a left-exact functor on (Art/W). Then $F(k[\epsilon]/(\epsilon^2))$ has the natural structure of a k-vector space. Indeed, we can set the zero element to be the image of $\{*\} = F(k) \to F(k[\epsilon]/(\epsilon^2))$. Any $\lambda \in k$ acts on $F(k[\epsilon]/(\epsilon^2))$ via the ring homomorphism $k[\epsilon]/(\epsilon^2) \to k[\epsilon]/(\epsilon^2)$, $a + b\epsilon \mapsto a + b\lambda\epsilon$. And addition is given by the map $F(k[\epsilon]/(\epsilon^2)) \times F(k[\epsilon]/(\epsilon^2)) = F(k[\epsilon]/(\epsilon^2) \times_k k[\epsilon]/(\epsilon^2)) \to F(k[\epsilon]/(\epsilon^2))$ via $k[\epsilon]/(\epsilon^2) \times_k k[\epsilon]/(\epsilon^2) \to k[\epsilon]/(\epsilon^2)$, $(a + b\epsilon, a + b'\epsilon) \mapsto a + (b + b')\epsilon$.

Theorem 2.3.4 (Schlessinger's Criterion)**.** *A functor $F : (\mathsf{Art}/W) \to (\mathsf{Sets})$ is pro-representable if (and only if) it is left-exact and $\dim_k F(k[\epsilon]/(\epsilon^2)) = m < \infty$. And it suffices to check left-exactness for fibre products of the form $R \times_{R'} T$ where $R \to R'$ is a small surjection.*

If in addition F is formally smooth, then it is pro-representable by $\mathcal{O} \cong W[[t_1, \ldots, t_m]]$.

Proof. [12, Theorem 2.11, Proposition 2.5]. □

Proof of Theorem 2.3.3. We will show that $\mathscr{M}$ satisfies the conditions listed in Theorem 2.3.4.

Left-exactness: $\mathscr{M}(k) = \{[(X_0, \mathrm{id}_{X_0})]\}$ by definition. Now suppose $R \to R'$ is a small surjection and $T \to R'$ is any morphism in (Art/W). Write $Q = R \times_{R'} T$ and label the morphisms as in the following diagram.

$$\begin{array}{ccc} Q & \xrightarrow{\chi} & R \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \longrightarrow & R' \end{array}$$

Then ρ is surjective since π is, and its kernel is $J = I \times \{0\}$ where $I = \ker \pi$. In particular, $\mathfrak{m}_Q \cdot J = (\mathfrak{m}_R \times_{\mathfrak{m}_{R'}} \mathfrak{m}_T) \cdot (I \times \{0\}) = 0$, i.e. ρ is a small surjection.

We need to show that the natural map $\mathscr{M}(Q) \to \mathscr{M}(R) \times_{\mathscr{M}(R')} \mathscr{M}(T)$ is a bijection. Since the source is nonempty (cf. Proposition 2.3.2), the target has to as well.

Choose any $([(Y, \psi)], [(X, \phi)]) \in \mathscr{M}(R) \times_{\mathscr{M}(R')} \mathscr{M}(T)$ and let $[(X', \phi)] = \mathscr{M}(\pi)([(X, \phi)])$. In view of Lemma 2.3.1, Theorem 1.4.1 (combined with Proposition 2.2.1), and Remark 1.4.3, we see that $\mathscr{M}(\chi)$ restricts to a bijection

$$\begin{array}{ccccc} \mathscr{M}(\rho)^{-1}([(Y,\psi)]) & \xrightarrow[\sim]{\kappa} & \mathscr{L}(Y, \mathrm{Spec}(\rho)) & \xrightarrow[\sim]{} & H^1(X_0, T_{X_0/k}) \otimes_k I \\ {\scriptstyle\mathscr{M}(\chi)}\downarrow & & & & \wr\downarrow{\scriptstyle \mathrm{id} \otimes \chi|_I} \\ \mathscr{M}(\pi)^{-1}([(X',\phi)]) & \xleftarrow[\sim]{\kappa} & \mathscr{L}(X', \mathrm{Spec}(\pi)) & \xleftarrow[\sim]{} & H^1(X_0, T_{X_0/k}) \otimes_k J \end{array}$$

which is precisely what we need.

Dimension of $\mathcal{M}(k[\epsilon]/(\epsilon^2))$: By Theorem 1.4.1, we have a linear isomorphism $\mathcal{M}(k[\epsilon]/(\epsilon^2)) \cong H^1(X_0, T_{X_0/k}) \otimes_k (k\epsilon) \cong H^1(X_0, T_{X_0/k})$.

Same as in the proof of Theorem 2.2.2, we have $T_{X_0/k} \cong \mathcal{O}_{X_0} \otimes_k V$ where $V = H^0(X_0, T_{X_0/k})$ has dimension $g = \dim X_0$. So $\dim_k \mathcal{M}(k[\epsilon]/(\epsilon^2)) = g \dim H^1(X_0, \mathcal{O}_{X_0}) = g^2$ by [9, p. 129, Corollary 2].

Formal smoothness: Proposition 2.3.2. □

2.4 Lifting in General

Let's finish what we left off and prove Corollary 2.2.3 more generally.

Theorem 2.4.1 (Grothendieck)**.** *Suppose $R \to R'$ is any surjection of (Noetherian rings) with nilpotent kernel. Then any abelian scheme over R' lifts to an abelian scheme over R.*

Similar to what we did in the proof of Corollary 2.1.4, the assertion can be reduced to the connected case: Suppose $i : S' \to S$ is a closed immersion of Noetherian schemes with nilpotent kernel. Then $S = \coprod_j S_j$ is a disjoint union of open subschemes by [13, Lemma 0819]. Write S'_j for $S_j \times_S S'$, then $S' = \coprod_j S'_j$ too is a disjoint union of open subschemes.

Suppose any abelian scheme over S'_j lifts to an abelian scheme over S_j for all j. For any abelian scheme $X' \to S'$, the open subscheme $X'_j = X' \times_{S'} S'_j$ is an abelian scheme over S'_j for each j and $X' = \coprod_j X'_j$. Lift $X'_j \to S'_j$ to $X_j \to S_j$. Then $X = \coprod_j X_j \to \coprod_j S_j = S$ is an abelian scheme lifting $X' \to S'$.

By Proposition 2.2.1, to show Theorem 2.4.1, it suffices to establish:

Theorem 2.4.2. *Let S be a connected Noetherian scheme and $X \to S$ be a proper smooth morphism equipped with a section $e = e_X : S \to X$. If, for some $s : \mathrm{Spec}(k) \to S$, the fibre X_s is an abelian variety with unit section $e \circ s$, then $X \to S$ can be made an abelian scheme with unit section e.*

For simple cases, this is entirely classical.

Theorem 2.4.3 (Koizumi)**.** *Theorem 2.4.2 is true if S is the spectrum of a valuation ring, and s is the generic point of S.*

Proof. [7, Theorem 3]. □

Let's now see how one might prove Theorem 2.4.2 in general.

To give $X \to S$ the structure of an abelian scheme with unit e_X is the same as to give a morphism $\mu : X \times_S X \to X$ such that $i_X(x) = \mu(e_X, x)$, $m_X(x, y) = \mu(x, i_X(y))$, and e_X together satisfy various group axioms as in Remark 2.1.1. These axioms translate to identities involving μ, e, diagonal, projections, and identity. In order to find μ, we seek the aid of a global deformation functor.

Definition 2.5. Let X, Y be S-schemes. Write (LNSch/S) for the category of locally Noetherian S-schemes. We define the functor $\underline{\mathrm{Mor}}_S(X, Y) : (\mathsf{LNSch}/S)^{\mathrm{op}} \to (\mathsf{Sets})$ by assigning to each S-scheme T the set

$$\underline{\mathrm{Mor}}_S(X, Y)(T) = \{T\text{-morphisms } X \times_S T \to Y \times_S T\}$$

and to each $T' \to T$ the function that takes $f \in \underline{\mathrm{Mor}}_S(X, Y)(T)$ to its base-change to T'.

When X is projective and flat and Y is quasi-projective, $\underline{\mathrm{Mor}}_S(X, Y)$ is representable by a quasi-projective scheme by the theory of Hilbert schemes (cf. [4, Theorem 5.23]). In general, $\underline{\mathrm{Mor}}_S(X, Y)$ is not always representable by a scheme, but we still have:

Theorem 2.4.4. *Suppose X is flat, proper, and of finite presentation over S, and Y is separated and of finite presentation over S, then $\underline{\mathrm{Mor}}_S(X, Y)$ is an algebraic space locally of finite presentation over S.*

Proof. [13, Proposition 0D1C]. □

We shall not discuss the theory of algebraic spaces in detail, since it is beyond the scope of this essay. A comprehensive reference can be found in [13, Part 0ELT].

Proposition 2.4.5. *Suppose we are in the situation of Theorem 2.4.2. Consider the functor $F : (\mathsf{LNSch}/S)^{\mathrm{op}} \to (\mathsf{Sets})$ sending each $f : T \to S$ to*

$$F(T) = \{\textit{structures of an abelian scheme on } X \times_S T \textit{ with unit } (e \circ f, \mathrm{id}_T)\}$$

and each S-morphism $T' \to T$ to the corresponding map given by base-change. Then F is representable by an open subscheme $U \subset S$.

Proof. As discussed above, to give a structure of an abelian scheme on $X_T = X \times_S T$ with unit $(e \circ f, \mathrm{id}_T)$ is the same as to give $\mu_T : X_T \times_T X_T \to X_T$ satisfying various identities involving μ_T, $e \circ f$, diagonal, projections, and identity.

These identities cut out a closed algebraic subspace $Z \hookrightarrow \underline{\mathrm{Mor}}_S(X, Y)$, with the property that $\mu_T \in \underline{\mathrm{Mor}}_S(X, Y)(T)$ gives a structure of an abelian scheme on X_T with unit $e \circ f$ if and only if the corresponding morphism $T \to \underline{\mathrm{Mor}}_S(X, Y)$ factors through Z. Hence Z represents F.

The morphism $\omega : Z \hookrightarrow \underline{\mathrm{Mor}}_S(X, Y) \to S$ is smooth by a version of Proposition 1.1.3 for algebraic spaces (cf. [13, Lemma 0APN]) and Theorem 2.2.2. We also know that $Z(T) = F(T)$ has at most one point for any locally Noetherian S-scheme T due to Corollary 2.1.6, so ω has to be a monomorphism.

But any flat monomorphism between algebraic spaces is representable by schemes (cf. [13, Lemma 0B8A])! So Z is in fact a scheme, and ω is an open immersion by [13, Theorem 025G]. □

Proof of Theorem 2.4.2. This open immersion $U \subset S$ is also universally closed by Theorem 2.4.3 and the valuative criterion (cf. [13, Proposition 01KF]). Since S is connected and $F(k) \neq \varnothing$, we must have $U = S$. In particular, $F(S) \neq \varnothing$. □

3 The Serre-Tate Theorem

3.1 Formal Completions

The description of the local moduli $\mathscr{M}$ in Theorem 2.3.3 is non-canonical, in the sense that we have no natural choice of coordinates $t_{1,1}, \ldots, t_{g,g}$. To understand $\mathscr{M}$ further, we would have to find a way to describe the liftings using some intrinstic data of abelian schemes. This is the content of Serre-Tate theory.

For the rest of this essay, whenever we mention a group of any kind, we will automatically mean a *commutative* group.

First, let's analyse infinitesimal behaviours of an fppf sheaf. Fix a locally Noetherian base scheme S. We write $(\mathsf{Sch}/S)_{\text{fppf}}$ for the big fppf site of schemes over S, and $(\mathsf{Aff}/S)_{\text{fppf}}$ for the big affine fppf site.

We first remark that the categories of sheaves on $(\mathsf{Sch}/S)_{\text{fppf}}$ and $(\mathsf{Aff}/S)_{\text{fppf}}$ are equivalent via the natural restriction functor (cf. [13, Lemma 021V]). So we may, and will, talk about sheaves on these two sites as if they were the same thing.

Definition 3.1. Suppose X is a sheaf of sets on $(\mathsf{Sch}/S)_{\text{fppf}}$ and Y is a subsheaf of X. The j-th infinitesimal neighbourhood of Y in X is the subsheaf $\mathrm{Inf}^j_Y(X)$ of X defined by

$$\begin{aligned}\Gamma(T, \mathrm{Inf}^j_Y(X)) = \{t \in \Gamma(T, X) : &\text{ there exists an fppf cover } \{T_i \to T\}_{i \in I} \text{ and} \\ &\text{closed subschemes } T'_i \hookrightarrow T_i \text{ with } \mathcal{I}^{j+1}_{T'_i/T_i} = 0 \text{ such that } t_{T'_i} \in \Gamma(T'_i, Y)\}\end{aligned}$$

It's clear that Inf^j_Y is a subsheaf of Inf^{j+1}_Y in the natural way. So what we have is a directed system of subsheaves $\mathrm{Inf}^1_Y(X) \hookrightarrow \mathrm{Inf}^2_Y(X) \hookrightarrow \mathrm{Inf}^3_Y(X) \hookrightarrow \cdots$ of X.

In familiar situations, this definition simplifies to the following:

Lemma 3.1.1. *When X is a scheme and Y is a closed subscheme of X, then $\mathrm{Inf}^j_Y(X)$ is representable by the closed subscheme of X cut out by the ideal $\mathcal{I}^{j+1}_{Y/X}$.*

Proof. [8, Ch. II, Lemma (1.02)]. □

Definition 3.2. The formal completion of an fppf sheaf X along a subsheaf Y is the subsheaf $\mathrm{Inf}_Y(X) = \varinjlim_j \mathrm{Inf}^j_Y(X)$ of X, where the colimit is taken in the category of fppf sheaves.

It is perhaps enlightening to work out what actually happens when a colimit of this form is taken. By the universal properties, the colimit of a diagram of fppf sheaves is simply the sheafification of the colimit taken in the presheaf category. In fact, the sheafification process does almost nothing:

Lemma 3.1.2. *Suppose $Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \cdots$ is a directed system of fppf sheaves. Then $\Gamma(T, \varinjlim_j Y_j) = \varinjlim_j \Gamma(T, Y_j)$ for any quasicompact S-scheme T.*

Proof. This is a special case of [13, Lemma 0738]. □

This in particular works for any affine T. Therefore such a description completely characterises the resulting colimit.

Definition 3.3. Suppose $\hat{X}$ is an fppf sheaf of groups. Its completion is $\hat{X} = \mathrm{Inf}_S(X)$ where S is regarded as a subsheaf of X via the inclusion of the identity.

Clearly $\hat{X}$ is a subgroup of X (cf. [8, Ch. II, Lemma (1.1.6)]).

Let us now take $X = G$ where G is a smooth separated group scheme over S (e.g. an abelian scheme over S). Since G is separated, its unit section is a closed immersion. So we are in the familiar situation of Lemma 3.1.1.

Recall that the completion of a scheme along a closed subscheme is representable by a formal scheme. This process preserves products, so $\hat{G}$ is representable by a formal group, i.e. a group in the category (FmlSch/S) of formal schemes over S.

More precisely (or rather, pedantically), there is a formal group $\mathfrak{G}$, obtained by completing G along the closed subscheme defined by its unit section, such that $\hat{G} \cong h_{\mathfrak{G}}|_{(\mathsf{Sch}/S)^{\mathrm{op}}}$.

It turns out that the smoothness of G gives rise to a surprisingly simple description of $\mathfrak{G}$ as a formal scheme.

Proposition 3.1.3. *S can be covered by open affines $S = \bigcup_i \operatorname{Spec}(R_i)$ such that, for each i, there is an isomorphism of formal R_i-schemes $\mathfrak{G} \times_S \operatorname{Spec}(R_i) \cong \operatorname{Spf}(R_i[[t_1, \dots, t_r]])$ for some r.*

Proof. Combine [2, IV_4, Corollaire (16.9.9)] and [2, IV_4, Théorème (17.12.1)(c')]. □

We therefore make the following definition.

Definition 3.4. A formal Lie group over a Noetherian ring R is a formal group over R which is isomorphic, as a formal scheme, to $\operatorname{Spf}(R[[t_1, \dots, t_r]])$ for some r.

An fppf sheaf H of groups over R is representable by a formal Lie group if $H \cong h_{\mathfrak{H}}|_{(\mathsf{Sch}/R)^{\mathrm{op}}}$ for some formal Lie group $\mathfrak{H}$ over R.

An fppf sheaf H of groups over a locally Noetherian base S is locally representable by a formal Lie group if S can be covered by open affines $S = \bigcup_i \operatorname{Spec}(R_i)$ such that $H \times_S \operatorname{Spec}(R_i)$, viewed as an fppf sheaf of groups over R_i, is representable by a formal Lie group.

So Proposition 3.1.3 implies that $\hat{G}$ is always locally representable by a formal Lie group.

The structure of a formal Lie group is quite easy to understand. Fix a Noetherian base ring R and a formal Lie group $\mathfrak{H}$ over R. Choose an isomorphism $\mathfrak{H} \cong \operatorname{Spf}(R[[t_1, \dots, t_r]])$. Then $\mathfrak{H} \times \mathfrak{H} \cong \operatorname{Spf}(R[[x_1, \dots, x_r, y_1, \dots, y_r]])$. So we have

$$\begin{aligned} \operatorname{Mor}_{(\mathsf{FmlSch}/R)}(\mathfrak{H} \times \mathfrak{H}, \mathfrak{H}) &= \operatorname{Mor}_{(\mathsf{FmlSch}/R)}(\operatorname{Spf}(R[[x_1, \dots, x_r, y_1, \dots, y_r]]), \operatorname{Spf}(R[[t_1, \dots, t_r]])) \\ &= \operatorname{Hom}_{\mathrm{cont.}}(R[[t_1, \dots, t_r]], R[[x_1, \dots, x_r, y_1, \dots, y_r]]) \end{aligned}$$

Hence the multiplication map $m : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ gives rise to a continuous homomorphism $m^\sharp : R[[t_1, \dots, t_r]] \to R[[x_1, \dots, x_r, y_1, \dots, y_r]]$.

Definition 3.5. The tuple of formal power series

$$F = F_{\mathfrak{H}} = (m^\sharp(t_1), \dots, m^\sharp(t_r)) \in R[[x_1, \dots, x_r, y_1, \dots, y_r]]^{\oplus r}$$

is called a formal group law associated to $\mathfrak{H}$.

For simplicity, we will write $\underline{x}$ for $(x_1, \dots, x_r)$, $\underline{y}$ for $(y_1, \dots, y_r)$, and so on. It is immediate that $F(\underline{x}, \underline{y}) \equiv \underline{x} + \underline{y} +$ (terms of degree ≥ 2), $F(\underline{x}, F(\underline{y}, \underline{z})) = F(F(\underline{x}, \underline{y}), \underline{z})$, $F(\underline{x}, \underline{y}) = F(\underline{y}, \underline{x})$, and $F(\underline{x}, \underline{0}) = F(\underline{0}, \underline{x}) = \underline{x}$. Conversely, given any tuple of formal power series satisfying these conditions, we recover a group structure on $\operatorname{Spf}(R[[\underline{t}]])$ by considering the corresponding $m^\sharp$.

For any S-scheme T, $h_{\mathfrak{H}}(T)$ has underlying set $\operatorname{Hom}_{\mathrm{cont.}}(R[[\underline{t}]], \Gamma(T, \mathcal{O}_T)) = \operatorname{Nil}(\Gamma(T, \mathcal{O}_T))^{\oplus r}$ where $\Gamma(T, \mathcal{O}_T)$ is endowed with the discrete topology. The group operation is given simply by $\underline{x} +_F \underline{y} = F(\underline{x}, \underline{y})$ for $\underline{x}, \underline{y} \in \operatorname{Nil}(\Gamma(T, \mathcal{O}_T))^{\oplus r}$.

Example 3.1.1. Let $\mathbb{G}_{m/R} = \operatorname{Spec}(R[t, t^{-1}])$ be the multiplicative group. Then its completion $\hat{\mathbb{G}}_{m/R}$ (the *formal* multiplicative group) has formal group law $F(x, y) = (x + 1)(y + 1) - 1 = x + y + xy$.

3.2 p-Divisible Groups

Fix a locally Noetherian base scheme S.

Definition 3.6. A morphism $f : X \to S$ is finite flat if it is finite and flat.

By [13, Lemma 02KB], this is equivalent to saying that f is finite locally free.

Since finite morphisms are affine, any finite flat $f : X \to S$ is of the form $\underline{\mathrm{Spec}}_{\mathcal{O}_S}(\mathcal{A}) \to S$ where $\mathcal{A} = f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-(Hopf)algebra which is finite locally free when viewed as an $\mathcal{O}_S$-module.

Definition 3.7. The rank $\mathrm{rank}_S(X) : S \to \mathbb{Z}_{\geq 0}$ of a finite flat morphism $f : X \to S$ is the rank of the finite locally free $\mathcal{O}_S$-module $\mathcal{A} = f_*\mathcal{O}_X$, considered as a locally constant function on S.

Definition 3.8. A finite flat group scheme G over S is a group scheme over S such that $G \to S$ is finite flat.

Example 3.2.1. For a group-valued functor X and an integer $n \in \mathbb{Z}$, we have a multiplication-by-n homomorphism $[n] = [n]_X : X \to X$. We denote its kernel by $X[n]$. Of course $X[n]$ is a scheme (resp. an fppf sheaf) if X is.

Suppose now that $X \to S$ is an abelian scheme of relative dimension g (cf. Definition 1.3). When S is the spectrum of a field, it follows from the theory of abelian varieties that $[n]$ is finite flat of rank n^{2g} (cf. [13, Lemma 0BFG]). It is in fact faithfully flat by [13, Proposition 03RP].

For general S, we see from this special case that $[n]$ has finite fibres. But $[n]$ is also proper since it is a morphism between proper schemes, so it must be finite (cf. [13, Lemma 02LS]). On the other hand, $[n]$ is also faithfully flat since this can be checked on fibres (cf. [13, Lemma 039E]). So $[n]$ is finite and faithfully flat. It has rank n^{2g} since we can check this on fibres as well.

The kernel $X[n] \to S$, as a base-change of $[n]$, is then a finite flat group scheme of rank n^{2g}.

From now on we fix a prime p.

Definition 3.9. A p-divisible group (or a Barsotti-Tate group) over S is an fppf sheaf of groups over S that takes the form $G = \varinjlim_j G_j$ where $0 = G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$ is a directed system of finite flat group schemes over S such that:

(i) $G_j \hookrightarrow G_{j+1}$ is the composition of the closed immersion $G_{j+1}[p^j] \hookrightarrow G_{j+1}$ and an isomorphism $G_j \to G_{j+1}[p^j]$ of group schemes. In particular, $[p^i] : G_j \to G_j$ factors through G_{j-i}.

(ii) For all $0 \leq i \leq j$, the sequence

$$0 \longrightarrow G_i \hookrightarrow G_j \xrightarrow{[p^i]} G_{j-i} \longrightarrow 0$$

is exact in the category of fppf sheaves of groups.

Of course, such a directed system may be recovered from the p-divisible group via $G_j = G[p^j]$.

Note that (ii) is really asserting the faithful flatness of $[p^i] : G_j \to G_{j-i}$ by [8, Ch. I, Lemma (1.5)(b)]. It also clearly suffices to check the cases where $i = 1$.

Example 3.2.2. For an abstract group Γ, we write $\underline{\Gamma}_S = \coprod_{g\in\Gamma} S$ for the constant group scheme over S associated to Γ. For any S-scheme T, $\underline{\Gamma}_S(T)$ is the abelian group of locally constant functions $|T| \to \Gamma$, with the latter given the discrete topology.

Then the p-divisible group associated to the system

$$0 \hookrightarrow \underline{\mathbb{Z}/p\mathbb{Z}}_S \xhookrightarrow{1\mapsto p} \underline{\mathbb{Z}/p^2\mathbb{Z}}_S \xhookrightarrow{1\mapsto p} \cdots$$

is representable by $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_S$. p-divisible groups isomorphic to a finite direct sum of copies of it are called *constant* p-divisible groups over S.

Example 3.2.3. For a ring R and an integer N, consider the group scheme $\boldsymbol{\mu}_{N/R} = \ker[N]_{\mathbb{G}_{m/R}}$. For any R-algebra A, $\boldsymbol{\mu}_{N/R}(A)$ is the abelian group $\{x \in A : x^N = 1\}$ under ring multiplication. The p-divisible group associated to the system

$$0 \hookrightarrow \boldsymbol{\mu}_{p/R} \hookrightarrow \boldsymbol{\mu}_{p^2/R} \hookrightarrow \cdots$$

is denoted $\boldsymbol{\mu}_{p^\infty/R}$. p-divisible groups isomorphic to a finite direct sum of copies of it are called *toroidal* p-divisible groups over R.

Now suppose p is nilpotent in R (say $p^r = 0$). Then $\boldsymbol{\mu}_{p^\infty/R}(A) = 1+\mathrm{Nil}(A)$ for any R-algebra A. Indeed, suppose $x^{p^t} = 1$, then $\bar{x}^{p^t} = 1$ where $\bar{x} = x + \mathrm{Nil}(A) \in A_0 = A/\mathrm{Nil}(A)$. But p is zero in A_0, so $\bar{x} = 1$ as A_0 is torsion-free, i.e. $x \in 1+\mathrm{Nil}(A)$. Conversely, suppose $x = 1+y \in 1+\mathrm{Nil}(A)$. Choose s such that $y^{p^s} = 0$, then $x^{p^{r+s}} = 1$. Thus $x \in \boldsymbol{\mu}_{p^\infty/R}(A)$.

In other words, $\boldsymbol{\mu}_{p^\infty/R}$ is representable by the formal Lie group $\hat{\mathbb{G}}_{m/R}$ (cf. Example 3.1.1).

To justify the terminology, we recall the following definition:

Definition 3.10. An fppf sheaf G of groups over S is N-divisible if $[N] : G \to G$ is a surjective morphism of fppf sheaves.

Example 3.2.4. Any p-divisible group is p-divisible by part (ii) of Definition 3.9.

Example 3.2.5. For any abelian scheme X and any N, $[N] : X \to X$ is fppf, therefore surjective as a map of fppf sheaves. So any abelian scheme is N-divisible.

Proposition 3.2.1. *Suppose G is a p-divisible fppf sheaf of groups over S such that $G[p^j]$ is representable by a finite flat group scheme for all $j \geq 1$. Then $G[p^\infty] = \varinjlim_j G[p^j]$ is a p-divisible group over S.*

Proof. The only nontrivial part is to show that $[p] : G[p^j] \to G[p^{j-1}]$ is a surjective morphism of sheaves. This follows from p-divisibility: For any S-scheme U and any $s \in \Gamma(U, G[p^{j-1}]) \subset \Gamma(U, G)$, we know that there is an fppf cover $\{U_i \to U\}$ and $t_i \in \Gamma(U_i, G)$ such that $[p]t_i = s|_{U_i}$ for each i. But then $[p^j]t_i = [p^{j-1}]s|_{U_i} = 0$, so $t_i \in \Gamma(U_i, G[p^j])$. □

Remark 3.2.1. In fact, we only need to assume that $G[p]$ is representable by a finite flat group scheme. Indeed, the exact sequence holds by our argument, so each $G[p^j]$ comes from a sequence of extensions of finite flat group schemes, which forces it to be a finite flat group scheme as well.

Remark 3.2.2. This construction is functorial: Any homomorphism $f : G \to G'$ of fppf sheaves of groups restricts to homomorphisms $G[p^j] \to G'[p^j]$, hence a homomorphism $f[p^\infty] : \varinjlim_j G[p^j] \to \varinjlim_j G[p^j]$.

There is an intimate connection between p-divisible groups and formal Lie groups, as characterised by the following theorem.

Theorem 3.2.2 (Grothendieck-Messing)**.** *Suppose $S = \mathrm{Spec}(R)$ is affine with p nilpotent in R. Then any p-divisible group G over S is formally smooth, and $\hat{G}$ is locally representable by a formal Lie group.*

Proof. [8, Ch. II, Theorem (3.3.13), Theorem (3.3.18)]. □

3.3 Drinfeld's Rigidity Lemma

Fix a Noetherian ring R. Let (Grp/R) be the category of sheaves of groups on $(\mathsf{Aff}/R)_{\mathrm{fppf}}$. Objects of (Grp/R) will be called R-groups.

For any ideal $I \leq R$ and any functor $G : (\mathsf{Aff}/R)^{\mathrm{op}} \to (\mathsf{Ab})$, we write G_I for the subgroup functor of G defined by $G_I(A) = \ker(G(A) \to G(A/IA))$ for any $R \to A$.

Suppose $NR = 0$ for some integer N and $I \leq R$ is an ideal with $I^{\nu+1} = 0$ for some $\nu \geq 1$.

Lemma 3.3.1. *Suppose G is an R-group which is locally representable by a formal Lie group, then $[N^\nu]G_I = 0$.*

Proof. Assume, without loss of generality, that G is representable by a formal Lie group. Let $F(\underline{x}, \underline{y}) = \underline{x} + \underline{y} + (\text{terms of degree } \geq 2) \in R[[\underline{x}, \underline{y}]]^{\oplus r}$ be its formal group law. Recall that $G(A)$ is the set $\mathrm{Nil}(A)^{\oplus r}$ equipped with the group operation $\underline{x} +_F \underline{y} = F(\underline{x}, \underline{y})$.

Now let J be any nilpotent ideal of A. As $JA \subset \mathrm{Nil}(A)$, $G_J(A)$ is just the subgroup $(JA)^{\oplus r}$ (under $+_F$). For any $\underline{x} \in G_J(A)$,

$$[N]\underline{x} = N\underline{x} + (\text{terms of degree } \geq 2 \text{ in } \underline{x}) = (\text{terms of degree } \geq 2 \text{ in } \underline{x}) \in G_{J^2}(A)$$

since $NR = 0$. So $[N]G_J \subset G_{J^2}$.

For any $a \geq 1$, taking $J = I^a$ shows that $[N]G_{I^a} \subset G_{I^{2a}} \subset G_{I^{a+1}}$. Consequently $[N^\nu]G_I = 0$ since $I^{\nu+1} = 0$. □

Corollary 3.3.2. *Suppose G is an R-group such that $\hat{G}$ is locally representable by a formal Lie group. Then $[N^\nu]G_I = 0$.*

Proof. Note that G_I is always a subfunctor of the ν-th infinitesimal neighbourhood of G, hence a subfunctor of $\hat{G}$. So $(\hat{G})_I(A) = \hat{G}(A) \cap G_I(A) = G_I(A)$ is annihilated by $[N^\nu]$. □

Lemma 3.3.3. *Suppose H is a formally smooth functor $(\mathsf{Aff}/R)^{\mathrm{op}} \to (\mathsf{Ab})$ such that $[P]H_I = 0$ for some integer P. Then for any R-algebra A and any set-theoretic section $s : H(A/IA) \to H(A)$ of the surjective map $H(A) \to H(A/IA)$, $\ulcorner P \lrcorner = [P] \circ s$ is a group homomorphism which does not depend on the choice of s. Furthermore, $\ulcorner P \lrcorner$ is functorial in the sense that, whenever $A \to B$ is an R-homomorphism, the diagram*

$$\begin{array}{ccc} H(A) & \xleftarrow{\ulcorner P \lrcorner} & H(A/IA) \\ \downarrow & & \downarrow \\ H(B) & \xleftarrow[\ulcorner P \lrcorner]{} & H(B/IB) \end{array}$$

commutes.

Proof. For any $x, y \in H(A/IA)$, $s(x) + s(y) - s(x + y)$ lives in $H_I(A)$ which is annihilated by $[P]$. So $\ulcorner P \lrcorner$ must be a homomorphism. Furthermore, if s' were another set-theoretic section, then $s - s'$ has image in $H_I(A)$, so $[P] \circ s = [P] \circ s'$.

To show functoriality, consider the commutative diagram

$$\begin{array}{ccccc} H(A) & \xrightarrow{\pi_A} & H(A/IA) & \longrightarrow & 0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \\ H(B) & \xrightarrow[\pi_B]{} & H(B/IB) & \longrightarrow & 0 \end{array}$$

Take any set-theoretic sections s^A of π^A and s^B of π^B. Then $\pi_B \circ f \circ s_A = \bar{f} \circ \pi_A \circ s_A = \bar{f} = \pi_B \circ s_B \circ \bar{f}$. This means that the image of $f \circ s_A - s_B \circ \bar{f}$ lives inside $H_I(B)$, which again is annihilated by $[P]$. Thus $f \circ \ulcorner P \lrcorner = [P] \circ f \circ s_A = [P] \circ s_B \circ \bar{f} = \ulcorner P \lrcorner \circ \bar{f}$. □

Write $R_0 = R/I$.

Theorem 3.3.4 (Drinfeld's Rigidity Lemma)**.** *Suppose G and H are R-groups such that G is N-divisible, $\hat{H}$ is locally representable by a formal Lie group, and H is formally smooth. Let R_0 be the restriction of G to $(\mathsf{Aff}/R_0)^{\mathrm{op}}$ (which is simply the base-change of G along $R \to R_0$), and H_0 be that of H.*

(i) The groups $\mathrm{Hom}_{(\mathsf{Grp}/R)}(G, H)$ and $\mathrm{Hom}_{(\mathsf{Grp}/R_0)}(G_0, H_0)$ have no N-torsion.

(ii) The natural map $\mathrm{Hom}_{(\mathsf{Grp}/R)}(G,H) \to \mathrm{Hom}_{(\mathsf{Grp}/R_0)}(G_0,H_0)$ *is injective.*

(iii) Whenever $f_0 : G_0 \to H_0$ *is a homomorphism,* $N^\nu f_0$ *can be lifted to a homomorphism* $\ulcorner N^\nu f \lrcorner : G \to H$ *(necessarily unique by (ii)).*

(iv) $f_0 : G_0 \to H_0$ *can be lifted to a homomorphism* $f : G \to H$ *(necessarily unique by (ii)) if and only if* $\ulcorner N^\nu f \lrcorner$ *annihilates* $G[N^\nu]$*.*

Proof. (i): Immediate from the N-divisibility of G and (hence) G_0.

(ii): Suppose $f : G \to H$ is in the kernel of this map, then the diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & G(A/IA) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle 0} \\ H(A) & \longrightarrow & H(A/IA) \end{array}$$

commutes, which shows that f factors through H_I. So $N^\nu f = 0$ since $[N^\nu]H_I = 0$ by Corollary 3.3.2. But G is N-divisible (hence N^ν-divisible), hence $f = 0$.

(iii) Simply take $\ulcorner N^\nu f \lrcorner$ to be the composite

$$G(A) \longrightarrow G(A/IA) \xrightarrow{f_0} H(A/IA) \xrightarrow{\ulcorner N^\nu \lrcorner} H(A)$$

for any R-algebra A. This gives a well-defined homomorphism $G \to H$ by Lemma 3.3.3. And it lifts f_0 since $IA = 0$ whenever A comes from an R_0-algebra.

(iv) If f lifts f_0, then $\ulcorner N^\nu f \lrcorner = N^\nu f$ (by (ii)) which annihilates $G[N^\nu]$. Conversely, suppose $\ulcorner N^\nu f \lrcorner$ annihilates $G[N^\nu]$, then $\ulcorner N^\nu f \lrcorner = F \circ [N^\nu] = N^\nu F$ for some $F : G \to H$ since $[N^\nu] : G \to G$ is surjective (by the N-divisibility of G).

The base-change $F_0 : G_0 \to H_0$ of F then must satisfy $N^\nu F_0 = N^\nu f_0$. But (i) tells us that this implies $F_0 = f_0$. So $f = F$ lifts f_0. ☐

3.4 The Serre-Tate Theorem

Suppose we are in the situation of the last section, except with N assumed to be a power of p.

Let (AbSch/R) be the category of abelian schemes over R, and $(\mathsf{Def}/(R \to R_0))$ the category of triples (X_0, G, ϵ), where X_0 is an abelian scheme over R_0, G a p-divisible group over R, and $\epsilon : G \otimes_R R_0 \to X_0[p^\infty]$ an isomorphism.

Theorem 3.4.1 (Serre-Tate)**.** *The functor* $(\mathsf{AbSch}/R) \to (\mathsf{Def}/(R \to R_0))$ *sending an abelian scheme* X *over* R *to the triple* $(X \otimes_R R_0, X[p^\infty], \epsilon)$ *(where* ϵ *is the natural isomorphism* $X[p^\infty] \otimes_R R_0 \to (X \otimes_R R_0)[p^\infty]$*) is an equivalence of categories.*

We shall devote the rest of this section to its proof.

Let's first show that this functor is fully faithful. Note that if G is representable by either an abelian schemes or a p-divisible groups over R, then G is p-divisible (Example 3.2.4, Example 3.2.5), formally smooth (Theorem 3.2.2, Proposition 1.1.2), and $\hat{G}$ is locally representable by a formal Lie group (Theorem 3.2.2, Proposition 3.1.3). So we may use the various conclusions of Theorem 3.3.4.

Faithfulness comes from part (ii) of Theorem 3.3.4. As for fullness, we need to show that whenever X, Y are abelian schemes, and $f_0 : X \otimes_R R_0 \to Y \otimes_R R_0$, $\phi : X[p^\infty] \to Y[p^\infty]$ are homomorphisms with $f_0[p^\infty] = \phi \otimes_R R_0$, then there is a homomorphism $f : X \to Y$ such that $f \otimes_R R_0 = f_0$, $f[p^\infty] = \phi$.

In view of part (iii) and (iv) of Theorem 3.3.4, we want to show that $\ulcorner N^\nu f \lrcorner$ annihilates $X[N^\nu]$. Since $\ulcorner N^\nu f \lrcorner$ lifts $N^\nu f_0$, $\ulcorner N^\nu f \lrcorner[p^\infty]$ lifts $N^\nu f_0[p^\infty] = N^\nu \phi \otimes_R R_0$. But ϕ already lifts $\phi \otimes_R R_0$, so we must have $N^\nu\phi = \ulcorner N^\nu f \lrcorner[p^\infty]$ by part (ii) of Theorem 3.3.4. In particular, $\ulcorner N^\nu f \lrcorner$ annihilates $X[N^\nu]$, so we can lift f_0 to some $f : X \to Y$. Moreover, $N^\nu\phi = N^\nu f[p^\infty]$ and thus $\phi = f[p^\infty]$ by part (i) of Theorem 3.3.4.

To show essential surjectivity, we import a well-known result about quotients.

Theorem 3.4.2. *Let S be a locally Noetherian base scheme. Suppose G is a finite flat subgroup scheme of a group scheme H such that, for any $s \in S$, the fibre H_s is contained in an open affine of G. Then the quotient fppf sheaf G/H is representable. Moreover, the quotient morphism $G \to G/H$ is finite locally free (in particular fppf).*

Proof. This is a special case of [13, Proposition 07S6]. □

Suppose now that (X_0, G, ϵ) is an object of $(\mathsf{Def}/(R \to R_0))$. We want to find an abelian scheme over R giving rise to these data.

By Theorem 2.4.1, we can always find an abelian scheme Y over R such that there is an isomorphism $\alpha_0 : Y_0 = Y \otimes_R R_0 \to X_0$. This gives rise to an isomorphism $\alpha_0[p^\infty] : Y_0[p^\infty] \to X_0[p^\infty]$. Since G lifts $X_0[p^\infty]$ (via ϵ), part (iii) of Theorem 3.3.4 tells us that we can form $\Phi : Y[p^\infty] \to G$ and $\Psi : G \to Y[p^\infty]$ which lift $N^\nu\alpha_0[p^\infty]$ and $N^\nu\alpha_0[p^\infty]^{-1}$, respectively.

By part (ii) of Theorem 3.3.4, $\Phi \circ \Psi = [N^{2\nu}]$ and $\Psi \circ \Phi = [N^{2\nu}]$. In particular, both Φ and Ψ are surjective morphisms of fppf sheaves, and $K = \ker \Phi$ is annihilated by $[N^{2\nu}]$. We write down the short exact sequence

$$0 \longrightarrow K \longrightarrow Y[p^\infty] \xrightarrow{\Phi} G \longrightarrow 0$$

which, by the Snake Lemma, induces an exact sequence

$$0 \longrightarrow K \longrightarrow Y[N^{2\nu}] \xrightarrow{\Phi[N^{2\nu}]} G[N^{2\nu}] \xrightarrow{\partial} K \longrightarrow 0. \qquad (*)$$

From the left-hand side of this exact sequence, we see that K is the kernel of a homomorphism of finite group schemes, hence is itself a finite group scheme. Then from the right-hand side of the exact sequence, $M = \ker \partial$ too is a finite group scheme. And we have a short exact sequence

$$0 \longrightarrow K \longrightarrow Y[N^{2\nu}] \xrightarrow{\pi} M \longrightarrow 0$$

where $\pi : Y[N^{2\nu}] \to M$ is the unique homomorphism that $\Phi[N^{2\nu}]$ factors through.

We claim that π is flat. Since $Y[N^{2\nu}]$ is flat, the fibrewise criterion for flatness (cf. [13, Lemma 039E]) shows that it suffices to check the flatness of $\pi \otimes_R R_0$. But $\Phi \otimes_R R_0$ is simply $[N^\nu]$ composed with the isomorphism $\alpha_0[p^\infty]$, so by passing (*) through the base-change we see that $\pi \otimes_R R_0$ is in fact the homomorphism $[N^\nu] : Y_0[N^{2\nu}] \to Y_0[N^\nu]$ composed with the same isomorphism, which is (faithfully) flat.

Consequently, K is a finite flat group scheme which lifts $Y_0[N^\nu]$. By Theorem 3.4.2 and Theorem 2.1.7, there is a group scheme X over S representing the fppf quotient Y/K equipped with an fppf morphism $Y \to X$. Since Y is flat, X must also be flat.

As K lifts $Y_0[N^\nu]$, X lifts the fppf quotient $Y_0/Y_0[N^\nu]$, which is representable by Y_0 itself since $[N^\nu] : Y_0 \to Y_0$ is a surjective morphism of fppf sheaves. So X is a flat lifting of $Y_0 \cong X_0$. Then X has to be smooth due to [13, Lemma 06AG]. Therefore X is an abelian scheme by Lemma 1.3.1.

We already know that X lifts X_0. But also by construction $X[p^\infty] \cong Y[p^\infty]/K \cong G$. This finishes the proof.

4 Lifting Ordinary Abelian Varieties

4.1 Ordinary Abelian Varieties

We now apply Theorem 3.4.1 to give a canonical description of the local moduli $\mathscr{M}$ of an ordinary abelian variety. Of course, this requires us to first define ordinary abelian varieties.

To start with, let's recall some basic notions surrounding the theory of finite flat group schemes.

Definition 4.1. Let $G = \mathrm{Spec}(A)$ be a finite flat group scheme over a ring R. Its Cartier dual is the finite flat group scheme $\mathrm{D}_R(G) = \mathrm{Spec}(A^\vee)$ where $A^\vee = \mathrm{Hom}_R(A, R)$ is the dual of the Hopf algebra A, swapping multiplication and comultiplication.

It's easy to see that $\mathrm{D}_R(G)$ represents the functor $(\mathsf{Aff}/R)^{\mathrm{op}} \to (\mathsf{Ab})$ sending an R-algebra B to $\mathrm{Hom}_{(\mathsf{Grp}/B)}(G \otimes_R B, \mathbb{G}_{m/B})$. From here, it's clear that $\mathrm{D}_R(G \times H) = \mathrm{D}_R(G) \times \mathrm{D}_R(H)$, and that D_R commutes with base-change.

In addition, $\mathrm{D}_R \circ \mathrm{D}_R$ is naturally isomorphic to the identity functor on the category (FFGrp/R) of finite flat group schemes over R. In particular, D_R provides an equivalence of categories $(\mathsf{FFGrp}/R) \cong (\mathsf{FFGrp}/R)^{\mathrm{op}}$.

Example 4.1.1. For any integer N, $\boldsymbol{\mu}_{N/R}$ and $\underline{\mathbb{Z}/N\mathbb{Z}}_R$ are dual to each other.

Suppose now that R is Artinian and local.

Definition 4.2. The connected part G° of a finite flat group scheme $G = \mathrm{Spec}(A)$ is the connected component of G through which e_G factors. The étale part $G^{\text{ét}}$ of G is $\mathrm{Spec}(A^{\text{ét}})$, where $A^{\text{ét}}$ is the maximal separable subalgebra of A.

It's easy to see that $G^{\text{ét}}$ is étale, and that any homomorphism $G \to H$ where H is a finite étale group scheme factors uniquely through $G \to G^{\text{ét}}$.

Theorem 4.1.1 (Connected-Étale Sequence)**.** *The sequence*

$$0 \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$$

is exact.

Proof. [1, p. 43]. ☐

Supose now that $R = k$ is an algebraically closed field. Then a finite group scheme over k is étale iff it's reduced iff it's constant, since the only finite étale scheme over k is a finite disjoint union of $\mathrm{Spec}(k)$. In particular, $G^{\text{ét}} = G^{\text{red}}$ is a constant group scheme.

Proposition 4.1.2. *Suppose $R = k$ is an algebraically closed field. Then the connected-étale sequence splits canonically.*

Proof. Suppose $G = \mathrm{Spec}(A)$ is a finite group scheme over k. Since A is finite over k, it decomposes into $A = \prod_i A_i$ with each A_i local. Write $G_i = \mathrm{Spec}(A_i)$. Then $G = \coprod_i G_i$, and the map $G \to G^{\text{ét}} = \coprod_i \mathrm{Spec}(k)$ is simply given by patching together the structure maps of G_i. But the residue field of A_i equals k as A_i is finite over k, which is algebraically closed. So each of these structure maps has a unique section given by the residue homomorphism, and they patch together to give a unique splitting. ☐

Corollary 4.1.3. *Suppose k is an algebraically closed field. Then every finite group scheme G over k decomposes canonically as $G = G^\circ \times G^{\text{ét}}$.*

Let's now use this together with Cartier duality. Let G be a finite group scheme over k. Then

$$\begin{aligned} G &= G^\circ \times G^{\text{ét}} = \mathrm{D}_k(\mathrm{D}_k(G^\circ)) \times \mathrm{D}_k(\mathrm{D}_k(G^{\text{ét}})) \\ &= \left[\mathrm{D}_k(\mathrm{D}_k(G^\circ)^\circ) \times \mathrm{D}_k(\mathrm{D}_k(G^\circ)^{\text{ét}})\right] \times \left[\mathrm{D}_k(\mathrm{D}_k(G^{\text{ét}})^\circ) \times \mathrm{D}_k(\mathrm{D}_k(G^{\text{ét}})^{\text{ét}})\right] \end{aligned}$$

So we may decompose $G = G^{\circ,\circ} \times G^{\circ,\text{ét}} \times G^{\text{ét},\circ} \times G^{\text{ét},\text{ét}}$ where $G^{\circ,\circ} = \mathrm{D}_k(\mathrm{D}_k(G^\circ)^\circ)$ is connected with connected dual, $G^{\circ,\text{ét}} = \mathrm{D}_k(\mathrm{D}_k(G^\circ)^{\text{ét}})$ is connected with étale dual, and so on.

By the classification of finite abelian groups, $G^{\text{ét},\text{ét}}$ is a product of constant group schemes of the form $\underline{\mathbb{Z}/\varpi^m\mathbb{Z}}_k$ for various primes ϖ and integers $m \geq 1$. Its dual, which is supposed to be étale, would then be the product of $\boldsymbol{\mu}_{\varpi^m/k}$ for these values of ϖ and m.

Suppose now that $\operatorname{char} k = p > 0$. Then $\boldsymbol{\mu}_{p^m/k}$ is not reduced, so none of these ϖ can ever equal p. Consequently, $p \nmid \operatorname{rank}_k(G^{\text{ét},\text{ét}})$. If $\operatorname{rank}_k(G)$ is a power of p (e.g. if G is the p-power torsion subgroup scheme of an abelian variety), then this means that $G^{\text{ét},\text{ét}} = 0$ is the trivial group scheme over k.

Now let X_0 be an abelian variety over k. $X_0[p]^{\text{ét}}$ is a constant group scheme annihilated by p, so it is isomorphic to $\underline{(\mathbb{Z}/p\mathbb{Z})^r}_k = (\underline{\mathbb{Z}/p\mathbb{Z}}_k)^r$ for some $r = r_{X_0}$. The exact sequence in part (ii) of Definition 3.9 reduces to

$$0 \longrightarrow X_0[p^i]^{\text{ét}} \hookrightarrow X_0[p^j]^{\text{ét}} \xrightarrow{[p^i]} X_0[p^{j-i}]^{\text{ét}} \longrightarrow 0$$

from which we conclude $X_0[p^j]^{\text{ét}} \cong \underline{(\mathbb{Z}/p^j\mathbb{Z})^r}_k$.

Let X_0^{t} be the dual abelian variety of X_0. Recall that we have the following fundamental theorem:

Theorem 4.1.4. *Suppose $f : X_0 \to Y_0$ is an isogeny (i.e. finite faithfully flat homomorphism between abelian varieties), then the kernel of its dual $f^{\mathrm{t}} : Y_0^{\mathrm{t}} \to X_0^{\mathrm{t}}$ is canonically isomorphic to $\mathrm{D}_k(\ker f)$.*

Proof. [9, p. 143]. □

In particular, $X_0^{\mathrm{t}}[p^j] \cong \mathrm{D}_k(X_0[p^j])$. Applying our discussions to X_0^{t}, we see that there is some $s = s_{X_0}$ such that $\mathrm{D}_k(X_0[p^j])^{\text{ét}} \cong \underline{(\mathbb{Z}/p^j\mathbb{Z})^s}_k$ for all j.

Putting these together, we conclude $X_0[p^j]^{\text{ét},\text{ét}} = 0$, $X_0[p^j]^{\text{ét},\circ} \cong \underline{(\mathbb{Z}/p^j\mathbb{Z})^r}_k$, and $X_0[p^j]^{\circ,\text{ét}} \cong \mathrm{D}_k(\underline{(\mathbb{Z}/p^j\mathbb{Z})^s}_k) \cong \boldsymbol{\mu}^s_{p^j/k}$.

Now, r_{X_0} is invariant under isogeny: Suppose $f : X_0 \to Y_0$ is an isogeny. It suffices to show that $r_{X_0} \leq r_{Y_0}$. f restricts to a homomorphism $X_0[p^j] \to Y_0[p^j]$ for all j, and therefore $p^{jr_{X_0}} = \#X_0[p^j](k) \leq \#(\ker f)(k) \cdot \#Y_0[p^j](k) = \#(\ker f)(k) \cdot p^{jr_{Y_0}}$ for all j. But this can only hold if $r_{X_0} \leq r_{Y_0}$.

Since X_0 is isogenous to X_0^{t}, we conclude that $r_{X_0} = r_{X_0^{\mathrm{t}}} = s_{X_0}$. We hence get a decomposition $X_0[p^j] = \underline{(\mathbb{Z}/p^j\mathbb{Z})^{r_{X_0}}}_k \times \boldsymbol{\mu}^{r_{X_0}}_{p^j/k} \times X_0[p^j]^{\circ,\circ}$.

Definition 4.3. r_{X_0} is called the p-rank of X_0. X_0 is ordinary if $r_{X_0} = g = \dim X_0$.

If X_0 is ordinary, then $X_0[p^j] \cong \underline{(\mathbb{Z}/p^j\mathbb{Z})^g}_k \times \boldsymbol{\mu}^g_{p^j/k}$ since $X_0[p^j]$ has rank p^{2jg}. So $X_0[p^\infty] \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_k \times \boldsymbol{\mu}^g_{p^\infty/k} \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_k \times \hat{\mathbb{G}}^g_{m/k}$. Let's call this the ordinary p-divisible group.

4.2 Lifting the Ordinary p-Divisible Group

Let X_0 be an ordinary abelian variety over an algebraically closed field k of characteristic $p > 0$. We want to find a canonical description of the local moduli functor $\mathscr{M} = \mathscr{M}_{X_0}$. Fix W as in Section 2.3.

Let R be an Artinian local W-algebra whose residue field is k. Then p is automatically nilpotent in R. Theorem 3.4.1 tells us that the liftings of X_0 to R are controlled precisely by the liftings of $X_0[p^\infty] \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_k \times \boldsymbol{\mu}^g_{p^\infty/k}$, where $g = \dim X_0$. So it suffices to compute the deformation of this p-divisible group.

Write $\mathfrak{C}_0$ for the constant factor $\underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_k$ and $\boldsymbol{\mu}_0$ the toroidal factor $\boldsymbol{\mu}^g_{p^\infty/k}$. Note that the formations of $\mathfrak{C}_0$ and $\boldsymbol{\mu}_0$ are canonical, since they come from the étale and connected parts of various $X_0[p^j]$.

To understand constant and toroidal p-divisible groups, we recall the following strengthening of Proposition 1.1.2 in the case of an étale morphism.

Proposition 4.2.1. *A morphism $X \to S$ is étale if and only if it is locally of finite presentation, and for any commutative diagram of solid arrows*

$$\begin{array}{ccc} X & \longleftarrow & Y' \\ \downarrow & \nwarrow & \downarrow \\ S & \longleftarrow & Y \end{array}$$

where $Y' \to Y$ is a closed immersion of (not necessarily affine) schemes with nilpotent ideal, there is a unique morphism filling in the dashed arrow.

Proof. [13, Lemma 02HM], [13, Lemma 04FD]. □

Corollary 4.2.2. *Suppose G, H are finite flat group schemes over R and G_0, H_0 their respective base-change to k. Then a homomorphism $f_0 : G_0 \to H_0$ lifts uniquely to a homomorphism $f : G \to H$ if either:*

(a) H is étale, or

(b) G is toroidal, i.e. $G \cong \boldsymbol{\mu}^g_{N/R}$ for some $N > 1$, $g \geq 1$.

Proof. (b) follows from (a) since $\mathrm{D}_R(G)$ would be constant.

To establish (a), observe that Proposition 4.2.1 means that f_0 lifts to a unique morphism $f : G \to H$. But then $F : G \times G \to H$ defined by $F(x, y) = f(x + y) - f(x) - f(y)$ lifts the identity $G_0 \times G_0 \to \mathrm{Spec}(k) \to H_0$ since f_0 is a homomorphism. The uniqueness part of Proposition 4.2.1 then shows that F is the identity $G \times G \to \mathrm{Spec}(R) \to H$. Hence f is a homomorphism. □

Corollary 4.2.3. *Suppose G, H are p-divisible groups over R and G_0, H_0 their respective base-change to k. Then a homomorphism $f_0 : G_0 \to H_0$ lifts uniquely to a homomorphism $f : G \to H$ if either:*

(a) H is a constant p-divisible group, or

(b) G is a toroidal p-divisible group.

Proof. Use Corollary 4.2.2 on each $f_0[p^j] : G_0[p^j] \to H_0[p^j]$. □

Corollary 4.2.4. *Suppose G is a p-divisible group over R which is either toroidal or constant. Then for any other p-divisible group H, any isomorphism $H \otimes_R k \cong G \otimes_R k$ lifts uniquely to an isomorphism $H \cong G$.*

Proof. Combine Corollary 4.2.3 with Lemma 1.3.3. □

We are now ready to compute the category $(\mathsf{Def}(X_0[p^\infty]))$ of p-divisible groups over R which lift $X_0[p^\infty] = \mathfrak{C}_0 \times \boldsymbol{\mu}_0$. By Corollary 4.2.4, each of $\mathfrak{C}_0$, $\boldsymbol{\mu}_0$ admit a canonical lifting $\mathfrak{C} \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_R$, $\boldsymbol{\mu} \cong \boldsymbol{\mu}^g_{p^\infty/R}$ to R.

We consider the category $(\mathsf{Ext}(\mathfrak{C}, \boldsymbol{\mu}))$ of pairs (E, ϵ) where E is an extension (in (Grp/R)) of $\mathfrak{C}$ by $\boldsymbol{\mu}$ and $\epsilon : \mathfrak{C}_0 \to E \otimes_R k$ is a splitting (in particular, $E \otimes_R k \cong X_0[p^\infty]$). Any such E is a p-divisible group by Proposition 3.2.1: Clearly $E = \varinjlim_j E[p^j]$ and E is p-divisible by the Five Lemma. The Snake Lemma gives short exact sequences of the form

$$0 \longrightarrow \boldsymbol{\mu}[p^j] \longrightarrow E[p^j] \longrightarrow \mathfrak{C}[p^j] \longrightarrow 0$$

and so each $E[p^j]$ is a finite flat group scheme.

So we get a functor $F : (\mathsf{Ext}(\mathfrak{C}, \boldsymbol{\mu})) \to (\mathsf{Def}(X_0[p^\infty]))$ sending (E, ϵ) to E.

Theorem 4.2.5. *F is an equivalence of categories.*

Proof. We construct an inverse to F as follows: Suppose E is a p-divisible group over R reducing to $X_0[p^\infty]$. Then Corollary 4.2.3 shows that we can find a unique lift $f : E \to \mathfrak{C}$ of the projection $f_0 : X_0[p^\infty] \to \mathfrak{C}_0$. Each $f[p^j] : E[p^j] \to \mathfrak{C}[p^j]$ is faithfully flat since this can be checked on fibres ([13, Lemma 039E] again). In particular, f is a surjective homomorphism of fppf sheaves.

Let $K = \ker f$. As K is a subsheaf of E, $K = \varinjlim_j K[p^j]$. Since each $f[p^j] : E[p^j] \to \mathfrak{C}[p^j]$ is faithfully flat, $K[p^j] = \ker f[p^j]$ is a finite flat group scheme. In addition, as E is p-divisible, we have an exact sequence

$$0 \longrightarrow K[p] \longrightarrow E[p] \xrightarrow{f[p]} \mathfrak{C}[p] \longrightarrow \operatorname{coker}[p]_K \longrightarrow 0$$

by the Snake Lemma. This shows that $\operatorname{coker}[p]_K = 0$, i.e. K is p-divisible. By Proposition 3.2.1, K is a p-divisible group.

Now $K \otimes_R k = \ker f_0 = \boldsymbol{\mu}_0$, so $K = \boldsymbol{\mu}$ by Corollary 4.2.4. $\square$

Of course the splitting $\epsilon : \mathfrak{C}_0 \to X_0[p^\infty]$ is unique by Proposition 4.1.2. Hence

Corollary 4.2.6. *The set of isomorphism classes of p-divisible groups over R lifting $X_0[p^\infty]$ is in natural bijection with* $\operatorname{Ext}^1(\mathfrak{C}, \boldsymbol{\mu})$.

Here, the Ext-functor is taken in the abelian category $(\mathsf{Grp/R})$.

4.3 An Extension Problem

Let $\operatorname{Tate}_p(X_0) = \varprojlim_j X_0[p^j](k)$ where the limit is taken with respect to the system

$$\cdots \xrightarrow{[p]} X_0[p^2](k) \xrightarrow{[p]} X_0[p](k) \xrightarrow{[p]} 0$$

As X_0 is ordinary, $X_0[p^j](k) \cong \underline{(\mathbb{Z}/p^j\mathbb{Z})^g}_k(k) \times \boldsymbol{\mu}^g_{p^j/k}(k) \cong (\mathbb{Z}/p^j\mathbb{Z})^g$. So $\operatorname{Tate}_p(X_0) \cong \mathbb{Z}_p^g$. And we may (naturally) identify $\mathfrak{C}_0 = \underline{\operatorname{Tate}_p(X) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)}_k$.

Fix an isomorphism $\operatorname{Tate}_p(X_0) \cong \mathbb{Z}_p^g$. We shall first describe $\operatorname{Ext}^1(\mathfrak{C}, \boldsymbol{\mu})$ under this choice of coordinates.

The fixed isomorphism gives rise to identifications $\mathfrak{C}_0 \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_k$, $\mathfrak{C} \cong \underline{(\mathbb{Q}_p/\mathbb{Z}_p)^g}_R$, and thus $\operatorname{Ext}^1(\mathfrak{C}, \boldsymbol{\mu}) \cong \operatorname{Ext}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}_R, \boldsymbol{\mu})^{\oplus g}$. So it suffices to understand the group $\operatorname{Ext}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}_R, \boldsymbol{\mu})$.

Consider the directed system

$$\underline{\mathbb{Z}}_R \xrightarrow{[p]} \underline{\mathbb{Z}}_R \xrightarrow{[p]} \underline{\mathbb{Z}}_R \xrightarrow{[p]} \cdots$$

which allows for a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_R \longrightarrow \varinjlim \underline{\mathbb{Z}}_R \longrightarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p}_R \longrightarrow 0$$

and hence a long exact sequence (writing Hom for $\mathrm{Hom}_{(\mathsf{Grp}/R)}$)

$$\mathrm{Hom}(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \longrightarrow \mathrm{Hom}(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \xrightarrow{\delta_R} \mathrm{Ext}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}_R, \boldsymbol{\mu}) \longrightarrow \mathrm{Ext}^1(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}).$$

Theorem 4.3.1. *δ_R is an isomorphism.*

Proof. Since the second argument of every term in the long exact sequence commutes with finite direct products, we may assume $g = 1$, i.e. $\boldsymbol{\mu} = \boldsymbol{\mu}_{p^\infty/R}$.

Note first that $\mathrm{Hom}(\underline{\mathbb{Z}}_R, -) \cong \Gamma(\mathrm{Spec}(R), -)$. In particular, $\mathrm{Hom}(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \cong \boldsymbol{\mu}(R) \cong 1 + \mathfrak{m}_R$. Choose $r, s \geq 1$ such that p^r is zero in R and $\mathfrak{m}_R^{p^s} = 0$. Then $(1 + \mathfrak{m}_R)^{p^{r+s}} = 1$. This shows that $\mathrm{Hom}(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}) = \varprojlim \mathrm{Hom}(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}) = 0$, i.e. δ_R is injective.

For surjectivity, we will show that $\mathrm{Ext}^1(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}) = 0$. Our first claim is that the natural map $\mathrm{Ext}^1(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \to \varprojlim \mathrm{Ext}^1(\underline{\mathbb{Z}}_R, \boldsymbol{\mu})$ is injective. We will prove this using the Grothendieck spectral sequence (cf. [14, Theorem 5.8.3]).

Let $\mathcal{C}$ be the category of sheaves of groups on $\mathbb{N}$, viewed as a topological space with open sets $\varnothing$, $\mathbb{N}$, and $\{0, \ldots, N\}$ for various $N \in \mathbb{N}$. Alternatively, $\mathcal{C}$ is the category of inverse systems of groups indexed by $\mathbb{N}$. Let $\Gamma : \mathcal{C} \to (\mathsf{Ab})$ be the global section functor, which simply corresponds to taking inverse limit.

So $G \mapsto \mathrm{Hom}(\varinjlim \underline{\mathbb{Z}}_R, G) = \varprojlim \mathrm{Hom}(\underline{\mathbb{Z}}_R, G)$ factors as $\Gamma \circ F$ where $F : (\mathsf{Grp}/R) \to \mathcal{C}$ is the functor that takes an R-group G to the system

$$\cdots \xrightarrow{[p]^*} \mathrm{Hom}(\underline{\mathbb{Z}}_R, G) \xrightarrow{[p]^*} \mathrm{Hom}(\underline{\mathbb{Z}}_R, G) \xrightarrow{[p]^*} \mathrm{Hom}(\underline{\mathbb{Z}}_R, G).$$

Suppose G is injective, then each $[p]^*$ is surjective by definition. Therefore $F(G)$ is a flasque sheaf on $\mathbb{N}$, which is Γ-acyclic. So this factorisation satisfies the conditions under which the Grothendieck spectral sequence applies. In particular, we get an exact sequence

$$0 \longrightarrow \mathbf{R}^1\Gamma(F(\boldsymbol{\mu})) \longrightarrow \mathrm{Ext}^1(\varinjlim \underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \longrightarrow \varprojlim \mathrm{Ext}^1(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}).$$

Under the identification $\mathrm{Hom}(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}) \cong 1 + \mathfrak{m}_R$, $F(\boldsymbol{\mu})$ is the system whose entries are $1 + \mathfrak{m}_R$ and whose transition maps are given by the operation of raising to the p-th power. Since $(1 + \mathfrak{m}_R)^{p^{r+s}} = 1$, such a system is Mittag-Leffler. Therefore $\mathbf{R}^1\Gamma(F(\boldsymbol{\mu})) = 0$ by [13, Lemma 0598], hence we have the injectivity of the natural map as claimed.

To complete the proof, it now suffices to show that $\mathrm{Ext}^1(\underline{\mathbb{Z}}_R, \boldsymbol{\mu}) = 0$. The isomorphism $\mathrm{Hom}(\underline{\mathbb{Z}}_R, -) \cong \Gamma(\mathrm{Spec}(R), -)$ shows that $\mathrm{Ext}^i(\underline{\mathbb{Z}}_R, -) \cong H^i_{\mathrm{fppf}}(\mathrm{Spec}(R), -)$ for all i. So we are left to establish the vanishing of $H^1_{\mathrm{fppf}}(\mathrm{Spec}(R), \boldsymbol{\mu})$.

Abbreviate $H^i(-) = H^i_{\mathrm{fppf}}(\mathrm{Spec}(R), -)$. For an R-group G and a natural number N, we write $H^i(G)[N]$ for the kernel of $[N]^* : H^i(G) \to H^i(G)$.

First note that $H^1(\boldsymbol{\mu}) = H^1(\varinjlim_j \boldsymbol{\mu}_{p^j/R}) = \varinjlim_j H^1(\boldsymbol{\mu}_{p^j/R})$ by [13, Lemma 0739]. Since $[p^j]$ annihilates $\boldsymbol{\mu}_{p^j/R}$, $[p^j]^*$ annihilates $H^1(\boldsymbol{\mu}_{p^j/R})$. Hence $H^1(\boldsymbol{\mu}) = \varinjlim_j H^1(\boldsymbol{\mu})[p^j]$. So it suffices to show that $H^1(\boldsymbol{\mu})[p^j] = 0$ for all j.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{\mu}_{p^j/R} & \longrightarrow & \boldsymbol{\mu} & \xrightarrow{[p^j]} & \boldsymbol{\mu} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{\mu}_{p^j/R} & \longrightarrow & \mathbb{G}_{m/R} & \xrightarrow[{[p^j]}]{} & \mathbb{G}_{m/R} & \longrightarrow & 0
\end{array}$$

with exact rows. The associated system of long exact sequences gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\mu}(R)/[p^j]\boldsymbol{\mu}(R) & \longrightarrow & H^1(\boldsymbol{\mu}_{p^j/R}) & \longrightarrow & H^1(\boldsymbol{\mu})[p^j] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & \mathbb{G}_{m/R}(R)/[p^j]\mathbb{G}_{m/R}(R) & \longrightarrow & H^1(\boldsymbol{\mu}_{p^j/R}) & \longrightarrow & H^1(\mathbb{G}_{m/R})[p^j] & \longrightarrow & 0 \end{array}$$

again with exact rows. Of course α is just the map $(1+\mathfrak{m}_R)/(1+\mathfrak{m}_R)^{p^j} \to R^\times/(R^\times)^{p^j}$, which is always surjective: For any $z \in R^\times$, we have $z + \mathfrak{m}_R \in k^\times$ and so there is some $y \in R^\times$ with $zy^{p^j} \in 1+\mathfrak{m}_R$ (as k is algebraically closed). It is also injective, as the only p^j-th root of unity in k is 1. Thus α is an isomorphism, which means that β has to be as well.

But $H^1(\mathbb{G}_{m/R}) = 0$. So the proof is completed. □

Hence $\mathrm{Ext}^1(\mathfrak{C}, \boldsymbol{\mu})$ is isomorphic to $\mathrm{Hom}(\underline{\mathbb{Z}}_R, \boldsymbol{\mu})^{\oplus g} \cong \boldsymbol{\mu}(R)^{\oplus g}$. Since every element of $\boldsymbol{\mu}(R)$ is a p^{r+s}-torsion, $\boldsymbol{\mu}(R)$ is naturally a $\mathbb{Z}/p^{r+s}\mathbb{Z}$-module, hence a $\mathbb{Z}_p$-module. And we can further identify $\boldsymbol{\mu}(R)^{\oplus g} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^g, \boldsymbol{\mu}(R))$.

The choice of a different isomorphism $\mathrm{Tate}_p(X) \cong \mathbb{Z}_p^g$ would correspond to the automorphism of $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^g, \boldsymbol{\mu}(R))$ given by the corresponding change-of-coordinates on $\mathbb{Z}_p^g$. Therefore:

Corollary 4.3.2. $\mathscr{M}_{X_0}(R) \cong \mathrm{Ext}^1(\mathfrak{C}, \boldsymbol{\mu}) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0), \boldsymbol{\mu}(R))$ *canonically.*

4.4 The Canonical Lifting

By Theorem 4.1.4, we have a canonical isomorphism $X_0[p^j] \cong \mathrm{D}_k(X_0^{\mathrm{t}}[p^j])$. This restricts to a canonical isomorphism $\boldsymbol{\mu}_0[p^j] \cong \mathrm{D}_k(\underline{X_0^{\mathrm{t}}[p^j](k)}_k)$ under the connected-étale decomposition, where we identify $\boldsymbol{\mu}_0[p^j] = X_0[p^j]^\circ$ and $\underline{X_0^{\mathrm{t}}[p^j](k)}_k = X_0^{\mathrm{t}}[p^j]^{\text{ét}}$.

Corollary 4.2.2 tells us that this isomorphism lifts to a unique isomorphism

$$\boldsymbol{\mu}[p^j] \cong \mathrm{D}_R(\underline{X_0^{\mathrm{t}}[p^j](k)}_R).$$

So we can identify $\boldsymbol{\mu}[p^j](R)$ with $\mathrm{Hom}_{\mathbb{Z}}(X_0^{\mathrm{t}}[p^j](k), \mathbb{G}_{m/R}(R)) = \mathrm{Hom}_{\mathbb{Z}}(X_0^{\mathrm{t}}[p^j](k), \boldsymbol{\mu}_{p^j/R}(R))$.

Taking (co)limits, we obtain an identification of $\boldsymbol{\mu}(R)$ with

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0^{\mathrm{t}}), \boldsymbol{\mu}_{p^\infty/R}(R)) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0^{\mathrm{t}}), \hat{\mathbb{G}}_{m/R}(R))$$

and therefore a canonical isomorphism

$$\begin{aligned} \mathscr{M}_{X_0}(R) &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0), \boldsymbol{\mu}(R)) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0), \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0^{\mathrm{t}}), \hat{\mathbb{G}}_{m/R}(R))) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0) \otimes_{\mathbb{Z}_p} \mathrm{Tate}_p(X_0^{\mathrm{t}}), \hat{\mathbb{G}}_{m/R}(R)) \end{aligned}$$

which is functorial in R since δ_R is.

Definition 4.4 (Tate's q-Construction)**.** For $X \in \mathscr{M}_{X_0}(R)$, we write

$$q_X(-,-) \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Tate}_p(X_0) \otimes_{\mathbb{Z}_p} \mathrm{Tate}_p(X_0^{\mathrm{t}}), \hat{\mathbb{G}}_{m/R}(R))$$

for the associated $\mathbb{Z}_p$-bilinear form.

We summarise the result as the following theorem:

Theorem 4.4.1 (Serre-Tate Local Moduli)**.** *Suppose k is an algebraically closed field of characteristic p, and R is an Artinian local ring with residue field k. Let X_0 be an ordinary abelian variety over k.*

Then the map $X \mapsto q_X$ establishes a canonical bijection between the set of abelian schemes X over R reducing to X_0 and the group of $\mathbb{Z}_p$-bilinear forms $\mathrm{Tate}_p(X_0) \times \mathrm{Tate}_p(X_0^{\mathrm{t}}) \to \hat{\mathbb{G}}_{m/R}(R)$ *Furthermore, this bijection is functorial in R.*

We note a few expected properties of q_X, which follow from explicit computations.

Proposition 4.4.2. *Let X_0 and Y_0 be ordinary abelian varieties over k.*

(i) Suppose $X \in \mathscr{M}_{X_0}(R)$. We write X^{t} for its dual abelian scheme, which is a member of $\mathscr{M}_{X_0^{\mathrm{t}}}(R)$. Then $q_X = q_{X^{\mathrm{t}}}^{\top}$, i.e. $q_X(x, x^{\mathrm{t}}) = q_{X^{\mathrm{t}}}(x^{\mathrm{t}}, x)$ for any $x \in \mathrm{Tate}_p(X_0)$, $x^{\mathrm{t}} \in \mathrm{Tate}_p(X_0^{\mathrm{t}})$.

(ii) Suppose $X \in \mathscr{M}_{X_0}(R)$ and $Y \in \mathscr{M}_{Y_0}(R)$. Then a homomorphism $f_0 : X_0 \to Y_0$ lifts to a homomorphism $f : X \to Y$ (necessarily unique by part (ii) of Theorem 3.3.4) if and only if $q_X(x, f_0^{\mathrm{t}}(y^{\mathrm{t}})) = q_Y(f_0(x), y^{\mathrm{t}})$ for any $x \in \mathrm{Tate}_p(X_0)$ and $y^{\mathrm{t}} \in \mathrm{Tate}_p(Y_0^{\mathrm{t}})$.

Proof. [6, Theorem 2.1]. □

Since we have identified $\mathscr{M}_{X_0}(R)$ with a group, there certainly should be some significance to the lifting that corresponds to the identity.

Definition 4.5. The canonical lifting $X_R^{\mathrm{can}} \in \mathscr{M}_{X_0}(R)$ of X_0 to R is such that $q_{X_R^{\mathrm{can}}} = 0$.

In more down-to-earth terms, X_R^{can} is simply the lifting whose p-divisible group is the trivial extension $\mathfrak{C} \times \boldsymbol{\mu}$.

Example 4.4.1. If R is in fact a k-algebra, then $X_R^{\mathrm{can}} = X_0 \otimes_k R$.

Corollary 4.4.3. *For ordinary abelian varieties X_0, Y_0 over k, the natural map*

$$\mathrm{Hom}_{(\mathsf{Grp}/R)}(X_R^{\mathrm{can}}, Y_R^{\mathrm{can}}) \to \mathrm{Hom}_{(\mathsf{Grp}/k)}(X_0, Y_0)$$

is an isomorphism.

Proof. Immediate from part (ii) of Proposition 4.4.2. □

Let's sketch how this theory gives rise to a canonical way of lifting an abelian variety over k to characteristic 0. Suppose $W = W(k)$ is the ring of Witt vectors over k. By functoriality, the inverse system

$$\cdots \longrightarrow \mathscr{M}_{X_0}(W_3(k)) \longrightarrow \mathscr{M}_{X_0}(W_2(k)) \longrightarrow \mathscr{M}_{X_0}(W_1(k))$$

consists of group homomorphisms. In particular, this gives a sequence of abelian schemes $X_{W_i(k)}^{\mathrm{can}} \to \mathrm{Spec}(W_i(k))$ with $X_{W_i(k)}^{\mathrm{can}} \otimes_{W_i(k)} W_j(k) = X_{W_j(k)}^{\mathrm{can}}$ for any $j \le i$. Taking directed limit of the system $X_{W_1(k)}^{\mathrm{can}} \to X_{W_2(k)}^{\mathrm{can}} \to \cdots$, we obtain a *formal* abelian scheme $\mathfrak{X}^{\mathrm{can}}$ over $W(k)$.

By lifting line bundles at the same time, one finds:

Theorem 4.4.4. *There is a (projective) abelian scheme X^{can} over $W(k)$ completing to $\mathfrak{X}^{\mathrm{can}}$.*

Proof. [8, Ch. V, Theorem (3.3)]. □

Corollary 4.4.5. *For ordinary abelian varieties X_0, Y_0 over k, the natural map*

$$\mathrm{Hom}_{(\mathsf{Grp}/W(k))}(X^{\mathrm{can}}, Y^{\mathrm{can}}) \to \mathrm{Hom}_{(\mathsf{Grp}/k)}(X_0, Y_0)$$

is an isomorphism.

Proof. Combine Corollary 4.4.3 and [2, III_1, Théorème 5.4.1]. □

In particular, X^{can} is canonical.

Definition 4.6. X^{can} is the canonical lifting of X_0 to $W(k)$.

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