The GAGA Principle of Serre

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0 Introduction

Given a smooth algebraic variety X over $\mathbb{C}$, we can usually associate with it a complex manifold X^h. For example, if $X \subset \mathbb{A}^2$ is a smooth plane curve, then its set of closed points identifies a subset $X^h = X(\mathbb{C}) \subset \mathbb{C}^2$, which can be given a conformal structure via coordinate projections.

It's very tempting to ask whether the algebraic geometry of X relates to the analytic geometry of X^h. This is not crazy at all, since we have many low-dimensional examples that illustrates an equivalence between the two. The most elementary of which is the fact that the complex analytic endomorphisms of $\mathbb{P}^1$ are precisely the rational functions. We also have the result that every complex torus is isomorphic to an elliptic curve via the Weierstrass $\wp$ function.

There are also very good reasons why an equivalence principle bewteen the two could be useful: It's usually easier to obtain rigidity results in the algebraic theory, whereas one has access to transcendental methods (e.g. singular homology) in the analytic theory. Allowing the interplay of the two would then provide new kinds of techniques on both sides.

Understanding how this plays out precisely has long been a subject of interest to algebraic geometers. A certain form of equivalence between smooth projective curves and compact Riemann surfaces is, allegedly, already known to Riemann. Generalising those results to higher dimensions, however, isn't easy.

One of the most important breakthroughs on this topic is the result of Serre in [Ser56], famously known as GAGA, where a precise equivalence principle between the algebraic and analytic geometry of a projective variety was formulated. The purpose of this article is to sketch a proof of it, focusing mainly on the cohomological arguments involved. To avoid divergence from the main point, we will only sketch the proof of some techincal yet uninspiring parts of the proof. References to the full proofs will be provided.

The reader is expected to be familiar with the language of schemes, coherent sheaves and sheaf cohomology on projective spaces.

1 Analytification of an Algebraic Scheme

Our first task is to associate an "analytic space" X^h to every finite-type $\mathbb{C}$-scheme X. When the X is smooth and projective, X^h should have the natural structure of a complex manifold.

We will construct X^h as a ringed space, and we will also equip it with a canonical map $\lambda : X^h \to X$ (which may be called the "analytification map"). This is to allow a convenient way to state GAGA, which is basically an equivalence between coherent sheaves on X and those on X^h in the occasion where X is a projective variety.

One should note that the analytification of a scheme is NOT the main focus of GAGA. Little is lost by thinking that *analytification does exactly what you think it does*, at least for smooth projective varieties.

First suppose $X = \operatorname{Spec}\mathbb{C}[X_1, \ldots, X_n]$ for some n. Then the set of closed points $X(\mathbb{C})$ is in bijection with $\mathbb{C}^n$ via $(X_1 - a_1, \ldots, X_n - a_n) \leftrightarrow (a_1, \ldots, a_n)$. Giving $\mathbb{C}^n$ the usual (Euclidean) topology and complex structure, we take X^h to be the ringed space whose underlying topological space is $\mathbb{C}^n$ and whose sheaf of rings is given by the sheaf of holomorphic functions on $\mathbb{C}^n$.

This construction comes equipped with a map of ringed spaces $\lambda : X^h \to X$. On the level of topological spaces, this is simply the inclusion map (recall that we have identified $X^h = \mathbb{C}^n = X(\mathbb{C})$) which is automatically continuous since polynomials are continuous in the complex topology. On the level of structure sheaves, this is the process of identifying a well-defined rational function on an open set as a analytic function.

Next, suppose $X = \operatorname{Spec} R/I$ where $R = \mathbb{C}[X_1, \ldots, X_n]$ and $I = (f_1, \ldots, f_m) \leq R$. The quotient map $R \to R/I$ induces the closed embedding $i : X \to X_0 = \operatorname{Spec} R$. Since we already know what X_0^h is, what we are looking for here is to identify X^h as an "analytic closed subscheme" of X_0^h.

Like before, the only choice of underlying set of X^h that can possibly make sense is the set of closed points $X(\mathbb{C})$ of X. The closed embedding $X \to X_0$ induces a map of sets $X(\mathbb{C}) \to X_0(\mathbb{C})$, so we can give X^h the subspace topology by viewing it as a (closed) subset of X_0^h.

There is a small subtlety when it comes to the structure sheaf. Inspired by the "analytic closed subscheme" idea, we consider the cokernel $\mathcal{O}$ of the sheaf morphism $\mathcal{O}_{X_0^h}^{\oplus m} \to \mathcal{O}_{X_0^h}$ given (on an open set U) by $g_1 \oplus \cdots \oplus g_m \mapsto f_1 g_1 + \cdots + f_m g_m$. We are done if we can show that $\mathcal{O}$ is supported in X.

This however requires us to know what values $\mathcal{O}$ actually takes. Morally, we'd want it to have $\mathcal{O}(U) = \mathcal{O}_{X_0^h}(U)/I\mathcal{O}_{X_0^h}(U)$, which is just not true in general. But we don't need it to be always true to deduce our claim about the support of $\mathcal{O}$: We get what we want as long as it is true for a collection of U that forms a basis for the topology on X_0^h. And there is a good reason to believe it: The algebraic analogue $i_*\mathcal{O}_X(U) = \mathcal{O}_{X_0}(U)/I\mathcal{O}_{X_0}(U)$, albeit untrue in general, is true if U is a distinguished open set (as $(A/I)_f \cong A_f/IA_f$ canonically).

Finding this basis inevitably requires a little bit of complex geometry, namely the following theorem:

Theorem 1.1. *$\mathcal{O}(U) = \mathcal{O}_{X_0^h}(U)/I\mathcal{O}_{X_0^h}(U)$ is true if U has the form $\Delta(g,w,r) = \{x \in \mathbb{C}^n : \forall i, |g_i(x) - w_i| < r_i\}$ where $g : \mathbb{C}^n \to \mathbb{C}^l$ is a polynomial map, $w \in \mathbb{C}^l$ and $r \in \mathbb{R}^l_{>0}$.*

Proof. It suffices to show the exactness of the functor $\Gamma(\Delta(g,w,r), -)$, which is given by Cartan's Theorem B (see e.g. [GR65, p. 243]). □

As usual, we get a map $\lambda : X^h \to X$ of ringed spaces.
For general X, we simply cover it by affines and glue together our construction above. Gluing the λ's together gives a map of ringed spaces $\lambda : X^h \to X$. One can show that none of these depends on the affine covering. The details of this (as well as the arguments above) can be found in [Nee07, Ch. 4–6].

2 Analytification of a Coherent Sheaf

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is coherent if:
1. X can be covered by open sets $\{U_i\}_i$ such that for every i, $\mathcal{M}|_{U_i}$ is finitely generated, i.e. a quotient of $\mathcal{O}_X|_{U_i}^{\oplus n}$ for some n.
2. For every open $V \subset X$, and every natural number n, every morphism $\mathcal{O}_X|_V^{\oplus n} \to \mathcal{F}|_V$ has finitely generated kernel.
We write (Coh_X) to denote the category of coherent sheaves on X.

Remark. In the case where X is a scheme, this coincides with the usual definition of a coherent sheaf. We advise the reader to assume that coherent sheaves on X^h (and in general ringed spaces) share similar properties with those on a scheme. In particular, (Coh_{X^h}) is abelian and closed under taking sheaf $\mathcal{H}om$.

There is only one sensible way to analytify a coherent sheaf.

Definition 2.2. Let X be a finite type $\mathbb{C}$-scheme and $\lambda : X^h \to X$ its analytification. For a coherent $\mathcal{O}_X$-module $\mathcal{M}$, its analytification is the coherent $\mathcal{O}_{X^h}$-module given by the module pullback $\lambda^{-1}\mathcal{M} \otimes_{\lambda^{-1}\mathcal{O}_X} \mathcal{O}_{X^h}$.

Unwinding the definitions gives an universal property:

Theorem 2.1. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, then for any morphism $D : \mathcal{M} \to \lambda_*\mathcal{F}$, with $\mathcal{F}$ an $\mathcal{O}_{X^h}$-module, there is a unique morphism $e : \mathcal{M}^h \to \mathcal{F}$ such that D factorises as*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{D} & \lambda_*\mathcal{F} \\ \downarrow & \nearrow_{\lambda_* e} & \\ \mathcal{M}^h & & \end{array}$$

It's immediate that analytification is functorial. Moreover,

Theorem 2.2. *(i) The operation $\mathcal{M} \mapsto \mathcal{M}^h$, regarded as a functor $(\mathsf{Coh}_X) \to (\mathsf{Mod}_{\mathcal{O}_{X^h}})$, is exact.*
(ii) For any coherent $\mathcal{O}_X$-module $\mathcal{M}$ and open $U \subset X$, the natural map $\Gamma(U, \mathcal{M}) \to \Gamma(\lambda^{-1}U, \mathcal{M}^h)$ is injective.

We will not include a full proof here. Interested reader may confer [Nee07, App. 1]. Since we essentially constructed the analytification as a tensor product, the use of flatness should be expected. Indeed, the point of the proof is the following result:

Theorem 2.3. *The ring of holomorphic functions on $\mathbb{C}^n$ is faithfully flat over the ring of polynomials.*

It's noteworthy that the GAGA paper [Ser56] is also the first piece of literature where the notion of flatness is used. Serre had invented it purely for algebraic reasons, but it has found itself useful in many more situations in algebraic geometry later on, insofar as it almost becomes a standard technical tool in the modern theory.

Corollary 2.4. *The functor in Theorem 2.2(i), in fact, lands in* (Coh_{X^h}).

We call $\mathcal{M} \mapsto \mathcal{M}^h, (\mathsf{Coh}_X) \to (\mathsf{Coh}_{X^h})$ the analytification functor.

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. We want to show that $\mathcal{M}^h$ is a coherent $\mathcal{O}_{X^h}$-module.
Being a coherent sheaf on a scheme, $\mathcal{M}$ is locally finitely presented, in the sense that we can cover X by open sets $\{U_i\}_i$ such that for each i, we have an exact sequence of the form

$$\mathcal{O}_X|_{U_i}^{\oplus m} \longrightarrow \mathcal{O}_X|_{U_i}^{\oplus n} \longrightarrow \mathcal{M}|_{U_i} \longrightarrow 0$$

for some m, n possibly depending on i. Analytifying this sequence gives an exact sequence

$$\mathcal{O}_{X^h}|_{\lambda^{-1}U_i}^{\oplus m} \longrightarrow \mathcal{O}_{X^h}|_{\lambda^{-1}U_i}^{\oplus n} \longrightarrow \mathcal{M}^h|_{\lambda^{-1}U_i} \longrightarrow 0$$

So $\mathcal{M}^h|_{\lambda^{-1}U_i}$ is a coherent $\mathcal{O}_{X^h}|_{\lambda^{-1}U_i}$-module, since it's a cokernel of a map between coherent $\mathcal{O}_{X^h}|_{\lambda^{-1}U_i}$-modules. As $\{\lambda^{-1}U_i\}_i$ is an open cover of $X^h = \lambda^{-1}X$, we conclude the coherence of $\mathcal{M}^h$. □

By expanding the definitions, we see that Theorem 2.3 also implies

Theorem 2.5. *Suppose $\mathcal{M}, \mathcal{N}$ are coherent $\mathcal{O}_X$-modules, then the map $\mathcal{H}om(\mathcal{M}, \mathcal{N})^h \to \mathcal{H}om(\mathcal{M}^h, \mathcal{N}^h)$ is an isomorphism.*

Lastly, we look at what happens to cohomology when we analytify.

Theorem 2.6. *For $i \geq 0$ and $\mathcal{M}$ coherent, the canonical maps $H^i(\lambda) : H^i(X, \mathcal{M}) \to H^i(X^h, \mathcal{M}^h)$ commute with the long exact sequences of cohomology. More precisely, if any morphism $\mathcal{M} \to \mathcal{N}$ of coherent $\mathcal{O}_X$-module induces a commuative diagram*

$$\begin{array}{ccc} H^i(X, \mathcal{M}) & \longrightarrow & H^i(X, \mathcal{N}) \\ {\scriptstyle H^i(\lambda)}\downarrow & & \downarrow{\scriptstyle H^i(\lambda)} \\ H^i(X^h, \mathcal{M}^h) & \longrightarrow & H^i(X^h, \mathcal{N}^h) \end{array}$$

Furthermore, if

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

is an exact sequence of coherent $\mathcal{O}_X$-modules, then Theorem 2.2(i) gives an exact sequence of coherent $\mathcal{O}_{X^h}$-modules

$$0 \longrightarrow \mathcal{L}^h \longrightarrow \mathcal{M}^h \longrightarrow \mathcal{N}^h \longrightarrow 0$$

And we assert that the diagram

$$\begin{array}{ccc} H^i(X,\mathcal{N}) & \xrightarrow{\delta} & H^{i+1}(X,\mathcal{L}) \\ {\scriptstyle H^i(\lambda)}\downarrow & & \downarrow{\scriptstyle H^{i+1}(\lambda)} \\ H^i(X^h,\mathcal{N}^h) & \xrightarrow[\delta]{} & H^{i+1}(X^h,\mathcal{L}^h) \end{array}$$

commutes, where the δ's denote the respective connecting homomorphisms.

Proof. These calculations can be done by passing to a Čech complex. See [Ser56, §11] for details. ☐

3 The First GAGA principle

From now on, we fix $X = \mathbb{P}^n$. It's a mere formality to deduce the analogous results for when X is a projective variety. We write $\mathcal{O} = \mathcal{O}_X$, $\mathcal{O}^h = \mathcal{O}_X^h = \mathcal{O}_{X^h}$.
We'll also use the notation $H^i(\mathcal{M}) = H^i(X,\mathcal{M})$ when $\mathcal{M} \in \mathrm{ob}(\mathsf{Coh}_X)$ and $H^i(\mathcal{M}) = H^i(X^h,\mathcal{M})$ when $\mathcal{M} \in \mathrm{ob}(\mathsf{Coh}_{X^h})$. It should be clear from context as to which one we mean.
In this section we prove the following result:

Theorem 3.1. *For $i \geq 0$, the morphisms $H^i(\lambda) : H^i(\mathcal{M}) \to H^i(\mathcal{M}^h)$ in Theorem 2.6 are always isomorphisms.*

By taking $\mathcal{M}$ to be a $\mathcal{H}om$-sheaf and setting $i = 0$, we deduce (with the help from Theorem 2.5) that

Corollary 3.2. *The functor $\mathcal{M} \to \mathcal{M}^h$ is full and faithful.*

To prove Theorem 3.1, we use a divide-and-conquer strategy: We first show the theorem in the special case where $\mathcal{M} = \mathcal{O}(m)$ for some m. Then we prove Serre's result that every coherent sheaf on X is a quotient of $\mathcal{O}(m)^{\oplus p}$ for some p, and use it to deduce the result in general.

Lemma 3.3. *Theorem 3.1 is true for $\mathcal{M} = \mathcal{O}$.*

Proof. We know that $H^0(\mathcal{O})$ and $H^0(\mathcal{O}^h)$ are both constants (the latter by open mapping theorem), which are fixed by $H^0(\lambda)$, so $H^0(\lambda)$ is an isomorphism. For $i > 0$, we in fact have $H^i(\mathcal{O}) = H^i(\mathcal{O}^h) = 0$. The vanishing of $H^i(\mathcal{O})$ is again clear. The vanishing of $H^i(\mathcal{O}^h)$ can be obtained via calculation, either from Čech cohomology or Dolbeault cohomology (making use of e.g. [Huy05, p. 109]). ☐

Lemma 3.4. *Theorem 3.1 is true for $\mathcal{M} = \mathcal{O}(m)$ for every m.*

Proof. Induction on n. For a hyperplane $E = \mathbb{V}(t)$ in $X = \mathbb{P}^n$, we have the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

where the second arrow is multiplication by the linear form t. Tensoring with $\mathcal{O}(k)$ is an equivalence of categories, since it has an inverse given by tensoring with $\mathcal{O}(-k)$. In particular, this operation is exact. We therefore have the exact sequence

$$0 \longrightarrow \mathcal{O}(k-1) \longrightarrow \mathcal{O}(k) \longrightarrow \mathcal{O}_E(k) \longrightarrow 0$$

Taking cohomologies, Theorem 2.6 gives the commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i(\mathcal{O}(k-1)) & \longrightarrow & H^i(\mathcal{O}(k)) & \longrightarrow & H^i(\mathcal{O}_E(k)) & \longrightarrow & H^{i+1}(\mathcal{O}(k-1)) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^i(\mathcal{O}(k-1)^h) & \longrightarrow & H^i(\mathcal{O}(k)^h) & \longrightarrow & H^i(\mathcal{O}_E(k)^h) & \longrightarrow & H^{i+1}(\mathcal{O}(k-1)^h) & \longrightarrow \cdots \end{array}$$

Suppose the induction hypothesis is true for $n-1$, then $H^i(\mathcal{O}_E(k)) \to H^i(\mathcal{O}_E(k)^h)$ is an isomorphism. The Five Lemma then implies that $H^i(\mathcal{O}_E(k)) \to H^i(\mathcal{O}_E(k)^h)$ is an isomorphism for all i iff $H^i(\mathcal{O}_E(k-1)) \to H^i(\mathcal{O}_E(k-1)^h)$ is an isomorphism for all i. But $H^i(\mathcal{O}_E(0)) \to H^i(\mathcal{O}_E(0)^h)$ is an isomorphism for all i by the preceding lemma, hence the result. □

To extend Theorem 3.1 to general coherent sheaves, we use the following result from projective algebraic geometry.

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{M}$ is generated by global sections if the image of $\Gamma(X, \mathcal{M}) \to \mathcal{M}_x$ generates $\mathcal{M}_x$ for all $x \in X$.

Theorem 3.5. *For any coherent sheaf $\mathcal{M}$ on $X = \mathbb{P}^n$, there is some $m_0 = m_0(\mathcal{M})$ such that $\mathcal{M}(m)$ is generated by global sections for all $m \geq m_0$.*

Before proving this, let's first see how it allows us to establish Theorem 3.1.

Lemma 3.6. *Suppose $(X, \mathcal{O}_X)$ is a ringed space whose underlying topological space is quasicompact. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module generated by global sections, then there is a surjection $\mathcal{O}_X^{\oplus p} \to \mathcal{M}$ for some integer p.*

Proof. Note that for any $\mathcal{O}_X$-module $\mathcal{M}$ we have the identification $\mathrm{Hom}(\mathcal{O}_X^{\oplus p}, \mathcal{M}) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{M})^p \cong \Gamma(X, \mathcal{M})^p$. This means that a morphism $\mathcal{O}_X^{\oplus p} \to \mathcal{M}$ can be identified as a p-tuple of global sections of $\mathcal{M}$, and that this morphism is surjective if and only if these global sections generate every stalk.

Now assume that $\mathcal{M}$ is generated by global sections. What's left to show is then that it is in fact generated by finitely many global sections.

Let $x \in X$. Since $\mathcal{M}$ is coherent, there is an open set $U \ni x$ on which a finite collection of sections $t_1, \ldots, t_r \in \Gamma(U, \mathcal{M})$ generates $\mathcal{M}_y$ for all $y \in U$. The coherence

of $\mathcal{M}$ also implies that $\mathcal{M}_x$ is a finite $\mathcal{O}_{x,X}$-module. The map $\Gamma(X,\mathcal{M}) \to \mathcal{M}_x$ has generating image, so by finiteness of $\mathcal{M}_x$ we can extract $s_1,\dots,s_q \in \Gamma(X,\mathcal{M})$ such that they generate $\mathcal{M}_x$. In particular, there is some open $V \subset U$ around x and $f_{ij} \in \Gamma(V,\mathcal{O}_X)$ such that $t_i(x) = \sum_j f_{ij}(x)s_j(x)$. So we can find some open $W \subset V$ around x such that $t_i|_W = \sum_j f_{ij}|_W s_j|_W$. Therefore $s_1,\dots,s_q$ generate $\mathcal{M}_y$ for all $y \in W$.
Quasicompactness allows us to collect finitely many global sections in this fashion and they necessarily generate $\mathcal{M}$ by construction. □

Corollary 3.7. *Every coherent sheaf $\mathcal{M}$ on $\mathbb{P}^n$ is a quotient of $\mathcal{O}(N)^{\oplus p}$ for some N,p.*

Proof. Theorem 3.5 together with the preceding lemma shows that $\mathcal{M}(m)$ is a quotient of $\mathcal{O}^{\oplus p}$ for some m,p. Tensoring with $\mathcal{O}(-m)$, we see that $\mathcal{M}$ is a quotient of $\mathcal{O}(-m)^{\oplus p}$. □

To finish the proof of Theorem 3.1, we take $\mathcal{L} = \mathcal{O}(N)^{\oplus p}$ as in the corollary and $\mathcal{K}$ the kernel of the surjection. We then have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0$$

of coherent $\mathcal{O}$-modules. We proceed by downward induction on the hypothesis that $H^i(\lambda)$ is an isomorphism for all coherent $\mathcal{O}$-module. This is true for $i > 2n$ since for these values of i we have $H^i(\mathcal{F}) = H^i(\mathcal{F}^h) = 0$ for all coherent $\mathcal{O}$-module $\mathcal{F}$.
For the induction step, Theorem 2.6 gives the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H^i(\mathcal{K}) & \longrightarrow & H^i(\mathcal{L}) & \longrightarrow & H^i(\mathcal{M}) & \longrightarrow & H^{i+1}(\mathcal{K}) & \longrightarrow & H^{i+1}(\mathcal{L}) \\ \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle \epsilon_3} & & \downarrow{\scriptstyle \epsilon_4} & & \downarrow{\scriptstyle \epsilon_5} \\ H^i(\mathcal{K}^h) & \longrightarrow & H^i(\mathcal{L}^h) & \longrightarrow & H^i(\mathcal{M}^h) & \longrightarrow & H^{i+1}(\mathcal{K}^h) & \longrightarrow & H^{i+1}(\mathcal{L}^h) \end{array}$$

for each i. Here, the ϵ's are the respective $H^i(\lambda), H^{i+1}(\lambda)$'s. By the induction hypothesis, ϵ_4, ϵ_5 are isomorphisms. Lemma 3.4 shows that Theorem 3.1 is true for $\mathcal{L}$. In particular, ϵ_2 is also an isomorphism. Consequently ϵ_3 must be surjective by the Five Lemma. In other words, $H^i(\lambda)$ is always a surjection.
But ϵ_1 is an $H^i(\lambda)$, hence a surjection by the result we just obtained. Applying the Five Lemma again shows that ϵ_3 is an isomorphism, so we conclude the induction step.

Proof of Theorem 3.5. Let $t_0,\dots,t_n$ be the standard coordinates on $\mathbb{P}^n$ and $U_i = D(t_i)$ be the standard affine opens which cover $\mathbb{P}^n$.
We'll use the fact that coherent sheaves on affine schemes are precisely the sheafifications of modules. This is merely to simplify the presentation of the proof, as the reader readily sees that we never make use of the full power of this result.
Since each U_i is affine, $\mathcal{M}|_{U_i}$ is generated by global sections. Therefore it suffices to show that for each i, every $s_i^* \in \Gamma(U_i,\mathcal{M})$ and all sufficiently large m, there is some $s \in \Gamma(\mathbb{P}^n,\mathcal{M}(m))$ that restricts to s_i^* on U_i.
An element of $\Gamma(\mathbb{P}^n,\mathcal{M}(m))$ can be described as a system of sections $(s_0,\dots,s_n), s_j \in$

$\Gamma(U_j, \mathcal{M})$ such that $s_k|_{U_j \cap U_k} = (t_j/t_k)^m s_j|_{U_j \cap U_k}$. We want to show that, for prescribed i and $s_i^* \in U_i$, such a tuple with $s_i = s_i^*$ exists for all sufficently large m.
For $j \neq i$, $U_i \cap U_j$ is a distinguished open set $D(t_i/t_j)$ on the affine scheme U_j. Since $\mathcal{M}|_{U_j}$ is the sheafification of a module, we conclude that there is some $s_j' \in \Gamma(U_j, \mathcal{M})$ restricting to $(t_i/t_j)^p s_i^*|_{U_i \cap U_j}$ on $U_i \cap U_j$ for all sufficiently large p. Take p large enough to work for all j at once (note that there are only finitely many of them) and set $s_i' = s_i^*$. We then have the formula $s_j'|_{U_i \cap U_j} = (t_i/t_j)^p s_i'|_{U_i \cap U_j}$ for all j.
Now, for any j, k, the section $s_j'|_{U_j \cap U_k} - (t_k/t_j)^p s_k'|_{U_j \cap U_k}$ on $U_j \cap U_k$ restricts to zero on $U_i \cap U_j \cap U_k$. As $U_i \cap U_j \cap U_k$ is a distinguished open set $D(t_i/t_j)$ of the affine scheme $U_i \cap U_j$, we use the fact that $\mathcal{M}|_{U_j \cap U_k}$ is the sheafification of a module to conclude that, for all sufficiently large $q \geq q_0$, $(t_i/t_j)^q (s_j'|_{U_j \cap U_k} - (t_k/t_j)^p s_k'|_{U_j \cap U_k}) = 0$ on $U_j \cap U_k$.
Take q_0 large enough to work for all j, k and $m_0 = q_0 + p$. For all $m = q + p \geq m_0$, the system $s_j = (t_i/t_j)^q s_j' \in \Gamma(U_j, \mathcal{M})$ gives the desired global section. □

4 The Second GAGA principle

Corollary 3.2 says that the analytification functor is fully faithful, so (Coh_X) may be regarded as a full subcategory of (Coh_{X^h}). Natually, we want to know what exactly is the discrepancy between the two. The answer, which is the very core of GAGA, is "none".

Theorem 4.1. *Suppose X is a projective variety over $\mathbb{C}$. For every coherent $\mathcal{O}_{X^h}$-module $\mathcal{F}$, there is a coherent $\mathcal{O}_X$-module $\mathcal{M}$ such that $\mathcal{M}^h = \mathcal{F}$.*

That is, the analytification functor is an equivalence of categories between the algebraic and analytic coherent sheaves.
Again, we'll only prove Theorem 4.1 for $X = \mathbb{P}^n$ and leave the reduction argument to the reader. The same notation convention as in the previous section will be used. We want to make use of the same divide-and-conquer strategy, except this time it's a little bit harder, since we are working with analytic instead of algebraic coherent sheaves. Before we do anything else, we first start an induction on n so that we may assume Theorem 4.1 for hyperplanes in X.
The main body of the proof will be to derive an analytic analogue of Theorem 3.5, namely the following:

Theorem 4.2. *For any coherent $\mathcal{O}_{X^h}$-module $\mathcal{M}$, there is some $m_0 = m_0(\mathcal{M})$ such that $\mathcal{M}(m)$ is generated by global sections for all $m \geq m_0$.*

Inevitably, the proof of this requires some prerequisites in both complex geometry and projective algebraic geometry. The following two theorems will be blackboxed.

Theorem 4.3 (Cartan-Serre). *For any coherent $\mathcal{O}^h$-module $\mathcal{M}$, $H^i(\mathcal{M})$ is a finite-dimensional complex vector space.*

Proof. [GR04, p. 186]. □

Theorem 4.4 (Serre). *For any coherent $\mathcal{O}$-module $\mathcal{M}$ and any sufficiently large m, we have $H^i(\mathcal{M}(m)) = 0$ for all $i > 0$.*

Proof. [Ser55, §65, Prop. 7]. □

Proof of Theorem 4.2. By the proof of Lemma 3.6, if $\mathcal{F}$ is a coherent $\mathcal{O}_{X^h}$-module and $\Gamma(X^h, \mathcal{F})$ generates $\mathcal{F}_x$, then there is an open neighbourhood $U \ni x$ such that $\Gamma(X^h, \mathcal{F})$ also generates $\mathcal{F}_y$ for all $y \in U$. Hence, since X^h is quasicompact, it suffices to fix $x \in X$ and find $m_0 = m_0(x, \mathcal{M})$ such that $\Gamma(X^h, \mathcal{M}(m))$ generates $\mathcal{M}(m)_x$ for all $m \geq m_0$.
If $\Gamma(X^h, \mathcal{M}(m))$ generates $\mathcal{M}(m)_x$ then $\Gamma(X^h, \mathcal{M}(m'))$ generates $\mathcal{M}(m')_x$ for all $m' \geq m$. Indeed, let k be such that $x \in U_k$, then the multiplication by $(t_k/t_i)^{m'-m}$ on U_i induces a map $\mathcal{M}(m) \to \mathcal{M}(m')$ which is an isomorphism on U_k. So the image of $\Gamma(X^h, \mathcal{M}(m'))$ in $\mathcal{M}(m')_x \cong \mathcal{M}(m)_x$ contains the image of $\Gamma(X^h, \mathcal{M}(m))$, which already generates everything. Hence $\Gamma(X^h, \mathcal{M}(m'))$ generates $\mathcal{M}(m')_x$.
Therefore we have reduced the problem to finding one m such that $\Gamma(X^h, \mathcal{M}(m))$ generates $\mathcal{M}(m)_x$.
Choose a hyperplane E in X passing through x. We have the short exact sequence

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{O}^h \longrightarrow \mathcal{O}^h_E \longrightarrow 0$$

as per usual, where $\mathcal{I}_E \cong \mathcal{O}^h(-1)$ is the analytic sheaf of ideals for E. Tensoring with $\mathcal{M}$ (a right-exact operation) gives an exact sequence

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{M}(-1) \longrightarrow \mathcal{M} \longrightarrow \mathcal{B} \longrightarrow 0$$

where $\mathcal{C}$ is the kernel of $\mathcal{M} \otimes_{\mathcal{O}^h} \mathcal{I}_E \to \mathcal{M} \otimes_{\mathcal{O}^h} \mathcal{O}^h = \mathcal{M}$ and $\mathcal{B} = \mathcal{M} \otimes_{\mathcal{O}^h} \mathcal{O}^h_E$. We justify this blasphemous introduction of notation by observing that $\mathcal{B}$ and $\mathcal{C}$ are both coherent sheaves on E: Indeed, $\mathcal{I}_E\mathcal{B} = \mathcal{I}_E\mathcal{C} = 0$ by definition.
Seeking information about $\mathcal{M}(m)$, we twist the sequence above to obtain

$$0 \longrightarrow \mathcal{C}(m) \longrightarrow \mathcal{M}(m-1) \longrightarrow \mathcal{M}(m) \longrightarrow \mathcal{B}(m) \longrightarrow 0$$

This is not a short exact sequence, which is unfortunate since we are looking for cohomological information about $\mathcal{M}(m)$. But fear not – if we introduce $\mathcal{P}_m = \ker(\mathcal{M}(m) \to \mathcal{B}(m)) = \operatorname{coker}(\mathcal{C}(m) \to \mathcal{M}(m-1))$, then the sequence splits into

$$0 \longrightarrow \mathcal{C}(m) \longrightarrow \mathcal{M}(m-1) \longrightarrow \mathcal{P}_m \longrightarrow 0$$

$$0 \longrightarrow \mathcal{P}_m \longrightarrow \mathcal{M}(m) \longrightarrow \mathcal{B}(m) \longrightarrow 0$$

We first look at the pieces

$$H^1(\mathcal{M}(m-1)) \longrightarrow H^1(\mathcal{P}_m) \longrightarrow H^2(\mathcal{C}(m))$$

$$H^1(\mathcal{P}_m) \longrightarrow H^1(\mathcal{M}(m)) \longrightarrow H^1(\mathcal{B}(m))$$

in the long exact sequences. By the induction hypothesis, $\mathcal{B},\mathcal{C}$ comes from analytification of algebraic coherent sheaves (on the algebraic hyperplane corresponding to E). Combining this with Theorem 3.1 and Theorem 4.4, we see that both $H^1(\mathcal{B}(m))$ and $H^2(\mathcal{C}(m))$ vanish when m is large enough. From now on we only consider large m so that this holds.
We then have the inequalities

$$\dim H^1(\mathcal{M}(m-1)) \geq \dim H^1(\mathcal{P}_m) \geq \dim H^1(\mathcal{M}(m))$$

These dimensions are finite by Theorem 4.3. Therefore $m \mapsto \dim H^1(\mathcal{M}(m))$ is a nonincreasing sequence of integers, which must eventually stabilise. That is, for large enough m, the inequalities are equalities. We again discard small values of m and only consider those where this already happened.
Then $H^1(\mathcal{P}(m)) \to H^1(\mathcal{M}(m))$ is a surjective map of vector spaces of the same finite dimension, so it can only be an isomorphism. This means that

$$\Gamma(X^h, \mathcal{M}(m)) = H^0(\mathcal{M}(m)) \to H^0(\mathcal{B}(m)) = \Gamma(X^h, \mathcal{B}(m))$$

must be surjective. The induction hypothesis combined with Theorem 3.5 shows that, after further discarding small values of m, $H^0(\mathcal{B}(m)) = \Gamma(X^h, \mathcal{B}(m))$ generates $\mathcal{B}(m)_x$.
And now we are satisfied with the magnitude of m. We shall show that for the large values of m that's left, $\mathcal{M}(m)$ is generated by global sections. Set $A = \mathcal{O}_x^h$, $M = \mathcal{M}(m)_x$ and $\mathfrak{p} = (\mathcal{I}_E)_x$. Let N be the A-submodule of M generated by $\Gamma(X^h, \mathcal{M}(m))$. We have $\mathcal{B}(m)_x = \mathcal{M}(m)_x \otimes_A \mathcal{O}_{x,E}^h = M \otimes_A A/\mathfrak{p} = M/\mathfrak{p}M$. Since $\Gamma(X^h, \mathcal{M}(m))$ surjects to $\Gamma(X^h, \mathcal{B}(m))$, the image of N under $M \to M/\mathfrak{p}M$ generates $M/\mathfrak{p}M$.
This shows that $M = N + \mathfrak{p}M$, which implies $M = N + \mathfrak{m}M$ where $\mathfrak{m}$ is the maximal ideal of A. Nakayama's lemma (noting M is finitely generated since $\mathcal{M}$ is coherent) then gives $N = M$. □

To finish the proof of Theorem 4.1, we note that Theorem 4.2 and Lemma 3.6, combined with the proof of Corollary 3.7, indicate an exact sequence of the form

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{L}_0^h \longrightarrow \mathcal{F} \longrightarrow 0$$

in (Coh_{X^h}), where $\mathcal{L}_0$ is a direct sum of sheaves of the form $\mathcal{O}(N)$ (noting $\mathcal{O}^h(N) = \mathcal{O}(N)^h$). Doing the same with $\mathcal{R}$ gives a surjection $\mathcal{L}_1^h \to \mathcal{R}$ for some $\mathcal{L}_1$ which too is a direct sum of $\mathcal{O}(N)'s$. So we arrive at an exact sequence

$$\mathcal{L}_1^h \longrightarrow \mathcal{L}_0^h \longrightarrow \mathcal{F} \longrightarrow 0$$

Corollary 3.2 tells us that $\mathcal{L}_1^h \to \mathcal{L}_0^h$ comes from a map $\theta : \mathcal{L}_1 \to \mathcal{L}_0$. Theorem 2.2(i) then shows that $\mathcal{F} = \mathcal{M}^h$ where $\mathcal{M} = \operatorname{coker}\theta$.

5 Chow's Theorem

There are many uses of the GAGA principle. We give one of the applications of the GAGA principle for $\mathbb{P}^n$.

Theorem 5.1 (Chow's Theorem). *Suppose $X \subset (\mathbb{P}^n)^h$ is a closed subset which is locally analytic, in the sense that it's locally the vanishing locus of finitely many analytic functions. Then X is in fact algebraic, i.e. closed in the restriction of the Zariski topology on $\mathbb{P}^n$ to $(\mathbb{P}^n)^h$.*

Proof. The hypothesis means that the sheaf of ideals of X is coherent. Theorem 4.1 tells us that this sheaf is the analytification of an algebraic coherent sheaf whose support, which is Zariski-closed, intersects $(\mathbb{P}^n)^h$ at X. $\square$

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