Beyond Serre Vanishing

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0 Introduction

Let k be an infinite field and $\mathbb{P}^n = \mathbb{P}^n_k$ the projective space over it. In [Ser55, §55, §66], Serre proved the following important result about the cohomology of $\mathbb{P}^n$.

Theorem 0.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$,*

(i) There is some $r_0 = r_0(\mathcal{F})$ such that for any $r \geq r_0$, $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for all $i > 0$.

(ii) There is some $d_0 = d_0(\mathcal{F})$ such that for any $d \geq d_0$, $\mathcal{F}(d)$ is globally generated.

This theorem is important in the sense that it gives a very easy procedure to make a coherent sheaf on $\mathbb{P}^n$ "nice", namely by twisting it. Needless to say, it is incredibly powerful in projective geometry.

The purpose of this note, however, is to address an unsatisfactory aspect of Theorem 0.1, namely our lack of knowledge of what r_0 and d_0 actually are. This piece of information isn't really necessary for most applications of this theorem, but knowing it can often give precious insights into the geometry of $\mathcal{F}$, as we will see later.

We shall first calculate values of r_0 for $\mathcal{O}_Z$ when Z is a geometrically integral curve. As an application of this, we give a bound on the arithmetic genus of a geometrically integral curve by the degree of its embedding in projective space.

We will then introduce a quantity that turns out to simultaneously control r_0 and d_0 for general $\mathcal{F}$, namely the Castelnuovo-Mumford regularity. Properties of this can be used to show that each component of the Hilbert functor is a subfunctor of the Grassmannian functor – an important step in proving the representability of the former.

The infinitude assumption on k is not absolute: Since the majority of the results covered here are cohomological, most of them also holds for finite fields by a base-change argument. We made the assumption so that there is a hyperplane avoiding any finite set of points, which allows easy induction arguments on projective spaces.

1 Hilbert Polynomial

We recall the definition and several crucial properties of the Hilbert polynomial of a coherent sheaf on $\mathbb{P}^n$. The same theory extends easily to projective varieties in general.

Definition 1.1. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. The Hilbert function of $\mathcal{F}$ is the map

$$p_{\mathcal{F}} : m \mapsto \chi(\mathbb{P}^n, \mathcal{F}(m)) = \sum_{i=0}^{\infty} (-1)^i h^i(\mathbb{P}^n, \mathcal{F}(m))$$

For a closed subscheme $Z \subset \mathbb{P}^n$, we often write $p_Z = p_{\mathcal{O}_Z}$.

Being defined using Euler characteristic, $p_{\mathcal{F}}$ enjoys the same linearity property, namely that a short exact sequence of the form

$$0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

gives rise to the identity $p_{\mathcal{F}''} + p_{\mathcal{F}'} = p_{\mathcal{F}}$. In particular $p_{\mathcal{I}_Z} + p_Z = p_{\mathbb{P}^n}$ for any closed subscheme $Z \subset \mathbb{P}^n$, so one can pretty easily work out one from the other.

Theorem 1.1. *$p_{\mathcal{F}}$ is a polynomial.*

Proof. [Har77, III, Ex. 5.2]. $\square$

We therefore call $p_{\mathcal{F}}$ the Hilbert polynomial of $\mathcal{F}$. When $\mathcal{F} = \mathcal{O}_Z$, we call p_Z the Hilbert polynomial of Z.

The Hilbert polynomial encodes important geometric information.

Theorem 1.2. $\deg p_{\mathcal{F}} = \dim \operatorname{Supp} \mathcal{F}$. *In particular,* $\deg p_Z = \dim Z$.

Proof. [Ser55, §81]. $\square$

Corollary 1.3. *$p_{\mathcal{F}}$ has rational coefficient. In fact,* $(\deg p_{\mathcal{F}})! p_{\mathcal{F}}(T) \in \mathbb{Z}[T]$.

Proof. Lagrange interpolation. $\square$

Definition 1.2. The degree of a closed subscheme $Z \subset \mathbb{P}^n$ is the leading coefficient of $(\deg p_Z)! p_Z(T)$.

It's not hard to check that this notion of degree is consistent with the degree of a hypersurface as well as the degree of a curve.

Example 1.1. 1. If $Z = \mathbb{P}^n$, then

$$p_{\mathbb{P}^n}(T) = \chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(T)) = \binom{n+T}{n}$$

2. If Z is a hypersurface of defined by a homogenous polynomial of degree d, then

$$p_Z(T) = p_{\mathcal{O}_{\mathbb{P}^n}}(T) - p_{\mathcal{O}_{\mathbb{P}^n}(-d)}(T) = p_{\mathbb{P}^n}(T) - p_{\mathbb{P}^n}(T-d) = \binom{n+T}{n} - \binom{n+T-d}{n}$$

In particular, Z has degree d (duh!).

3. If Z is supported in dimension 0, then $p_Z = h^0(\mathbb{P}^n, \mathcal{O}_Z) = h^0(Z, \mathcal{O}_Z)$ is a non-negative integer. If Z is reduced, then this integer is simply the number of (distinct) points that Z consists of.

4. If Z is a geometrically integral curve whose hyperplane section has degree d, then it has degree d since $p_Z(T) - p_Z(T-1) = d$ by linearity. Consequently, $p_Z(T) = dT + 1 - g$ where $g = H^1(Z, \mathcal{O}_Z)$ is the arithmetic genus of Z.

2 Geometrically Integral Curves

We shall prove the following theorem:

Theorem 2.1. *Suppose C is a geometrically integral curve in $\mathbb{P}^n$ of degree d, then we can take $r_0 = d-2$, i.e. $H^i(\mathbb{P}^n, \mathcal{O}_C(r)) = H^i(C, \mathcal{O}_C(r)) = 0$ for all $r \geq d-2$ and $i > 0$.*

Corollary 2.2. *If $C \subset \mathbb{P}^n$ is a geometrically integral curve of degree d and genus g, then $g = (d-1)^2 - h^0(C, \mathcal{O}_C(d-2)) \leq (d-1)^2$.*

Proof. We have $h^0(C, \mathcal{O}_C(d-2)) = h^0(\mathbb{P}^n, \mathcal{O}_C(d-2)) = p_C(d-2) = d(d-2) + 1 - g = (d-1)^2 - g$. □

Since the theorem is cohomological, we may assume WLOG that k is algebraically closed. As we want a result that has something to do with the degree of the curve, a natural starting point would be to take a general hyperplane which should intersect C (but avoid its associated points) at a dimension 0 closed subscheme Z with $h^0(Z, \mathcal{O}_Z) = d$.

In view of the long exact sequence of cohomology, it's somewhat tempting to understand the space $H^0(Z, \mathcal{O}_Z(r))$ for various values of r. So let's do just that.

Lemma 2.3. *Suppose Z is a 0-dimensional closed subscheme of $\mathbb{P}^n$ with $h^0(Z, \mathcal{O}_Z) = d$. Then the map $F_r : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(Z, \mathcal{O}_Z(r))$ is surjective whenever $r \geq d-1$.*

Proof. Choose a hyperplane disjoint from Z. This is possible as k is infinite. We then dehomogenise the space by removing this hyperplane. Under this setting, F_r is surjective if and only if $f_r : P_r \to H^0(Z, \mathcal{O}_Z)$ is surjective, where P_r is the space of polynomials in n variables $x_1, \ldots, x_n$ (the coordinates on $\mathbb{A}^n$ obtained from dehomogenisation) whose degree is at most r.

Consider the chain $0 \leq f_0(P_0) \leq f_1(P_1) \leq \cdots$ of k-vector subspaces of $H^0(Z, \mathcal{O}_Z)$. Since $H^0(Z, \mathcal{O}_Z)$ is a finite k-vector space of dimension d, there is some $0 \leq i \leq d-1$

such that $f_i(P_i) = f_{i+1}(P_{i+1})$. Now, whenever $f_r(P_r) = f_{r+1}(P_{r+1})$, we must have

$$
\begin{aligned}
f_{r+2}(P_{r+2}) &= f_{r+1}(P_{r+1}) + \sum_{j=1}^{n} f_{r+1}(x_j) f_{r+1}(P_{r+1}) \\
&= f_{r+1}(P_{r+1}) + \sum_{j=1}^{n} f_{r+1}(x_j) f_r(P_r) \\
&= f_{r+1}(P_{r+1}) + \sum_{j=1}^{n} f_{r+1}(x_j P_r) = f_{r+1}(P_{r+1})
\end{aligned}
$$

So $f_i(P_i) = f_{i+1}(P_{i+1}) = f_{i+2}(P_{i+2}) = \cdots$. But we know that f_r is surjective for large enough r by Theorem 0.1, so we must in fact have $f_i(P_i) = f_{i+1}(P_{i+1}) = f_{i+2}(P_{i+2}) = \cdots = H^0(Z, \mathcal{O}_Z)$. Since $i \leq d-1$, this means that f_r is surjective whenever $r \geq d-1$. $\square$

Proof of Theorem 2.1. For each r, we have a short exact sequence

$$
0 \longrightarrow \mathcal{O}_C(r) \longrightarrow \mathcal{O}_C(r+1) \longrightarrow \mathcal{O}_Z(r+1) \longrightarrow 0
$$

giving the long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0(C, \mathcal{O}_C(r)) \arrow[r] & H^0(C, \mathcal{O}_C(r+1)) \arrow[r, "\epsilon"] & H^0(Z, \mathcal{O}_Z(r+1)) \arrow[dll, "\delta"] \\
& H^1(C, \mathcal{O}_C(r)) \arrow[r] & H^1(C, \mathcal{O}_C(r+1)) \arrow[rr] & & 0
\end{tikzcd}
$$

For any $r \geq d-2$, ϵ is surjective by the preceding lemma since F_{r+1} factors through it. So δ must the zero map for these values of r, giving an isomorphism $H^1(C, \mathcal{O}_C(r)) \cong H^1(C, \mathcal{O}_C(r+1))$. But we know from Theorem 0.1 that $H^1(C, \mathcal{O}_C(r)) = 0$ for sufficiently large r, so we must have $H^1(C, \mathcal{O}_C(r)) = 0$ for any $r \geq d-2$. $\square$

3 Castelnuovo-Mumford Regularity

Definition 3.1 (Castelnuovo-Mumford regularity). A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if $H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$ for all $i > 0$.

This is a bit of a weird definition – we are somehow interested in the vanishing of a "shifted diagonal" in the array of cohomology groups. The next proposition tells us that we haven't really left where we started from.

Proposition 3.1. *Suppose $\mathcal{F}$ is m-regular, then:*

(i) The natural map $H^0(\mathbb{P}^n, \mathcal{F}(d)) \otimes_k H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, \mathcal{F}(d+1))$ is a surjection for any $d \geq m$.

(ii) $H^i(\mathbb{P}^n, \mathcal{F}(d)) = 0$ *for any* $i > 0, d > m - i$. *Equivalently,* $\mathcal{F}$ *is* m'*-regular for any* $m' \geq m$.

(iii) $\mathcal{F}(d)$ *is globally generated for any* $d \geq m$.

In particular, m can be taken to be r_0 and d_0 simultaneously.

Proof. Theorem 0.1 combined with (i) implies (iii). We shall show (i) and (ii) together by induction on n. Since k is infinite, there is a hyperplane P avoiding every associated point of $\mathcal{F}$. We therefore have an exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_P \longrightarrow 0$$

where the second arrow is multiplication by the equation defining H. We twist it and take the long exact sequence of cohomology as usual. The part

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \longrightarrow H^i(P, \mathcal{F}_P(m-i)) \longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{F}(m-i-1))$$

gives the m-regularity of $\mathcal{F}_P$, thus the part

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{F}(m-i+1)) \longrightarrow H^i(P, \mathcal{F}_P(m-i+1))$$

gives the $(m+1)$-regularity of $\mathcal{F}$ by induction hypothesis. Repeating this shows that $\mathcal{F}$ is m'-regular for any $m' \geq m$, which is (ii). As for (i), we consider the commutative diagram

$$\begin{tikzcd}
H^0(\mathbb{P}^n, \mathcal{F}(d)) \otimes_k H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \arrow[r, "r_1"] \arrow[d, "s_1"'] & H^0(P, \mathcal{F}_P(d)) \otimes_k H^0(P, \mathcal{O}_P(1)) \arrow[d, "s_2"] \\
H^0(\mathbb{P}^n, \mathcal{F}(d+1)) \arrow[r, "r_2"'] & H^0(P, \mathcal{F}_P(d+1))
\end{tikzcd}$$

The induction hypothesis gives the surjectivity of s_2, and r_1 is surjective by (ii). These give the surjectivity of s_1 since $\ker r_2 \subset \operatorname{Im} s_1$, hence completing the proof. $\square$

What's good in introducing a weird quantity? Perhaps it gives rise to a nice theorem.

Theorem 3.2. *For any polynomial* p, *there is an integer* $m_0 = m_0(p)$ *such that* $\mathcal{I}_Z$ *is* m_0*-regular whenever it has Hilbert polynomial* p.

So one can control both r_0 and d_0 knowing only the Hilbert polynomial. We will see in a minute why this is very useful. But first, let's prove it.

Proof. Induction on n again (surprise surprise). Take a general hyperplane P and consider the short exact sequence

$$0 \longrightarrow \mathcal{I}(-1) \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}_P \longrightarrow 0$$

where $\mathcal{I} = \mathcal{I}_Z$ and $\mathcal{I}_P = (\mathcal{I}_Z)_P$. By induction hypothesis, there is some m_1 (WLOG $m_1 > 1$), depending only on $p_{\mathcal{I}_P}(T) = p(T) - p(T-1)$, such that $\mathcal{I}_P$ is m_1-regular. For $i > 1$ and $d \geq m_1 - i$, we have $H^{i-1}(P, \mathcal{I}_P(d+1)) = H^i(P, \mathcal{I}_P(d+1)) = 0$, therefore $H^i(\mathbb{P}^n, \mathcal{I}(d)) \cong H^i(\mathbb{P}^n, \mathcal{I}(d+1))$. Theorem 0.1 then shows that $H^i(\mathbb{P}^n, \mathcal{I}(d)) = 0$ whenever $i > 1$ and $d \geq m_1 - i$.

This is sadly not enough: We do not necessarily have the vanishing of $H^1(\mathbb{P}^n, \mathcal{I}(m_1 - 1))$, and it may be necessary to take some even larger $m_0 \geq m_1$ to rectify this problem. How do we control how large it should be? The claim is that the sequence $\{h^1(\mathbb{P}^n, \mathcal{I}(m))\}_{m \geq m_1 - 1}$ in fact strictly decreases to zero.

Indeed, the vanishing of $H^1(P, \mathcal{I}_P(m))$ for $m \geq m_1 - 1$ gives the exact sequence

$$H^0(\mathbb{P}^n, \mathcal{I}(m)) \xrightarrow{g_m} H^0(P, \mathcal{I}_P(m)) \longrightarrow H^1(\mathbb{P}^n, \mathcal{I}(m-1)) \longrightarrow H^1(\mathbb{P}^n, \mathcal{I}(m)) \longrightarrow 0$$

whenever $m \geq m_1 - 1$. This shows that the sequence is nonincreasing and that, for $m \geq m_1$, $h^1(\mathbb{P}^n, \mathcal{I}(m-1)) = h^1(\mathbb{P}^n, \mathcal{I}(m))$ if and only if g_m is surjective. But the diagram we've seen before

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{F}(m)) \otimes_k H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \longrightarrow & H^0(P, \mathcal{F}_P(m)) \otimes_k H^0(P, \mathcal{O}_P(1)) \\ \downarrow & & \downarrow \\ H^0(\mathbb{P}^n, \mathcal{F}(m+1)) & \longrightarrow & H^0(P, \mathcal{F}_P(m+1)) \end{array}$$

indicates that the surjectivity of g_m must imply the surjectivity of g_{m+1}. Theorem 0.1 then tells us that such a situation can only occur if we already have $h^1(\mathbb{P}^n, \mathcal{I}(m-1)) = h^1(\mathbb{P}^n, \mathcal{I}(m)) = \cdots = 0$, leading to the claim.

We therefore know that any $m_0 \geq m_1 + h^1(\mathbb{P}^n, \mathcal{I}(m_1 - 1))$ would work. Now let's pick one that depends only on p.

We have the bound $h^1(\mathbb{P}^n, \mathcal{I}(m_1 - 1)) \leq h^0(\mathbb{P}^n, \mathcal{O}_Z(m_1 - 1))$ from the vanishing of $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_1 - 1))$ (recall $m_1 > 1$). On the other hand, all higher cohomology groups of $\mathcal{O}_Z(m_1 - 1)$ vanishes since $h^i(\mathbb{P}^n, \mathcal{I}(m_1 - 1))$ vanishes for $i \geq 2$. Therefore

$$h^1(\mathbb{P}^n, \mathcal{I}(m_1 - 1)) \leq h^0(\mathbb{P}^n, \mathcal{O}_Z(m_1 - 1)) = p_Z(m_1 - 1) = \binom{n + m_1 - 1}{n} - p(m_1 - 1)$$

which gives a value of m_0 depending only on p. □

What's the point of all these? Well, suppose we have a Hilbert polynomial p of a sheaf of ideal on $\mathbb{P}^n$. Let $m = m_0 = m_0(p)$ be as in the lemma. For any sheaf of ideal $\mathcal{I}$ (defining a closed subscheme Z, say) with $p_{\mathcal{I}} = p$, the vanishing of $H^1(\mathbb{P}^n, \mathcal{I}(m))$ means that $H^0(\mathbb{P}^n, \mathcal{O}_Z(m))$ is a quotient of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$. Moreover, we have $h^0(\mathbb{P}^n, \mathcal{O}_Z(m)) = p_Z(m) = \binom{n+m}{n} - p(m)$. Call this dimension $D(m)$.

We therefore have a map from the set of closed subschemes whose ideal have Hilbert polynomial p to $\mathrm{Grass}^{D(m)}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)))(k)$, by sending $\mathcal{I}$ to $H^0(\mathbb{P}^n, \mathcal{O}_Z(m))$. The map is furthermore injective since $\mathcal{I}(m)$ is generated by global sections.

The way our theorem made its crucial contribution in this construction is that it ensures the existence of some m that works as r_0 and d_0 for all the ideals with prescribed Hilbert polynomial.

The same argument can be easily globalised (where we consider instead a flat family of closed subschemes with a fixed Hilbert polynomial). And some standard definition yoga translates this result to the statement that the Hilbert functor with fixed Hilbert polynomial is a subfunctor of a Grassmannian, a very important step towards showing the existence of Hilbert schemes. Detailed discussions of the existence of Hilbert schemes is beyond the scope of this note. Interested readers may confer [FGI+05, Ch. 5]

References

[FGI+05] Barbara Fantechi, Lothar Gottsche, Luc lllusie, Steven Lawrence Kleiman, Michael G. Eas twood, Michael Loss, and Angelo Vistoli. *Fundamental algebraic geometry: Grothendieck's FGA explained.* Mathematical surveys and monographs ; no. 123. American Mathematical Society, Providence, R.I., 2005.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[Ser55] Jean-Pierre Serre. Faisceaux algebriques coherents. *Annals of Mathematics*, 61(2):197–278, 1955.