Twenty-Seven Lines on a Smooth Cubic Surface

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0 Introduction

Algebraic geometry is the study of geometrical objects defined by polynomial equations. The rigidity of polynomials means that we can often find interesting combinatorial phenomena in this setting, e.g. "two distinct lines on the plane must either be parallel or intersect at a unique point".
One of the first nontrivial combinatorial results in this field is the following theorem, first discovered by A. Cayley and G. Salmon in [Cay49] (in the case $k = \mathbb{C}$):

Theorem 0.1. *Let k be an algebraically closed field with $\operatorname{char} k \neq 2$. Then any smooth cubic surface in $\mathbb{P}^3_k$ contains exactly 27 lines.*

This document sketches a proof of this theorem. We'll assume that the reader is familiar with the (classical) concept of projective spaces, as well as properties of linear and quadratic forms.
As tempting as it is, we'll *not* make use of the language of schemes.

1 Preliminaries

From here on out we abbreviate $\mathbb{P}^n = \mathbb{P}^n_k$. We'll write $(X : Y : Z : W)$ to denote the homogeneous coordinates on $\mathbb{P}^3$.

Definition 1.1. A cubic surface in $\mathbb{P}^3$ is a subset $S \subset \mathbb{P}^3$ of the form

$$S = \{(X : Y : Z : W) \in \mathbb{P}^3 : G(X, Y, Z, W) = 0\}$$

where G is a homogeneous polynomial of degree 3.
We say S is smooth if, in addition, G can be chosen to be irreducible with $\nabla G|_p \neq 0$ at any $p \in S$.

Remark. The notion of smoothness presented here is not quite how one might define smoothness in general. It is nonetheless sufficient for our purpose.

The expression of S is usually denoted by $\mathbb{V}(G)$. In general, if $f_1, \dots, f_m$ is a set of homogeneous polynomials in $X_0, \dots, X_n$, the homogenous coordinates on $\mathbb{P}^n$, we write

$$\mathbb{V}(f_1, \dots, f_m) = \{(X_0 : \cdots : X_n) \in \mathbb{P}^n : \forall j, f_j(X_0, \dots, X_n) = 0\}$$

It's clear that $\mathbb{V}(f_1, \dots, f_m) = \bigcap_j \mathbb{V}(f_j)$ and that $\mathbb{V}(f_1 \cdots f_m) = \bigcup_j \mathbb{V}(f_j)$.
We can describe lines and planes using this notation.

Definition 1.2. A plane in $\mathbb{P}^3$ is a subset $P \subset \mathbb{P}^3$ of the form $P = \mathbb{V}(L)$ for a nonzero linear form L. A line in $\mathbb{P}^3$ is a subset $\ell \subset \mathbb{P}^3$ of the form $\ell = \mathbb{V}(L_1, L_2)$ for some independent linear forms L_1, L_2.

We make three immediate observations about lines and planes in $\mathbb{P}^3$.

Proposition 1.1. *Suppose* $P = \mathbb{V}(\lambda Z - \mu W)$ *(where* $\lambda, \mu \in k$ *are not both zero) is a plane in* $\mathbb{P}^3$*, then there exists a bijection* $b : \mathbb{P}^2 \to P$ *given by* $(X : Y : T) \mapsto (X : Y : \mu T : \lambda T)$ *(where* $(X : Y : T)$ *is the homogeneous coordinates on* $\mathbb{P}^2$*). Furthermore,* b *induces a bijection between lines on* $\mathbb{P}^2$ *and lines in* $\mathbb{P}^3$ *contained in* P.

Remark. There is nothing special about planes of the form $\mathbb{V}(\lambda Z - \mu W)$. In fact, this is part of the next observation.

Proposition 1.2. $\mathrm{GL}_{n+1}(k)$ *acts on various collections of geometric objects in* $\mathbb{P}^n$ *by its action on* k^{n+1}*. This action is transitive in the following cases:*
(i) The action of $\mathrm{GL}_4(k)$ *is transitive on:*

- *The set of points in* $\mathbb{P}^3$.
- *The set of lines in* $\mathbb{P}^3$.
- *The set of planes in* $\mathbb{P}^3$.
- *The set of triplets of concurrent, non-coplanar lines in* $\mathbb{P}^3$.

(ii) The action of $\mathrm{GL}_3(k)$ *is transitive on:*

- *The set of points in* $\mathbb{P}^2$.
- *The set of lines in* $\mathbb{P}^2$.
- *The set of pairs of distinct lines in* $\mathbb{P}^2$.
- *The set of triplets of distinct concurrent lines in* $\mathbb{P}^2$.
- *The set of triplets of non-concurrent lines in* $\mathbb{P}^2$.

Remark. Such an action is essentially a change of coordinates. The proposition means that we can usually change our coordinates until the lines and planes we are interested in have nice expressions.

Proposition 1.3. *Any two planes in* $\mathbb{P}^3$ *intersect; any two lines in* $\mathbb{P}^2$ *intersect; and therefore, in* $\mathbb{P}^3$*, every line intersects every plane.*

The proofs are clear.

2 From One to Twenty-Seven

In this section we prove the folowing:

Theorem 2.1. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. If S contains one line, then it contains exactly 27 lines.*

The basic idea is the following: For every line ℓ contained in S, we can choose a plane P containing it, and ℓ would sit inside the intersection $P \cap S$. So we should be able to extract information about the lines in S by studying the intersections of the form $P \cap S$ where P is a plane.

Suppose ℓ is a line on S and P a plane containing it. By Proposition 1.2(i), we can assume without loss of generality that $\ell = \mathbb{V}(Z, W)$. So $P = \mathbb{V}(\lambda Z - \mu W)$ for some $\lambda, \mu \in k$, not both zero.

We found ourselves in the situation of Proposition 1.1. In particular, under the bijection stated in the proposition, $P \cap S \subset P$ would correspond to $\mathbb{V}(g) \subset \mathbb{P}^2$ where $g(X, Y, T) = G(X, Y, \mu T, \lambda T)$. To study lines in $\mathbb{P}^3$ contained in $P \cap S$ is then the same as to study lines in $\mathbb{P}^2$ contained in $\mathbb{V}(g)$, i.e. linear factors of g.

For S to contain ℓ, every monomial appearing in G must contain either Z or W, since $G(X, Y, 0, 0)$ have to be identically zero. This means that we can write G in the form

$$G = AX^2 + BXY + CY^2 + DX + EY + F$$

where $A, B, C, D, E, F \in k[Z, W]$ are homogenous polynomials of appropriate degree. Consequently, $g = Tq$ where q is the quadratic form

$$q = aX^2 + bXY + cY^2 + dTX + eTY + fT^2$$

where $a = A(\mu, \lambda), b = B(\mu, \lambda)$, and so on. So $\mathbb{V}(g) = \mathbb{V}(T) \cup \mathbb{V}(q) = \ell \cup \mathbb{V}(q)$.

If q is irreducible, then $\mathbb{V}(q)$ does not contain any lines. If q is reducible, g would factorise as the product of three linear factors. Our hope would be for the factors to define distinct lines, i.e. cases like $g = TX^2, g = T^3$ do not occur. This is indeed true and uses the smoothness of S. We dedicate the following lemma to this purpose:

Lemma 2.2. *Suppose S is a cubic surface and P is any plane in $\mathbb{P}^3$ such that $S \cap P$ is the union of at most 2 lines. Then S is not smooth.*

Proof. Suppose $S = \mathbb{V}(G)$. Proposition 1.2(i) allows us to assume, without loss of generality, that $P = \mathbb{V}(W)$. If the said situation occurs, then $G(X, Y, Z, 0) = L_1^2 L_2$ for some linear forms L_1, L_2 in X, Y, Z. Proposition 1.2(ii) then allows us to assume $L_1 = Z$.

Then $G = Z^2 G_1 + W G_2$ for some $G_1, G_2 \in k[X, Y, Z, W]$ where G_1 is a linear form and G_2 a quadratic form. Choose $(X_0, Y_0) \neq (0, 0)$ with $G_2(X_0, Y_0, 0, 0) = 0$, then $\nabla G|_p = 0$ where $p = (X_0 : Y_0 : 0 : 0) \in S$. □

How would one, then, decide when q is reducible? Since it is a quadratic form, it's reducible precisely when its discriminant

$$\Delta = \Delta(\mu, \lambda) = \det \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

vanishes. Now, Δ is a homogenous polynomial in μ, λ, and is a sum of homogenous polynomials of degree 5. If it were identically zero, then by completing squares we conclude that F must be reducible, which should not happen as S is smooth. Therefore Δ is a homogenous polynomial of degree 5 in μ, λ, and hence it has at most 5 solutions $(\mu : \lambda) \in \mathbb{P}^1$. In fact, it has exactly 5 solutions, i.e. the roots of Δ are simple. This is again due to the smoothness of S.

Lemma 2.3. *Δ only has simple roots.*

Proof. By Proposition 1.2(i), it suffices to show that if $(1:0)$ is a root of Δ, then it is simple. Suppose this is the case. The root $(\mu : \lambda) = (1:0)$ corresponds to the plane $P = \mathbb{V}(W)$. Using Proposition 1.2(ii), we may assume without loss of generality that one of two situations occur (recall that $T \mid g$):
(a) $g(X,Y,T) = G(X,Y,T,0) = XYT$.
(b) $g(X,Y,T) = G(X,Y,T,0) = XT(X-T)$.
(a): $G(X,Y,Z,W) = XYZ + WG_0$ for some quadratic form G_0. So λ must divide a,c,d,e,f, so we obtain $4\Delta \equiv -b^2 f \equiv -\mu^2 f \pmod{\lambda^2}$. It then suffices to show that $\lambda^2 \nmid f$, or equivalently $W^2 \nmid F$. But if $W^2 \mid F$, then (noting $W \mid A,C,D,E$ since $\lambda \mid a,c,d,e$) we have $\nabla G|_p = 0$ where $p = (0:0:1:0) \in S$, contradiction.
(b): $G(X,Y,Z,W) = XZ(X-Z) + WG_0$ for some quadratic form G_0. So $\lambda \mid b,c,e,f$, consequently $4\Delta \equiv -cd^2 \equiv -\mu^4 c \pmod{\lambda^2}$. Like in case (a), it's then reduced to show that $W^2 \nmid C$. This is equivalent to $C = 0$ since otherwise C is linear. But if $C = 0$ then $p = (0:1:0:0) \in S$ yet $\nabla G|_p = 0$. □

What we've got so far can be packaged into the following proposition:

Proposition 2.4. *Suppose ℓ is a line contained in a smooth cubic surface S. If a plane P contains ℓ, then $P \cap S$ either contains only one line (namely ℓ) or contains exactly three distinct lines. Furthermore, the latter happens for exactly 5 choices of P.*

Start with any line ℓ_1 in S and any plane P containing the line such that $P \cap S = \ell_1 \cup \ell_2 \cup \ell_3$ is the union of three lines. For each i, there are a total of $2 \times 4 = 8$ lines in S other than ℓ_1, ℓ_2, ℓ_3 sharing a plane with it (or, equivalently by Proposition 1.3, intersecting it) by the proposition. This gives us a total of $8 \times 3 + 3 = 27$ lines. The only remaining thing to do is to show that they do exhaust all the lines in S, and that we did not double-count.
For any line ℓ contained in S, $\ell \cap P \neq \varnothing$ by Proposition 1.3. So $\ell \cap \ell_i \neq \varnothing$ for some i since $\ell \subset S$. Hence every line contained in S is one of our 27 lines.
Suppose there is a line ℓ intersecting both ℓ_1 and ℓ_2, we shall show that it must be contained in P. Indeed, suppose not, then ℓ, ℓ_1, ℓ_2 are concurrent but not coplanar, therefore by Proposition 1.2 we may assume $\ell = \mathbb{V}(X,Y), \ell_1 = \mathbb{V}(Y,Z), \ell_2 = \mathbb{V}(X,Z)$. So every monomial in G has at least two of X,Y,Z. But then $\nabla G|_p = 0$ where $p = (0:0:0:1) \in S$, contradiction.
Therefore we did not double-count, and Theorem 2.1 is proved.

3 From Zero to One

To prove Theorem 0.1, it remains to show that any smooth cubic surface contains a line. In fact, every cubic surface contains a line.

Showing this turns out to be the hardest part of the proof. Although elementary arguments do exist (e.g. [Rei88, p. 110]), we have decided to include the sketch of a proof that's, strictly speaking, not elementary.

We'll only give the heuristics instead of the full proof, but hopefully it will provide the reader with some insights into these kinds of arguments. The proof in full generality can be found in most introductory books to scheme theory, e.g. [Mum76, p. 174].

There are a total of 20 monomials of degree 3 in 4 variables; and any two set of monomial coefficients determine the same cubic surface if and only if they can be related by a nonzero constant. This establishes a one-to-one correspondence between the set $\mathcal{C}$ of cubic surfaces and $\mathbb{P}^{19}$.

On the other hand, the set $\mathcal{G}$ of lines in $\mathbb{P}^3$ is "a geometric object of dimension 4". We will not go into what this means precisely, but it shouldn't be hard to see that there are 4 degrees of freedom in choosing a line in $\mathbb{P}^3$: One has 3 degrees of freedom in choosing a plane, hence 6 degrees of freedom in choosing two planes. Intersections of two planes are mostly lines, and there are 2 degrees of freedom in choosing two planes passing through the same (fixed) line. Hence $\mathcal{G}$ should have dimension $6-2=4$.

Now consider

$$X = \{(\ell, S) \subset \mathcal{G} \times \mathcal{C} : \ell \subset S\}$$

We have the projection maps $\pi : X \to \mathcal{G}$ and $\sigma : X \to \mathcal{C}$.

For each $\ell \in \mathcal{G}$, the fibre $\pi^{-1}(\ell)$ is the set of all cubics containing ℓ, which has dimension 15. Indeed, every fibre has the same dimension by Proposition 1.2, so we can assume $\ell = \mathbb{V}(Z, W)$. Then $\pi^{-1}(\ell)$ consists of all cubics not having Z^3, Z^2W, ZW^2, W^3 as monomials, hence has dimension $19-4=15$.

π is also surjective since there exists cubic surfaces containing lines (exercise: find all 27 lines on the Fermat cubic $S = \mathbb{V}(X^3+Y^3+Z^3+W^3)$). So we've got a surjective map to something of dimension 4, with the fibres having dimension 15. Just like in the rank-nullity theorem in linear algebra (or the preimage theorem in differential geometry), this should, on a good day, indicate $\dim X = 15+4 = 19$. So let's assume that this is the case.

Let's now look at σ. One observation is that it is a map of equidimensional objects: $\dim X = 19 = \dim \mathcal{C}$. Assuming that σ has equidimensional fibres, then the dimension of the fibres must be 0 since the Fermat cubic contains only finitely many lines. If our analogy with rank-nullity theorem works again (or, if one wish, one can also think about the fact that open mappings between compact Hausdorff spaces are surjective), then σ should be surjective, which exactly means that every cubic surface contains a line.

4 Final Remarks

One can obviously ask whether smooth surfaces of higher degrees too contains a fixed number of lines. Sadly,

Theorem 4.1. *A surface of degree $d > 3$ generally contains no line.*

This can be proved using the same kind of dimension-counting argument we used in the last section. The word "generally" is important here: There certainly exist smooth surfaces of higher degree containing lines, e.g. $\mathbb{V}(X^d + Y^d + Z^d + W^d)$. The theorem means that "most" surfaces of degree d don't contain lines. More precisely, (if the reader knows some algebraic geometry) the set of degree $d > 3$ surfaces containing no lines is Zariski-dense in $\mathbb{P}^{N-1}$ where $N = \binom{d+3}{3}$ is the number of degree d monomials in 4 variables.
This is not the end of the story, one can still ask the following question: Suppose a degree d surface does contain finitely many lines, is the number of lines bounded? The answer turns out to be positive. In fact, if we let N_d to be the maximum number of lines on a degree d surface with finitely many lines, then it's shown in [Seg43] that

Theorem 4.2 (Segre). $N_4 = 64$, $N_d \leq (d-2)(11d-6)$.

References

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