$\mathcal{D}$-Modules and Crystals

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Fix a field k of characteristic 0. By a $\mathcal{D}$-module over a smooth scheme X, we will always mean a *left* $\mathcal{D}_X$-module over X that is quasicoherent over $\mathcal{O}_X$.

0 Motivation

Let X be a smooth variety over k. Usually, $\mathcal{D}$-module on X are defined as modules over the noncommutative $\mathcal{O}_X$-algebra $\mathcal{D}_X = \varinjlim_n \mathcal{D}_X^{\leq n}$. Formulated this way, it is geometrically concrete in the sense that explicit computations are usually possible. However, we (I at least) still seek another way to think about them.

Such pursuit was fueled by the feeling that some parts of the theory appear to, well, "not arise from nature". This causes some insufficiency of intuition, especially around Kashiwara's lemma. The sheaf of differential operators $\mathcal{D}_X$ is defined in quite a "practical" way, namely by collecting everything that looks like a differential operator. Kashiwara's lemma, in a shocking way however, reveals intristic compatibility between modules over this ring and closed immersions – which is not even found in modules over $\mathcal{O}_X$! One is then lead to suspect that something is going on behind the scenes.

Another reason of trying to describe $\mathcal{D}$-modules in a separate, more "natural" way is the case of singular varieties. Using Kashiwara's lemma, one can define the category of $\mathcal{D}$-modules over singular varieties roughly as follows: Suppose X is an affine variety. Then it admits a closed embedding $i : X \to \mathbb{A}^m$ for some m. We then define a $\mathcal{D}$-module on X to be a $\mathcal{D}$-module on $\mathbb{A}^m$ supported in X. The resulting category is independent of the choice of this closed embedding: Indeed, suppose $j : X \to \mathbb{A}^n$ is another closed embedding. Then the identity on X extends to morphisms $I : \mathbb{A}^n \to \mathbb{A}^m$ and $J : \mathbb{A}^m \to \mathbb{A}^n$, and the diagram of closed embeddings

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & \mathbb{A}^m \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \mathrm{graph}_J} \\ \mathbb{A}^n & \xrightarrow[\mathrm{graph}_I]{} & \mathbb{A}^{m+n} \end{array}$$

commutes. So Kashiwara's lemma shows that i and j gives the same category of $\mathcal{D}$-modules on X. A gluing argument is required for non-affine X.

But this is a horribly extrinsic construction: One has to choose local embeddings into ambient spaces, and glue them back. We want to find a way to obtain the same data in an intrinsic way.

The construction that I will talk about, namely that of a *crystal*, provides an intrinsic description of $\mathcal{D}$-modules that resolves these issues to some extent. It has the additional advantage (or disadvantage, depending on how you look at it) of having quite a simplistic definition.

The core idea of the construction is the following: One might think of a $\mathcal{D}$-module as a quasicoherent sheaf E equipped with a flat connection $E \to E \otimes \Omega_X$. If we are in the differential-geometric setting and E happens to be a vector bundle, then this data gives rise to the notion of parallel transport along curves in X. And there is a way to recover the connection from any suitable notion of parallel transport.

This of course cannot be taken word-for-word in the algebraic context. After all, the differential-geometric construction of parallel transport is way too transcendental. Nonetheless, we are inspired to ask the following question: If, by some miracle, we are given an algebraic way to move between infinitesimally close points on X, does that give us an equivalent way of describing $\mathcal{D}$-modules?

The answer is yes.

1 Quasicoherent Sheaves on the de Rham Space

If one accepts the use of abstract nonsense, then crystals can be defined in quite a simplistic way.

Definition 1.1. Let X be a space (i.e. a functor $(\mathsf{CommAlg}/k) \to (\mathsf{Sets})$). A generalised quasicoherent sheaf $\mathcal{F}$ on X is the assignment that takes each $u \in X(R)$ to an R-module $\mathcal{F}(u)$, and each homomorphism $\phi : R \to R'$ to a family of isomorphisms $\alpha_{\phi,u} : \mathcal{F}(u) \otimes_R R' \cong \mathcal{F}(X(\phi)(u))$ indexed by $u \in X(R)$ (note that $X(\phi)(u)$ is an element of $X(R')$).

These isomorphisms are required to be natural in the sense that for any $\phi : R \to R'$, $\psi : R' \to R''$ and $u \in X(R)$, the diagram

$$\begin{array}{ccc} (\mathcal{F}(u) \otimes_R R') \otimes_{R'} R'' & \xrightarrow{\cong} & \mathcal{F}(u) \otimes_R R'' \\ {\scriptstyle \alpha_{\phi,u} \otimes_{R'} R''} \downarrow & & \downarrow {\scriptstyle \alpha_{\psi \circ \phi, u}} \\ \mathcal{F}(X(\phi)(u)) \otimes_{R'} R'' & \xrightarrow[\alpha_{\psi, X(\phi)(u)}]{} & \mathcal{F}(X(\psi \circ \phi)(u)) \end{array}$$

commutes.

Any such thing is, in particular, a sheaf on the site $\operatorname{Spec}(k)_{\mathrm{Zar}}/X$. This is due to the fact that, for any R-module M, the rule that assigns each affine open $\operatorname{Spec} A \subset \operatorname{Spec} R$ the A-module $M \otimes_R A$ gives rise to a sheaf on $\operatorname{Spec} R$ (which is just $\tilde{M}$). So affine-locality arguments are valid.

Example 1.1. If X was a scheme, then a generalised quasicoherent sheaf is really just a quasicoherent sheaf. Indeed, if we are given a quasicoherent sheaf $\mathcal{F}$, then we can assign, for any $u : \operatorname{Spec} R \to X$, the R-module corresponding to the quasicoherent $\mathcal{O}_{\operatorname{Spec} R}$-module $u^*\mathcal{F}$. Conversely, if we have a generalised quasicoherent sheaf on X, then its values on affine opens of X gives a sheaf $\mathcal{F}$ on X, which is quasicoherent because $\mathcal{F}(\operatorname{Spec} A)_f = \mathcal{F}(\operatorname{Spec} A) \otimes_A A_f = \mathcal{F}(\operatorname{Spec} A_f)$ for any affine open $\operatorname{Spec} A \subset X$. It is easy to check that pullbacks of $\mathcal{F}$ recovers the values of the orginal generalised quasicoherent sheaf.

We will write $\operatorname{Nil}(R)$ to denote the nilradical of R.

Definition 1.2. Let X be a space. The de Rham space (otherwise known as the de Rham stack in literatures) associated to X is the space X^{dR} sending each k-algebra R to $X(R/\operatorname{Nil}(R))$.

Example 1.2. If $X = \mathbb{A}^1$, then $X^{\mathrm{dR}}(R) = R/\operatorname{Nil}(R)$.

Definition 1.3. A crystal on X is a generalised quasicoherent sheaf on X^{dR}.

To draw connections to our motivation, let's try to decode this definition into more familiar terms in the situation where X is actually a smooth scheme. In this case, recall:

Lemma 1.1. *Suppose X is a smooth scheme over k. Then $X(R) \to X(R/\operatorname{Nil}(R))$ is surjective for any commutative k-algebra R.*

So $X(R/\operatorname{Nil}(R))$ is obtained from R by gluing together "infinitesimally close" R-points. Let's make a definition out of it.

Definition 1.4. For $z \in X(R)$, we write $\bar{z}$ for the image of z under $X(R) \to X(R/\operatorname{Nil}(R))$. Two R-points $u, v \in X(R)$ are called infinitesimally close if $\bar{u} = \bar{v}$.

Example 1.3. Suppose $X = \mathbb{A}^1$ and consider $u, v \in X(k[\epsilon]/(\epsilon^2))$, where u comes from the ring map $k[x] \to k[x]/(x^2) = k[\epsilon]/(\epsilon^2)$ and v from the composition $k[x] \to k[x]/(x) = k \to k[\epsilon]/(\epsilon^2)$. Then $u \neq v$ but $\bar{u} = \bar{v}$.

A crystal $\mathcal{F}$ on X gives rise to a quasicoherent sheaf on X, which we shall temporarily call $\mathcal{F}^\dagger$. This is constructed in the following way: For any $u \in X(R)$, let $u^\dagger \in X^{\mathrm{dR}}(R)$ be the element corresponding to $\bar{u}$. Then set $\mathcal{F}^\dagger(u) = \mathcal{F}(u^\dagger)$. It is easy to check that $\mathcal{F}^\dagger$ is a generalised quasicoherent sheaf on X.

This is an instance of a more general notion of pullback one can define. Suppose $\theta : Y \to Z$ is a morphism of spaces and $\mathcal{F}$ is a generalised quasicoherent sheaf on Z, then one may define a quasicoherent sheaf on Y by assigning to $u \in Y(R)$ the module $\mathcal{F}(\theta_R(u))$.

Since X is a scheme, $\mathcal{F}^\dagger$ is actually just an honest quasicoherent sheaf. An affine-locality argument shows that we have a natural isomorphism $\mathcal{F}^\dagger(u) \cong \mathcal{F}(u)$ for any nilpotent-free R and $u \in X(R)$.

For any $u, v \in X(R)$ infinitesimally close, we have an isomorphism $\eta_{u,v} : \mathcal{F}^\dagger(u) \to \mathcal{F}^\dagger(v)$ (the "parallel transport") given by the sequence of identifications $\mathcal{F}^\dagger(u) = \mathcal{F}(u^\dagger) = \mathcal{F}(v^\dagger) = \mathcal{F}^\dagger(v)$. It obviously has the following properties (PT):

(PT0) $\eta_{u,u} = \mathrm{id}$.

(PT1) Suppose $\phi : R \to S$ is a ring homomorphism and $u, v \in X(R)$ are infinitesimally close, then $\eta_{X(\phi)(u),X(\phi)(v)} = \eta_{u,v} \otimes_R 1_S$.

(PT2) Suppose $u, v, w \in X(R)$ are infinitesimally close, then $\eta_{u,w} = \eta_{v,w} \circ \eta_{u,v}$.

Conversely, suppose we have such a quaiscoherent sheaf $\mathcal{F}^\dagger$ and a collection $\eta_{u,v}$ of maps satisfying (PT) (note that combining (PT0) and (PT2) shows that each η is an isomorphism). We can recover the quasicoherent sheaf $\mathcal{F}$ on X^{dR} by simply setting $\mathcal{F}(u) = \mathcal{F}^\dagger(\tilde{u})$ where $\tilde{u}$ is any lifting of $u \in X^{\mathrm{dR}}(R) = X(R/\operatorname{Nil}(R))$ to $X(R)$ (exists by Lemma 1.1).

We hence obtain a more "geometric" definition of crystals.

Definition 1.5. Suppose X is a smooth scheme. A crystal on X is a quasicoherent sheaf $\mathcal{F}$ on x together with maps $\eta_{u,v} : \mathcal{F}(u) \to \mathcal{F}(v)$ for every pair of infinitesimally close u, v satisfying (PT).

2 An Equivalence of Categories

From now on, we fix a smooth scheme X.

Theorem 2.1. *The category of crystals on X is equivalent to the category of $\mathcal{D}_X$-modules.*

Why is this nice? Well, it first of all gives intuition for some classical results in the theory of $\mathcal{D}$-modules. Take Kashiwara's lemma as an example. It fails in the $\mathcal{O}_X$-module case precisely because of nilpotents: Indeed, every module has a scheme-theoretic support, but that need not be the reduced induced subscheme. Removing nilpotents, which is what X^{dR} is doing, can be seen as getting rid of this issue, so Kashiwara's lemma should hold true. Indeed, Kashiwara's lemma does hold for crystals, and this result does reduce to the usual Kashiwara's lemma under the equivalence in Theorem 2.1.

Another nice thing is that the de Rham space can be defined for any scheme, not necessarily smooth. So one can consider quasicoherent sheaves over the de Rham space of a singular variety and hope it behaves well enough to give a satisfactory theory of $\mathcal{D}$-modules. In fact, the category of $\mathcal{D}$-modules on a (possibly singular) variety X constructed using Kashiwara's lemma is equivalent to the category of quasicoherent sheaves on X^{dR}.

Finally, if one replace "de Rham space" by "crystalline space", one can use this idea to define arithmetic $\mathcal{D}$-modules. But that is much much more involved.

Remark. What we defined are technically called *left* crystals. There is also a mirror theory of *right* crystals, where we replace *-pullbacks by $^!$-pullbacks, generalised quasicoherent sheaves by ind-coherent sheaves, and Theorem 2.1 becomes an equivalence between right crystals and right $\mathcal{D}_X$-modules.

Enough info-dumping, let's now turn to the proof, which uses a "de-synthesis" argument: Starting with Definition 1.5, we will rewrite the characterisation of a crystal into more concrete terms, until we reach a stage where some easy computations yield what we want.

Proof. Let's start with decoding what being infinitesimally close actually means. Two R-points $u, v \in X(R)$ we infinitesimally close if and only if $(u, v) \circ \operatorname{Spec}(R \to R/\operatorname{Nil}(R))$ factors through $\Delta_X \hookrightarrow X \times X$.

In general, a morphism $f : X \to Z$ factors through a closed subscheme $Y \hookrightarrow Z$ if and only if the inverse image ideal $f^*\mathcal{I}_Y \cdot \mathcal{O}_X$ is zero (this corresponds to the fact that a ring map $\phi : A \to B$ factors through A/I if and only if $\phi(I)B = 0$). Hence u, v are infinitesimally close if and only if the ideal generated by $(u, v)^*\mathcal{I}_{\Delta_X}$ is contained in $\operatorname{Nil}(R)$.

But X is a smooth scheme over k (hence locally Noetherian), so this happens exactly when $(u, v)^*\mathcal{I}_{\Delta_X}^{n+1}$ generates the zero ideal in R for large enough n, which in turn means that (u, v) factors through $X^{(n)}$, the closed subscheme corresponding to $\mathcal{I}_{\Delta_X}^{n+1}$.

To give a collection η of parallel transports satisfying (PT1), we then need to find a compatible family of morphisms $\pi_{1,n}^*\mathcal{F} \to \pi_{2,n}^*\mathcal{F}$ (known as "n-connections"), where $\pi_{i,n} : X^{(n)} \to X$ are the restrictions of the first and second projections. One can also consider the completion $\hat{\Delta} = \varinjlim X^{(n)}$. Giving such a compatible family is the same as giving a morphism $\pi_1^*\mathcal{F} \to \pi_2^*\mathcal{F}$ where $\pi_i : \hat{\Delta} \to X$ are again the projections.

Now, giving $\pi_{1,n}^*\mathcal{F} \to \pi_{2,n}^*\mathcal{F}$ is the same as giving $\mathcal{F} \to (\pi_{1,n})_*\pi_{2,n}^*\mathcal{F}$. On the level of topological spaces, $\pi_{1,n}$ and $\pi_{2,n}$ both restrict to the homeomorphism $\Delta_X \to X$, so $(\pi_{1,n})_*\pi_{2,n}^*$ is really not that scary: It is simply $\mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} -$, where $\mathcal{O}_{X^{(n)}}$ denotes any of the two abstractly isomorphic $\mathcal{O}_X$ modules $(\pi_{1,n})_*\mathcal{O}_{X^{(n)}}$, $(\pi_{2,n})_*\mathcal{O}_{X^{(n)}}$.

So the data of a crystal can be described as the data of a compatible family of morphisms $\mathcal{F} \to \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}$. It's finally time for concrete computations. Recall that $\mathcal{D}_X$ is an $(\mathcal{O}_X, \mathcal{O}_X)$-bimodule. So for any left $\mathcal{O}_X$-module $\mathcal{M}$, $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is naturally a left $\mathcal{O}_X$-module.

The key idea in this theory is a perfect pairing $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(n)}} \to \mathcal{O}_X$. Let's first discuss in details what happens when $X = \mathbb{A}^m$ (i.e. "in an étale neighbourhood") before constructing the pairing in general. In this case, the pairing is defined as follows: Any $\partial \in D_{\mathbb{A}^m} = \Gamma(\mathcal{D}_{\mathbb{A}^m})$ may be regarded as an operator on $\Gamma(\mathcal{O}_{\mathbb{A}^m \times \mathbb{A}^m})$ via "partial differentiation on the first variable". Any $g \in \Gamma(\mathcal{O}_{X^{(n)}})$ may be regarded as a function in $\Gamma(\mathcal{O}_{\mathbb{A}^m \times \mathbb{A}^m})$ modulo $(x_1 - y_1, \ldots, x_m - y_m)^{n+1}$. The pairing takes any $(\partial, g) \in D_{\mathbb{A}^m}^{\leq n} \times \mathcal{O}_{X^{(n)}}$ to $\partial\tilde{g}(x, x) \in \Gamma(\mathcal{O}_{\mathbb{A}^m})$ for any lifting $\tilde{g}$ of g. This is obviously well-defined and perfect (recall the dual basis constructed in past seminars).

For a general affine X, the case is entirely similar. When X is affine, $(\pi_1)_*$ is exact, so $\mathcal{O}_{X^{(n)}}$ is a quotient of $(\pi_1)_*\mathcal{O}_{X\times X}$ with kernel $(\pi_1)_*\mathcal{I}^{n+1}$. Now $\mathcal{D}_X$ acts on the first component of $(\pi_1)_*\mathcal{O}_{X\times X} = \mathcal{O}_X \otimes_k \mathcal{O}_X$. Compose the action $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} (\pi_1)_*\mathcal{O}_{X\times X} \to (\pi_1)_*\mathcal{O}_{X\times X}$ with $(\pi_1)_*\mathcal{O}_{X\times X} \to (\pi_1)_*(\Delta_X)_*\mathcal{O}_X = \mathcal{O}_X$ gives an $\mathcal{O}_X$-linear $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} (\pi_1)_*\mathcal{O}_{X\times X} \to \mathcal{O}_X$ whose right radical contains $(\pi_1)_*\mathcal{I}^{n+1}$. Therefore we obtain a pairing $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(n)}} \to \mathcal{O}_X$.

This pairing is functorial in affine schemes since (as we have proved in past seminars) the construction of $\mathcal{D}_X$ is functorial. So, like $\mathcal{D}_X$, these pairings glue nicely over affines and give a global perfect pairing. This establishes $\mathcal{O}_X$-duality between (the right $\mathcal{O}_X$-module structure on) $\mathcal{D}_X$ and (the left $\mathcal{O}_X$-module structure on) $\mathcal{O}_{X^{(n)}}$.

Therefore the data of a morphism $\mathcal{F} \to \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}$ is the same as the data of a morphism $\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$. The data of a compatible family of the former is then the same as the data of a morphism $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$. Looks familiar?

Now, (PT0) amounts to the fact that $1 \in \mathcal{D}_X$ acts as the identity, and (PT2) is the compatibility between the multiplicaion on $\mathcal{D}_X$ and composition of the action, i.e. the commutativity of the diagram

$$\begin{array}{ccccc} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{F} & = & \mathcal{D}_X \otimes_{\mathcal{O}_X} (\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) & \longrightarrow & \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \\ \downarrow & & & & \downarrow \\ \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} & & \longrightarrow & & \mathcal{F} \end{array}$$

This completes the proof. □